HEDGING: THE EXACT CASE

SETUP: ON DISCOUNTED SCALE

\( \tilde{S}_t \): stock price \( \tilde{\eta} \): payoff at \( T \) \( \tilde{C}_t \): price at \( t \)

SELF FINANCING STRATEGY (SFS):

\[
\tilde{\eta} = \tilde{C}_T
\]

\[
d\tilde{C}_t = \theta_t d\tilde{S}_t \quad \theta_t = \text{“delta”}
\]

QUADRATIC VARIATION \([\cdot, \cdot]_t\):

\[
d[\tilde{C}, \tilde{S}]_t = \theta_t d[\tilde{S}, \tilde{S}]_t
\]

OR:

\[
\theta_t = \frac{d[\tilde{C}, \tilde{S}]_t}{d[\tilde{S}, \tilde{S}]_t}
\]

CONCLUSION: IF YOU KNOW HOW TO CALCULATE OPTIONS PRICES \( \tilde{C}_t \), AND IF SFS EXISTS, THEN YOU CAN CALCULATE \( \theta_t \).

WHAT DO YOU GET FROM (1) IF SFS DOES NOT EXIST? NEXT PAGE ON ORIGINAL SCALE
NUMERAIRE INVARiance: SFS IS

\[ \eta = C_T \]

\[ dC_t = \theta_t^{(0)} d\Lambda_t + \theta_t^{(1)} dS_t \quad (1) \]

\[ C_t = \theta_t^{(0)} \Lambda_t + \theta_t^{(1)} S_t \quad (2) \]

IF \( \Lambda_t = \exp \left( \int_0^t r_u du \right) = \text{MONEY MARKET BOND:} \)

\( \Lambda_t \) IS \( dt \) TERM, SO

\[ dC_t = \theta_t^{(1)} dS_t + dt \text{ TERM} \]

OR: \[ \theta_t^{(1)} = \frac{d[C, S]_t}{d[S, S]_t} \]

\( \theta_t^{(0)} \) IS GIVEN BY (2)

SINCE

\[ C_t = e^{rt} \tilde{C}_t, \quad \text{OR} \quad dC_t = e^{rt} d\tilde{C}_t + dt \text{ TERM} \]

\[ S_t = e^{rt} \tilde{S}_t, \quad \text{OR} \quad dS_t = e^{rt} d\tilde{S}_t + dt \text{ TERM} \]

ONE OBTAINS

\[ d[C, S] = (e^{rt})^2 d[\tilde{C}, \tilde{S}] \quad d[S, S] = (e^{rt})^2 d[S, S]_t \]

WHENCE

\[ \theta_t^{(1)} = \frac{(e^{rt})^2 d[\tilde{C}, \tilde{S}]_t}{(e^{rt})^2 d[\tilde{S}, \tilde{S}]_t} = \theta_t \quad \text{FROM (1)} \]

AS BEFORE
BLACK-SCHOLES MODEL:

\[ dS_t = rS_t \, dt + \sigma S_t \, dB_t \]

UNDER \( P^* \)

\( \Lambda_t = \) MONEY MARKET BOND, EUROPEAN OPTIONS

\[ C_t = f(S_t, T - t) = \text{B-S formula} \]

\[ dC_t = f'_S(S_t, T - t) \, dS_t + dt \text{ terms} \]

so \( d[C, S]_t = f'_S(S_t, T - t) \, d[S, S]_t \)

OR \( \theta_t = \frac{d[C, S]_t}{d[S, S]_t} = f'_S(S_t, T - t) \)

= THE REGULAR B-S DELTA

SHOOTING SPARROWS WITH CANNON
AN ASIAN LIABILITY

\[ \eta = \int_0^T g( Su ) du \]

TAKE \( r = 0 \):

\[ C_t = E \left( \int_0^T g(S_u) du \mid \mathcal{F}_t \right) \]

\[ = E \left( \int_0^t g(S_u) du \mid \mathcal{F}_t \right) + E \left( \int_t^T g(S_u) du \mid \mathcal{F}_t \right) \]

\[ = \int_0^t g(S_u) du + \int_t^T f(S_t, u - t) du \]

WHERE \( f(s, t) = \) BS price at zero for payoff \( g(S_t) \) at \( t \).

SET \( h(s, t) = \int_0^t f(s, u) du \):

\[ C_t = \int_0^t g(S_u) du + h(S_t, T - t) \]

THEN \( dC_t = h'_S(S_t, T - t) + dt - \) TERMS

\[ \theta_t = \frac{d[C, S]}{d[S, S]_t} = h'_S(S_t, T - t) \]
MORE INTERESTING: A LOOKBACK OPTION

\[ \eta = (M_T - K)^+ \quad \text{WHERE} \quad M_t = \max_{0 \leq u \leq t} S_u \]

\[ C_t = e^{-r(T-t)}E^*(\eta \mid \mathcal{F}_t) = \text{PRICE AT } t \]

IN FALL, WE LEARNED (?) TO CALCULATE \( C_0 \) FOR FIXED \( r, \sigma^2 \), SET

\[ C_0 = f(s_0, K, T) \]

