HEDGING: THE EXACT CASE

SETUP: ON DISCOUNTED SCALE

$\widetilde{S}_t$: stock price $\widetilde{\eta}$: payoff at T $\widetilde{C}_t$: price at t

SELF FINANCING STRATEGY (SFS):

$$\widetilde{\eta} = \widetilde{C}_T$$

$$d\widetilde{C}_t = \theta_t d\widetilde{S}_t \quad \theta_t = \text{ “delta”}$$

QUADRATIC VARIATION $[\cdot, \cdot]_t$:

$$d[\widetilde{C}, \widetilde{S}]_t = \theta_t d[\widetilde{S}, \widetilde{S}]_t$$

OR:

$$\theta_t = \frac{d[\widetilde{C}, \widetilde{S}]_t}{d[\widetilde{S}, \widetilde{S}]_t} \tag{$\widetilde{1}$}$$

CONCLUSION: IF YOU KNOW HOW TO CALCULATE OPTIONS PRICES $\widetilde{C}_t$, AND IF SFS EXISTS, THEN YOU CAN CALCULATE θ_t.

WHAT DO YOU GET FROM ($\widetilde{1}$) IF SFS DOES NOT EXIST? NEXT PAGE ON ORIGINAL SCALE

NUMERAIRE INVARIANCE: SFS IS

$$\eta = C_T$$

$$dC_t = \theta_t^{(0)} d\Lambda_t + \theta_t^{(1)} dS_t \tag{1}$$

$$C_t = \theta_t^{(0)} \Lambda_t + \theta_t^{(1)} S_t \tag{2}$$

IF $\Lambda_t = \exp\left(\int_0^t r_u du\right) =$ MONEY MARKET BOND:

Λ_t IS dt TERM, SO

$$dC_t = \theta_t^{(1)} dS_t + dt \text{ TERM}$$

OR: $$\theta_t^{(1)} = \frac{d[C,S]_t}{d[S,S]_t}$$

$\theta_t^{(0)}$ IS GIVEN BY (2)

SINCE

$$C_t = e^{rt}\widetilde{C}_t, \quad \text{OR} \quad dC_t = e^{rt} d\widetilde{C}_t + dt \text{ TERM}$$

$$S_t = e^{rt}\widetilde{S}_t, \quad \text{OR} \quad dS_t = e^{rt} d\widetilde{S}_t + dt \quad \text{TERM}$$

ONE OBTAINS

$$d[C,S] = (e^{rt})^2 d[\widetilde{C},\widetilde{S}] \quad d[S,S] = (e^{rt})^2 d[S,S]_t$$

WHENCE

$$\theta_t^{(1)} = \frac{(e^{rt})^2 d[\widetilde{C},\widetilde{S}]_t}{(e^{rt})^2 d[\widetilde{S},\widetilde{S}]_t} = \theta_t \quad \text{FROM } (\widetilde{1})$$

AS BEFORE

BLACK-SCHOLES MODEL:

$$dS_t = rS_t dt + \sigma S_t dB_t$$

UNDER P^*

$\Lambda_t =$ MONEY MARKET BOND, EUROPEAN OPTIONS

$$C_t = f(S_t, T-t) = \text{B-S formula}$$

$$dC_t = f'_S(S_t, T-t)dS_t + dt \text{ terms}$$

so $d[C,S]_t = f'_S(S_t, T-t)d[S,S]_t$

OR $\theta_t = \dfrac{d[C,S]_t}{d[S,S]_t} = f'_S(S_t, T-t)$

= THE REGULAR B-S DELTA

SHOOTING SPARROWS WITH CANNON

AN ASIAN LIABILITY

$$\eta = \int_0^T g(Su)du$$

TAKE $r = 0$:

$$\begin{aligned} C_t &= E\left(\int_0^T g(S_u)du \mid \mathcal{F}_t\right) \\ &= E\left(\int_0^t g(S_u)du \mid \mathcal{F}_t\right) + E\left(\int_t^T g(S_u)du \mid \mathcal{F}_t\right) \\ &= \int_0^t g(S_u)du + \int_t^T f(S_t, u-t)du \end{aligned}$$

WHERE $f(s,t) =$ BS price at zero for payoff $g(S_t)$ at t.

