STATIONARY AND NON-GAUSSIAN DISTRIBUTIONS

$$X_{t_i} = \begin{cases} \text{return } (\log S_{t_{i+1}} - \log S_{t_i}), \text{ or} \\ \text{volatility } (\sigma^2_{t_i}), \text{ or} \\ \text{interest rate } (r_{t_i}), \text{ etc} \end{cases}$$

Stationary distribution of X_{t_i}:

$$\begin{aligned} F(x) &= P(X_{t_i} \le x) \quad \text{if same for all } i \\ \text{or } &= \lim_{n \Rightarrow \infty} \frac{1}{n} \sum I(X_{t_i} \le x) \quad \text{if meaningful} \\ \text{or } &= \lim_{n \Rightarrow \infty} P(X_{t_i} \le x) \quad \text{if meaningful} \end{aligned}$$

Returns of Geometric Brownian motion:

$$X_{t_i} = \mu \Delta t_i + \sigma \Delta W_{t_i} \sim N(\mu \Delta t, \sigma^2 \Delta t)$$

Hence

$$F = N(\mu \Delta t, \sigma^2 \Delta t)$$

If

$$d \log S_t = \mu dt + \sigma dW_t$$

then

$$\frac{\log S_t}{t} \Rightarrow \mu$$

$\log S_t$, S_t has no stationary distribution.

The Vasicek (or Ornstein-Uhlenbeck) Model

$$dr_t = K(\alpha - r_t)dt + \sigma dW_t$$

Set $u_t = r_t - \alpha$, get

$$du_t = -Ku_t dt + \sigma dW_t$$

Itô:

$$\begin{aligned} de^{Kt}u_t &= e^{Kt}du_t + Ke^{Kt}u_t dt \\ &= e^{Kt}\sigma dW_t \end{aligned}$$

$$\begin{aligned} e^{Kt}u_t - e^{Ks}u_s &= \sigma^2 \int_s^t e^{Ku} dWu \\ &= N\left(0, \sigma^2 \int_s^t e^{2Ku} du\right) \\ &= N\left(0, \frac{\sigma^2}{2K}(e^{2Kt} - e^{2Ks})\right) \end{aligned}$$

or

$$u_t = e^{-K(t-s)}u_s + N\left(0, \frac{\sigma^2}{2K}(1 - e^{-2K(t-s)})\right)$$

Stationary distribution: $t \to \infty$

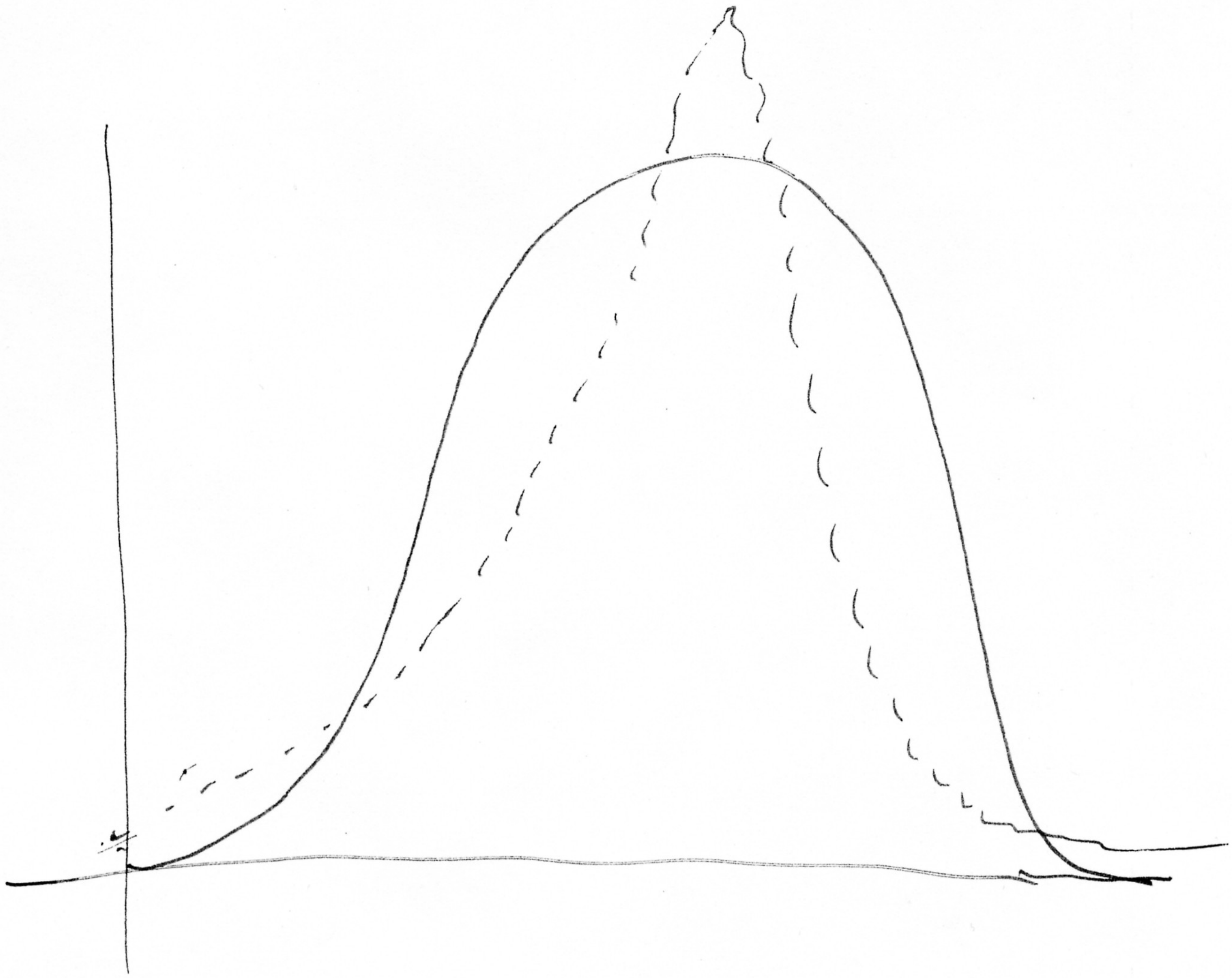
$$u_t \to N\left(0, \frac{\sigma^2}{2K}\right)$$