FIRST OBTAIN \( C_t \)

SET: \( K_t = \max(K, M_t) \)

NOTE: \((M_T - K) = (M_T - K_t) + (K_t - K)\)

\( M_T \geq K_t \geq K \) WHEN \( M_T \geq K \):

\[ (M_T - K)^+ = (M_T - K_t)^+ + (K_t - K)^+ \]

SO

\[ C_t = e^{-r(T-t)}E^* \left[ (M_T - K_t)^+ \mid \mathcal{F}_t \right] \]
\[ + e^{-r(T-t)}E^+ \left[ (K_t - K)^+ \mid \mathcal{F}_t \right] \]
\[ = f(S_t, K_t, T-t) + e^{-r(T-t)}(K_t - K)^+ \]
LOOKBACK (CONTINUED)

\[ C_t = f(S_t, K_t, T - t) + e^{-r(T-t)}(K_t - K)^+ \]

FINDING THE HEDGE

\( K_t \) IS NONDECREASING (BUT NOT A \( dt \) TERM):

\[ dC_t = f'_S(S_t, K_t, T - t)dS_t \]
\[ + \text{NONDECREASING (INCL } dt \text{) TERMS} \]

SO: \( d[C, S]_t = f'_S(S_t, K_t, T - t)d[S, S]_t \)

FROM EITHER ABOVE EQUATION:

\[ \theta_t = f'_s(S_t, K_t, T - T) \]

THE DELTA IS STILL DERIVATIVE W.R.T. \( S \), BUT HARDER TO SEE DIRECTLY.

ALSO: \( K_t = \text{FUNCTION OF PATH OF } S_t. \)
MORE COMPLEX LOOKBACKS

PAYOFF $g(M_T)$ at $T$

RECALL FROM LECTURE 8:

$$V_0 = f(s_0, T; g) = e^{-rT} E[g(M_T)]$$

$$= e^{-rT} \int_{b \geq a} \int \int g(S_0 \exp(\sigma b)) \exp\left(\nu a - \frac{1}{2} \nu^2 T\right) f_{X, M_X}(a, b) da db$$

where $\nu = \frac{1}{\sigma} \left(r - \frac{1}{2} \sigma^2\right)$

and

$$f_{X, M_X}(a, b) = \begin{cases} 2 \frac{(2b - a)}{\sqrt{2\pi T}^3} \exp\left(-\frac{(2b - a)^2}{2T}\right) & \text{if } b \geq a \\ 0 & \text{otherwise} \end{cases}$$
APPLICATION:
EVALUATION OF THE EURO-RUSSIAN OPTION

\[ V_0 = e^{-rT} E^* f \left( \max_{0 \leq t \leq T} S_t \right) \]

\[ = e^{-rT} E^* f \left( \exp \left( \max_{0 \leq t \leq T} \log S_t \right) \right) \]

\[ = e^{-rT} E^* f \left( \exp \left( \max_{0 \leq t \leq T} \log S_t \right) \right) \]

\[ = e^{-rT} E^* f \left( \exp \left( \log S_0 + \sigma \max_{0 \leq t \leq T} (\nu t + B_t^*) \right) \right) \]

\[ \sigma \nu = r - \frac{1}{2} \sigma^2 \]

\[ = e^{-rT} E_Q f \left( \exp(\log S_0 + \sigma \max_{0 \leq t \leq T} X_t) \right) \frac{dP^*}{dQ} \]

\[ P^* : \quad X_t = \nu t + B_t^* \]

\[ Q : \quad X_t \text{ is Brownian motion.} \]

\[ \frac{dP^*}{dQ} = \exp \{ \nu X_T - \frac{1}{2} \nu^2 T \}. \]

Problem reduced to one involving maxima of Brownian motion.
\[ M_T^X = \max_{0 \leq t \leq T} X_t \]

\[ V = e^{-rT} E_Q f (S_0 \exp(\sigma M_T)) \exp \left( \nu X_T - \frac{1}{2} \nu^2 T \right) \]

\[ f_{X,M}(a, b) = 2 \frac{(2b - a)}{\sqrt{2\pi T^3}} \exp \left\{ -\frac{(2b - a)^2}{2T} \right\} \]

\[ b \geq a, 0 \text{ (Karatzas and Shreve, p. 95)} \]

\[ V = e^{-rT} \int \int f (S_0 \exp(\sigma b)) \exp \left( \nu a - \frac{1}{2} \nu^2 T \right) f_{X,M}(a, b) dadb \]