SET $h(s,t) = \int_0^t f(s,u)du$:

$$C_t = \int_0^t g(S_u)du + h(S_t, T-t)$$

THEN $dC_t = h'_S(S_t, T-t) + dt -$ TERMS

$$\theta_t = \frac{d[C,S]}{d[S,S]_t} = h'_S(S_t, T-t)$$

MORE INTERESTING: A LOOKBACK OPTION

$$\eta = (M_T - K)^+ \quad \text{WHERE} \quad M_t = \max_{0 \le u \le t} S_u$$

$$C_t = e^{-r(T-t)} E^*(\eta \mid \mathcal{F}_t) = \text{ PRICE AT } t$$

IN FALL, WE LEARNED (?) TO CALCULATE C_0 FOR FIXED r, σ^2, SET

$$C_0 = f(s_0, K, T)$$

FIRST OBTAIN C_t

SET: $K_t = \max(K, M_t)$

NOTE: $(M_T - K) = (M_T - K_t) + (K_t - K)$

$M_T \ge K_t \ge K$ WHEN $M_T \ge K$:

$$(M_T - K)^+ = (M_T - K_t)^+ + (K_t - K)^+$$

SO

$$\begin{aligned} C_t &= e^{-r(T-t)} E^* \left[(M_T - K_t)^+ \mid \mathcal{F}_t \right] \\ &\qquad + e^{-r(T-t)} E^+ \left[(K_t - K)^+ \mid \mathcal{F}_t \right] \\ &= f(S_t, K_t, T-t) + e^{-r(T-t)} (K_t - K)^+ \end{aligned}$$

LOOKBACK (CONTINUED)

$$C_t = f(S_t, K_t, T-t) + e^{-r(T-t)}(K_t - K)^+$$

FINDING THE HEDGE

K_t IS NONDECREASING (BUT NOT A dt TERM):

$$dC_t = f'_S(S_t, K_t, T-t)dS_t$$
$$+ \text{NONDECREASING (INCL } dt\text{) TERMS}$$

SO: $d[C,S]_t = f'_S(S_t, K_t, T-t)d[S,S]_t$

FROM EITHER ABOVE EQUATION:

$$\theta_t = f'_s(S_t, K_t, T-T)$$

THE DELTA IS STILL DERIVATIVE W.R.T. S, BUT HARDER TO SEE DIRECTLY.

ALSO: $K_t =$ FUNCTION OF PATH OF S_t.

MORE COMPLEX LOOKBACKS

PAYOFF $g(M_T)$ at T

RECALL FROM LECTURE 8:

$$\begin{aligned} V_0 &= f(s_0, T; g) \\ &= e^{-rT} E[g(M_T)] \\ &= e^{-rT} \int\int_{b \geq a} g(S_0 \exp(\sigma b)) \\ &\qquad \exp\left(\nu a - \frac{1}{2}\nu^2 T\right) f_{X,M^X}(a,b) da db \end{aligned}$$

where $\nu = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$

and

$$f_{X,M^X}(a,b) = \begin{cases} 2\,\dfrac{(2b-a)}{\sqrt{2\pi T^3}}\,\exp\left(-\dfrac{(2b-a)^2}{2T}\right) & \text{if } b \geq a \\ 0 & \text{otherwise} \end{cases}$$

