Uncertain Volatility and Interest

\( \Lambda_t = \) zero coupon bond, \( \Lambda_T = 1 \)

\( \tilde{S}_t = S_t / \Lambda_t \)

\( d\tilde{S}_t = \mu_t \tilde{S}_t dt + \sigma_t \tilde{S}_t dW_t \)

Suppose

\[ \sigma^- \leq \sigma_t \leq \sigma^+ \] (1)

(Pointwise bound assumption) or

\[ \equiv^- \leq \int_0^T \sigma_t^2 dt \leq \equiv^+ \] (2)

(Integral bound assumption)

What is the “ask” price \( A \) for payoff

\[ \eta = f(S_T) = f(\tilde{S}_T) \]
Case: Pointwise bounds

\[ \sigma^- \leq \sigma_t \leq \sigma^+ \quad (1) \]

(Avellaneda, Levy, Paras (1995), Lyons (1995)).

Let \( V(s, t) \) solve

\[
\begin{cases}
    V_t(s, t) + \frac{1}{2} s^2 \max_{(1)} (\sigma^2 V_{SS}(s, t)) = 0 \\ 
    V(s, T) = f(s) 
\end{cases}
\]

(3): Berenblatt equation

Ito + (3)

\[
dV(\tilde{S}_t, t) = V_s(\tilde{S}, t)d\tilde{S}_t + V_t(\tilde{S}, t)dt + \frac{1}{2} \tilde{S}_t^2 \sigma_t^2 V_{SS}(\tilde{S}_t, t)dt
\]
\[
= V_S(\tilde{S}, t)d\tilde{S}_t - d\tilde{D}_t
\]

\[
d\tilde{D}_t = -(V_t(\tilde{S}, t) + \frac{1}{2} \tilde{S}_t \sigma_t^2 V_{SS}(\tilde{S}, t))dt
\]
\[
= \frac{1}{2} \tilde{S}_t^2 \max_{(1)} [\sigma^2 V_{SS}(\tilde{S}_t, t)] - \sigma_t^2 V_{SS}(\tilde{S}_t, t))dt 
\geq 0
Interpretation 1:

\[ V(\tilde{S}_t, t) = \text{super-replication of } f(S_T) \]

\[ D_t = \text{dividend} \]

Interpretation 2:

\[ V(\tilde{S}_t, t) = P^* - \text{supermartingale} \]

\[ V(\tilde{S}_T, T) \geq f(S_T) \]

In any case: \( \tilde{A} \leq V(\tilde{S}_0, 0) \)

However, if

\[ \sigma_t^2 = \begin{cases} 
(\sigma^+)^2 & \text{if } V_{SS}(\tilde{S}_t, t) \geq 0 \\
(\sigma^-)^2 & \text{if } V_{SS}(\tilde{S}_t, t) < 0 
\end{cases} \]

(one choice of \( P^* \) under (1))

then \( D_t \equiv 0 \), so \( V(\tilde{S}_t, t) \) exact replication,

so \( \tilde{A} \geq V(\tilde{S}_0, 0) \)

Conclusion:

\[ \tilde{A} = V(\tilde{S}_0, 0) \]

\( f \) convex: use \( V(s, t) = \text{BS with } \sigma = \sigma^+ \)

\( f \) concave: use \( V(s, t) = \text{BS with } \sigma = \sigma^- \)
Case: Integral bounds (2)

Recall:

\( \Lambda_t = \) zero coupon bond, maturity \( T \)

\( \tilde{S}_t = S_t / \Lambda_t \)

\[ d\tilde{S}_t = \mu_t \tilde{S}_t dt + \sigma_t \tilde{S}_t dW_t \]

Suppose

\[ \equiv^- \leq \int_0^T \sigma_u^2 du \leq \equiv^+ \] (2)

What is the ask price \( A \) for payoff?

\[ \eta = f(S_T) = f(\tilde{S}_T) \]

\[ A = \sup_{P^* \sim (2)} E^* \Lambda_0 f(\tilde{S}_T) \]

Computation of \( A \):

Under \( P^* \sim (2) \):

\[ d\tilde{S}_t = \sigma_t \tilde{S}_t dW_t^* \]

Or

\[ d\log \tilde{S}_t = -\frac{\sigma_t^2}{2} dt + \sigma_t dW_t^* \]
We set
\[ \tau_t = \int_0^t \sigma_u^2 du \]
(require \( t \to \tau_t \) to be 1-1 \( (\sigma_u^2 > 0 \ \text{a.s.}) \)) and define \( \hat{S} \) by
\[ \hat{S}_t = \hat{S}_{\tau_t} \]
Define \( M_t = \int_0^t \sigma_u dW^*_u = \hat{M}_{\tau_t} \)
If \( \mathcal{F}_t = \hat{\mathcal{F}}_{\tau_t} \), then
\( \hat{M}_t \) is \((\hat{\mathcal{F}}_t) - \text{MG} \) (show below)
Also
\[ [\hat{M}, \hat{M}]_{\tau_t} = [M, M]_t = \int_0^t \sigma_u^2 du = \tau_t \]
so
\[ [\hat{M}, \hat{M}]_t = t \Rightarrow \hat{M}_t \text{ is std BM } \cdot (\hat{\mathcal{F}}_t) \]
\[
\log \hat{S}_{\tau_t} = \log \hat{S}_t \\
= \log \hat{S}_0 - \int_0^t \frac{\sigma_u^2}{2} du + \int_0^t \sigma_u dW^*_u \\
= \log \hat{S}_0 - \frac{1}{2} \tau_t + \hat{M}_{\tau_t}
\]
or
\[
\log \hat{S}_t = \log \hat{S}_0 - \frac{1}{2} t + \hat{M}_t
\]
or
\[ d\hat{S}_t = \hat{S}_t d\hat{M}_t \]
Why is \( (\hat{M}_t) \) an \( (\hat{\mathcal{F}}_t) \) – MG?