Also,

$$\frac{1}{n}\sum_{i=1}^{n} I(u_{t_i} \leq x) \sim N\left(0, \frac{\sigma^2}{2K}\right)(x)$$

In practice, to get stationary density $f = F'$:

- Collect X_{t_i} over many periods $t_i \to t_{i+1}$
- Make histogram of result



APPROACHES TO NON GAUSSIAN STATIONARY DISTRIBUTIONS

- Jumps
- State dependent drift volatility — for example:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

Stationary density

$$f(x) = C\frac{1}{\sigma(x)^2}\exp\left(2\int_{x_0}^{x}\frac{\mu(y)}{\sigma^2(y)}dy\right)$$

- Stochastic volatility — for example:

$$dX_t = \mu_t dt + \sigma_t dW_t$$
$$d\sigma_t^2 = K(\alpha - \sigma_t^2)dt + \gamma dB_t$$

perhaps correlation between dW, dB

Jump Processes

The simplest case: the Poisson process.

N_t is a *Poisson process with rate* λ if:

- N_t only evolves by jumping (and stays still between jumps), and the jump size is always 1
- $N_t - N_s$ is independent of $\mathcal{F}_u$ for $t \geq s \geq u$
- $N_0 = 0$
- $E(N_t) = \lambda t$

Some main properties of a Poisson process:

- $N_t - \lambda t$ is a martingale
- $P(\text{jump in } (t, t+\delta) \mid \mathcal{F}_t) = \lambda\delta + o_p(\delta)$
- $P(\text{more than one jump in } (t, t+\delta) \mid \mathcal{F}_t) = o_p(\delta)$
- the distribution of $N_t - N_s$ given $\mathcal{F}_u$ is Poisson$(\lambda(t-s))$:

$$P(N_t - N_s = k \mid \mathcal{F}_u) = \frac{((\lambda(t-s))^k}{k!} \exp\{-\lambda(t-s)\}$$

A frequent stock price model: the compound Poisson process:

$$\log S_t - \log S_0 = \sum_{i=1}^{N_t} Z_i$$

More generally: jump-diffusion

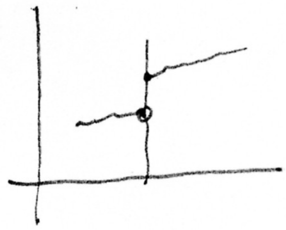
$$X_t = X_0 + \mu t + \sigma W_t + \sum_{i=1}^{N_t} Z_i$$

The Z's, N, W are independent

Itô's formula

$$df(X_t,t) = f'_x(X_{t-},t)dX_t + f'_t(X_{t-},t)dt$$
$$+ \tfrac{1}{2} f'_x(X_{t-},t)d[X^c,X^c]_t$$
$$+ \Delta f(X_t) - f'_x(X_{t-})\Delta X_t$$

$$X_t^c = \text{ continuous part of } X_t\text{, say,}$$
$$= \mu t + \sigma W_t$$
$$X_{t-} = \lim_{s \uparrow t} X_s$$
$$\Delta X_t = X_t - X_{t-}$$
$$= \text{ jump at time } t\text{, say,}$$
$$= Z_i \quad \text{if} \quad N_{t-} = i-1, N_t = i$$
$$dX_t = dX_t^c + \Delta X_t$$



where N_t is a Poisson process, and the Z_i are i.i.d. and independent of N_t.