APPLICATION: EVALUATION OF THE EURO-RUSSIAN OPTION

$$
\begin{aligned}
V_0 &= e^{-rT} E^* f(\max_{0\le t\le T} S_t) \\
&= e^{-rT} E^* f\left(\exp(\max_{0\le t\le T} \log S_t)\right) \\
&= e^{-rT} E^* f\left(\exp(\max_{0\le t\le T} \log S_t)\right) \\
&= e^{-rT} E^* f\left(\exp\left(\log S_0 + \sigma \max_{0\le t\le T} (\nu t + B_t^*)\right)\right) \\
&\qquad \sigma\nu = r - \frac{1}{2}\sigma^2 \\
&= e^{-rT} E_Q f\left(\exp(\log S_0 + \sigma \max_{0\le t\le T} X_t)\right) \frac{dP^*}{dQ}
\end{aligned}
$$

$$P^* : \quad X_t = \nu t + B_t^*$$

$$Q : \quad X_t \text{ is Brownian motion.}$$

$$\frac{dP^*}{dQ} = \exp\{\nu X_T - \frac{1}{2}\nu^2 T\}.$$

Problem reduced to one involving maxima of Brownian motion.

$$M_T^X = \max_{0 \le t \le T} X_t$$

$$V = e^{-rT} E_Q f\left(S_0 \exp(\sigma M_T)\right) \exp\left(\nu X_T - \frac{1}{2}\nu^2 T\right)$$

$$f_{X,M}(a,b) = 2\frac{(2b-a)}{\sqrt{2\pi T^3}} \exp\left\{-\frac{(2b-a)^2}{2T}\right\}$$

$b \ge a, 0$ (Karatzas and Shreve, p. 95)

$$V = e^{-rT} \int\int f(S_0 \exp(\sigma b)) \exp\left(\nu a - \frac{1}{2}\nu^2 T\right) f_{X,M}(a,b)\, da db$$

Derivation lecture 5.7 (Reflection principle).

PRICE AT TIME t

Let $m = M_t$, $s = S_t$, $N = \max_{t \le u \le T} S_u$

$$M_T = \begin{cases} m \text{ on set } N < m \\ N \text{ on set } N \ge m \end{cases}$$

$$g(M_T) = g(m)I(N < m) + g(N)I(N \ge m)$$

$$\begin{aligned}
&E^*(g(M_T) \mid \mathcal{F}_t) \\
&\quad = g(m)P^*(N < m \mid S_t = s) + E^*[g(N)I(N \ge m) \mid S_t = s] \\
&\quad = g(m)P^*(M_{T-t} < m \mid S_0 = s) + \\
&\qquad E^*[g(M_{T-t})I(M_{T-t} \ge m) \mid S_0 = s] \\
&\quad = e^{r(T-t)}[g(m)f(s, T-t; I(\cdot < m)) \\
&\qquad + f(s, T-t, g(\cdot)I(\cdot \ge m))]
\end{aligned}$$

Set

$$
\begin{aligned}
h(m,s,T) &= g(m)f(s,T-t,I(\cdot<m)) + f(s,T-t,g(\cdot)I(\cdot\geq m)) \\
&= e^{-r(T-t)}\Bigg\{g(m)\int\int_{b\geq a} I(S_0\exp(\sigma b)<m) \\
&\qquad \exp\left(va-\frac{1}{2}v^2T\right) f_{X,M}(a,b)dadb \\
&\quad + \int\int_{b\geq a} g(S_0\exp(\sigma b))I(S_0\exp(\sigma b)\geq m) \\
&\qquad \exp\left(va-\frac{1}{2}v^2T\right) f_{X,M}(a,b)dadb\Bigg\}
\end{aligned}
$$

Then

$$V_t = h(M_t, S_t, T-t)$$

$$dV_t = h'_S dS_t + \text{terms with } d \text{ (increasing quantities)}$$

Again

$$\text{delta} = f'_S(M_t, S_t, T-t)$$

Most general form: Malliavin calculus

WHAT IF YOU DO NOT KNOW FORM OF V_t?