Set \( \lambda_t : \tau_{\lambda_t} = \lambda_{\tau_t} = t \)

If \( t \to \tau_t \) is 1 - 1
then \( \lambda_t \) is continuous

since \( M_t = \hat{M}_{\tau_t} \), get \( \hat{M}_t = M_{\lambda_t} \)

by \( \mathcal{F}_t = \hat{\mathcal{F}}_{\tau_t} \) we mean \( \hat{\mathcal{F}}_t = \mathcal{F}_{\lambda_t} \)

Use optional stopping:

\[
E(\hat{M}_t \mid \hat{\mathcal{F}}_s) = E(M_{\lambda_t} \mid \mathcal{F}_{\lambda_s})
\]
\[
= M_{\lambda_s}
\]
\[
= \hat{M}_s
\]

Recall

\[
\mathcal{F}_\lambda = \{ A \in \mathcal{F} : \forall t, \ A \cap \{ \lambda \leq t \} \in \mathcal{F}_t \}
\]
Back to original problem:

\[
\tilde{S}_T = \hat{S}_{\tau_T} \quad \tau_T = \int_0^T \sigma_u^2 \, du
\]

\[P^* \sim (2) \iff \equiv^- \leq \int_0^T \sigma_u^2 \, du \leq \equiv^+
\]

\[\iff \equiv^- \leq \tau_T \leq \equiv^+
\]

It follows:

\[A = \sup_{P^* \sim (2)} \Lambda_0 E^* f(\tilde{S}_T)
\]

\[= \sup_{\equiv^- \leq \tau_T \leq \equiv^+} \Lambda_0 E^* f(\hat{S}_{\tau_T})
\]

Also, \(\hat{S}\) is the same for all \(P^*\)

\[\hat{S}_0 = \tilde{S}_0 = \frac{1}{\Lambda_0} S_0 \quad d\hat{S}_t = \hat{S}_t d\hat{M}_t
\]

If \(f = \text{ convex, Jensen's inequality } \Rightarrow \)

\[E[f(\hat{S}_{\equiv^+}) \mid \mathcal{F}_{\tau_T}] \geq f(\hat{S}_{\tau_t}) \quad \text{since } \tau_T \leq \equiv^+
\]

so

\[A = \Lambda_0 E^+ f(\hat{S}_{\equiv^+}) = BS(S_0, -\log \Lambda_0, \equiv^+)
\]

If \(f = \text{ concave}:

\[E[f(\hat{S}_{\tau_T}) \mid \mathcal{F}_{\equiv^-}] \geq f(\hat{S}_{\equiv^-})
\]

\[A = \Lambda_0 E^* f(\hat{S}_{\equiv^-}) = BS(S_0, -\log \Lambda_0, \equiv^-)
\]
Integral bound + traded options?

Ask price $A$ for payoff $f(\tilde{S}_T) = (\tilde{S}_T - K)^+$ under call prices

$$\Lambda_0 E^*(\tilde{S}_T - K_i)^+ = C_0^{(i)} \quad i = 1, \ldots, p \quad (5)$$

Set

$$h_\lambda(s) = (s - K)^+ + \sum_{i=1}^{P} \lambda_i [(s - K_i)^+ - \tilde{C}_0^{(i)}]$$

$$A \leq \inf_{\lambda} \sup_{P^* \sim (2)} \Lambda_0 E^* h_\lambda(\tilde{S}_T)$$

since one can replicate $h_\lambda(\tilde{S}_T)$ instead of $f(\tilde{S}_T)$. Any $\lambda \Rightarrow \sup$ over $P^*$ satisfying (5):

$$A \leq \sup_{P^* \sim (2),(5)} \Lambda_0 E^* (\tilde{S}_T - K)^+$$

Some regularity conditions: $A =$ this supremum
Time change:

\[ A = \sup \Lambda_0 E^* (\hat{S}_\tau - K)^+ \]

where \( \tau = \) stopping time:

\[ \equiv^- \leq \tau \leq \equiv^+ \] \hspace{1cm} (2)'

\[ \Lambda_0 E^* (\hat{S}_\tau - K_i)^+ = C_0^{(i)} \] \hspace{1cm} (5)'

\( \hat{S} \) as before: \( d\hat{S}_t = \hat{S}_td\hat{M}_t \)

If \( \lambda \) solves problem (in place of \( \tau \)), set

\[ \tau = \inf \{ t \geq \lambda \ \hat{S}_t = K_i \} \wedge \equiv^+, \]

then

\[ E^* [(\hat{S}_\tau - K_i)^+ | \mathcal{F}_\lambda] = (S_\lambda - K_i)^+ \]

and (Jensen)

\[ E^* (\hat{S}_\tau - K)^+ \geq E^* (\hat{S}_\lambda - K)^+ \]

Can take

\[ \hat{S}_\tau = K_i \] for some \( i \), or \( \tau = \equiv^+ \)
Simple case: all calls have some implied volatility $\equiv$:

$$\Lambda_0 E^* (\hat{S}_{\Xi} - K_i)^+ = C_0^{(i)}$$

Can take

$$\lambda = \Xi$$

This gives optimal $A$. 