Itô's formula S_t scale: $S_t = \exp(X_t)$

$$\Delta \log S_t = \begin{cases} Z_i & \text{if jump} \\ 0 & \text{otherwise} \end{cases}$$

Hence, if jump:

$$\begin{aligned}
\Delta S_t &= S_t - S_{t-} \\
&= S_{t-} \exp(Z_i) - S_{t-} \\
&= (\exp(Z_i) - 1) S_{t-} \\
d \exp(X_t) &= \exp(X_{t-}) dX_t + \tfrac{1}{2} \exp(X_{t-}) d[X, X]_t \\
&\quad + \Delta \exp(X_t) - \exp(X_{t-}) \Delta X_t \\
dS_t &= S_t dX_t + \tfrac{1}{2} S_{t-} \sigma^2 dt \\
&\quad + \Delta S_t - S_{t-} \Delta X_t \\
&= S_{t-} dX_t^c + \tfrac{1}{2} S_{t-} \sigma^2 dt + \Delta S_t \\
\frac{\Delta S_t}{S_{t-}} &= dX_t + \tfrac{1}{2} \sigma^2 dt + \frac{\Delta S_t}{S_t} - \Delta X_t
\end{aligned}$$

$$
\begin{aligned}
df(S_t,t) &= f'_x(S_{t-},t)dS_t \qquad \text{hedgable} \\
&+ (f'_t(S_{t-},t) + \tfrac{1}{2}f''_{ss}(S_{t-1}t)\sigma^2)dt \qquad \text{hedgable} \\
&+ \Delta f(S_t,t) - f'_s(S_{t-},t)\Delta S_t
\end{aligned}
$$

last line

$$
\begin{aligned}
&= f(S_t,t) - f(S_{t-},t) - f'_s(S_{t-},t)\Delta S_t \\
&= \tfrac{1}{2}f''_{ss}(S_{t-},t)\Delta S_t^2 + \ldots \qquad (1)
\end{aligned}
$$

not hedgable unless $f''_{ss}(s,t) = 0$ or unless

$$\Delta S_t = Z = \text{ fixed}$$

in which case

$$(1) = \tfrac{1}{2}f''_{ss}(S_{t-},t)Z\Delta S_t + \frac{1}{3!}f'''_{sss}(S_{t-},t)Z^2\Delta S_t + \ldots$$

In this case

$$\begin{aligned} df(S_t, t) = &\ \text{hedgable terms} \\ &+ g(S_{t-}, t)\Delta S_t \end{aligned}$$

If two options $f_i(S_t, t)$, $i = 1, 2$

$$df_1(S_t, t) - \frac{g_1(S_{t-}, t)}{g_2(S_{t-}, t)} df_2(S_t, t)$$
$$= dS_t + dt \text{ terms.}$$

One option completes market.

In general: p diffusions, q jump sizes:

need $p + q - 1$ options to complete market.

Variation processes for jump processes. For processes of this type, $\langle \cdot, \cdot \rangle$ is not the same as $[\cdot, \cdot]$. We use the following definitions: $[X, \cdots, X]$ is given by

$$\underbrace{[X, \cdots, X]}_{p \text{ times}}{}_t = \lim \sum (X_{t_{i+1}} - X_{t_i})^p$$

$$= \begin{cases} X_t \text{ if } p = 1 \\ \lim \sum (X_{t_{i+1}} - X_{t_i})^2 \text{ if } p = 2 \\ \sum \Delta X_u^p \text{ if } p > 2 \end{cases}$$

On the other hand, $\langle X, \cdots, X \rangle$ is the predictable component in the Doob-Meyer decomposition of $[X, \cdots, X]$:

$$[X, \cdots, X] = \underbrace{\langle X, \cdots, X \rangle}_{\text{predictable}} + \text{martingale} .$$

Important examples of variation processes. For the Poisson process,

$$[N, \cdots, N]_t = N_t$$

and

$$\langle N, \cdots, N \rangle_t = \lambda t.$$

For the compound Poisson:

$$[\log S, \cdots, \log S]_t = \sum_{i=1}^{N_t} Z_i^p$$

and

$$\langle \log S, \cdots, \log S \rangle_t = E(Z^p)\lambda t.$$

From Itô's formula:

$$df(X_t, t) = f'(X_{t-})dX_t + \tfrac{1}{2} f''(X_{t-}d[X, X]_t$$
$$+ \frac{1}{3'} f'''(X_{t-})d[X, X, X]_t + \ldots$$