MONTE CARLO SIMULATION (LECTURE 9)

FOR EXAMPLE, SUPPOSE YOU KNOW

$V_t = f(M_t, S_t, T - t)$

$M_t =$ maximum, or any other quantity without

quadratic variation

SIMULATE n COPIES $(V_t^{(i)}, M_t^{(i)}, S_t^{(i)}) \quad i = 1, \ldots, n$

$f(m, s, T - t) =$ NONPARAMETRIC REGRESSION

of $V_t^{(i)}$ on $M_t^{(i)}, S_t^{(i)}$

MOST SIMPLY: KERNEL REGRESSION

KERNEL REGRESSION
NADARAYA-WATSON ESTIMATOR

$$K(m,s) = \text{“KERNEL”}$$
$$= \text{any density in } m, s$$
$$\widehat{f}(m,s,T-t) = \frac{\sum_i V_t^{(i)} K(M^{(i)} - m, S^{(i)} - s)}{\sum_i K(M^{(i)} - m, S^{(i)} s)}$$
$$\text{DELTA} = \widehat{f}'_S(m,s,T-t)$$

FOR EXAMPLE,

$$K(m,s) = \frac{1}{h_1 h_2} \phi\left(\frac{m}{h_1}\right) \phi\left(\frac{s}{h_2}\right)$$

FOR $n \to \infty$, OPTIMAL $h_1, h_2 \to 0$

- MUCH THEORY AVAILABLE
- OTHER ESTIMATORS AVAILABLE
- NEED TO REDUCE # ARGUMENTS TO MINIMUM

HOW DO YOU KNOW YOU HAVE THE RIGHT ARGUMENTS?

$\widehat{f}'(M_t, S_t, T-t) =$ CORRECT DELTA FOR PAYOFF $g(M_T)$

$\widehat{f}'(S_t, T-t) =$ WRONG DELTA FOR PAYOFF $g(M_T)$

TO TEST THIS:

- OBTAIN $\widehat{f}'(m, s, T-t)$ FOR SEVERAL t (REUSE SAMPLE FOR DIFFERENT t, OR REDUCE DIMENSION)
- DO NEW SIMULATION TO SEE DISTRIBUTION OF

$$g(M_T^{(new,i)}) - \sum_{t_j} \widehat{f}'(M_{t_j}^{(new,i)} S_{t_j}^{(new,i)} T - t_j) \Delta S_{t_j}^{(new,i)}$$

IF $\asymp 0$: YOU'RE OK

$\widehat{f}(s, T-t)$ FAILS TEST FOR $g(M_T)$

EUROPEAN OPTION PAYOFF $g(S_T)$

$$C(s, R, \equiv) = \exp(-R) E_Q g(s \times \exp(R - \equiv /2 + \sqrt{\equiv} Z) \tag{1}$$

Black-Scholes price $= C(S_t, r(T-t), \sigma^2(T-t))$

If $\equiv^+ \geq \int_0^T \sigma_u^2 du$

$$\begin{aligned} & d\widetilde{C}\left(\widetilde{S}_t, 0, \equiv^+ - \int_0^t \sigma_u^2 du\right) \\ & = \widetilde{C}'_s d\widetilde{S}_t + \frac{1}{2}\widetilde{C}''_{SS} d[S,S]_t - \widetilde{C}'_{\equiv}\sigma_t^2 dt \\ & = \underbrace{\widetilde{C}'_S d\widetilde{S}_t}_{\text{HEDGEABLE}} + \underbrace{\left(\frac{1}{2}\widetilde{C}''_{SS}\widetilde{S}_t^2 - \widetilde{C}'_{\equiv}\right)}_{=0} \sigma_t^2 dt \end{aligned}$$

BECAUSE, FOR ALL $s, \equiv$,

$$C'_{\equiv}(s, 0, \equiv) = \frac{1}{2} C''_{SS}(s, 0, \equiv) s^2$$

BY DIFFERENTIATION OF (1)