Derivation lecture 5.7 (Reflection principle).
PRICE AT TIME $t$

Let $m = M_t$, $s = S_t$, $N = \max_{t \leq u \leq T} S_u$

$$M_T = \begin{cases} m & \text{on set } N < m \\ N & \text{on set } N \geq m \end{cases}$$

$$g(M_T) = g(m)I(N < m) + g(N)I(N \geq m)$$

$$\mathbb{E}^*(g(M_T) \mid \mathcal{F}_t) = g(m)P^*(N < m \mid S_t = s) + \mathbb{E}^*[g(N)I(N \geq m) \mid S_t = s]$$

$$= g(m)P^*(M_{T-t} < m \mid S_0 = s) +$$

$$\mathbb{E}^*[g(M_{T-t})I(M_{T-t} \geq m) \mid S_0 = s]$$

$$= e^{r(T-t)}[g(m)f(s, T-t; I(\cdot < m))$$

$$+ f(s, T-t, g(\cdot)I(\cdot \geq m))]$$
Set

\[ h(m, s, T) = g(m) f(s, T - t, I(\cdot < m)) + f(s, T - t, g(\cdot) I(\cdot \geq m)) \]

\[ = e^{-r(T-t)} \left\{ g(m) \int \int_{b \geq a} I(S_0 \exp(\sigma b) < m) \exp \left( va - \frac{1}{2} v^2 T \right) f_{X,M}(a, b) dadb \right. \]

\[ + \int \int_{b \geq a} g(S_0 \exp(\sigma b)) I(S_0 \exp(\sigma b) \geq m) \exp \left( va - \frac{1}{2} v^2 T \right) f_{X,M}(a, b) dadb \right\} \]

Then

\[ V_t = h(M_t, S_t, T - t) \]

\[ dV_t = h'_S dS_t + \text{terms with } d \text{ (increasing quantities)} \]

Again

\[ \text{delta} = f'_S(M_t, S_t, T - t) \]

Most general form: Malliavin calculus
WHAT IF YOU DO NOT KNOW FORM OF $V_t$?

MONTE CARLO SIMULATION (LECTURE 9)

FOR EXAMPLE, SUPPOSE YOU KNOW

$V_t = f(M_t, S_t, T - t)$

$M_t = \text{maximum, or any other quantity without quadratic variation}$

SIMULATE $n$ COPIES $(V_t^{(i)}, M_t^{(i)}, S_t^{(i)})$ \(i = 1, \ldots, n\)

$f(m, s, T - t) = \text{NONPARAMETRIC REGRESSION}$

of $V_t^{(i)}$ on $M_t^{(i)}, S_t^{(i)}$

MOST SIMPLY: KERNAL REGRESSION
KERNEL REGRESSION
NADARAYA-WATSON ESTIMATOR

\[ K(m, s) = \text{"KERNEL"} \]
\[ = \text{any density in } m, s \]
\[ \hat{f}(m, s, T - t) = \frac{\sum_i V_t^{(i)} K(M^{(i)} - m, S^{(i)} - s)}{\sum_i K(M^{(i)} - m, S^{(i)} s)} \]
\[ \text{DELTA} = \hat{f}'(m, s, T - t) \]

FOR EXAMPLE,

\[ K(m, s) = \frac{1}{h_1 h_2} \phi \left( \frac{m}{h_1} \right) \phi \left( \frac{s}{h_2} \right) \]

FOR \( n \to \infty \), OPTIMAL \( h_1, h_2 \to 0 \)

- MUCH THEORY AVAILABLE
- OTHER ESTIMATORS AVAILABLE
- NEED TO REDUCE \# ARGUMENTS TO MINIMUM
HOW DO YOU KNOW YOU HAVE THE RIGHT ARGUMENTS?

\[ \hat{f}'(M_t, S_t, T-t) = \text{CORRECT DELTA FOR PAYOFF } g(M_T) \]

\[ \hat{f}'(S_t, T-t) = \text{WRONG DELTA FOR PAYOFF } g(M_T) \]

TO TEST THIS:

- OBTAIN \( \hat{f}'(m, s, T-t) \) FOR SEVERAL \( t \) (REUSE SAMPLE FOR DIFFERENT \( t \), OR REDUCE DIMENSION)
- DO NEW SIMULATION TO SEE DISTRIBUTION OF

\[
g(M_{T}^{(\text{new},i)}) - \sum_{t_j} \hat{f}'(M_{t_j}^{(\text{new},i)}, S_{t_j}^{(\text{new},i)}, T-t_j) \Delta S_{t_j}^{(\text{new},i)}
\]

IF \( \neq 0 \): YOU’RE OK

\( \hat{f}(s, T - t) \) FAILS TEST FOR \( g(M_T) \)
EUROPEAN OPTION PAYOFF $g(S_T)$

$$C(s, R, \Xi) = \exp(-R)E_Q g(s \times \exp(R - \Xi / 2 + \sqrt{\Xi} Z)$$

Black-Scholes price $= C(S_t, r(T - t), \sigma^2(T - t))$

If $\Xi^+ \geq \int_0^T \sigma_u^2 du$

$$d\tilde{C} \left( S_t, 0, \Xi^+ - \int_0^t \sigma_u^2 du \right)$$

$$= \tilde{C}'_s dS_t + \frac{1}{2} \tilde{C}''_{SS} d[S, S]_t - \tilde{C}'_{\Xi} \sigma_t^2 dt$$

$$= \underbrace{\tilde{C}'_S dS_t}_{\text{HEDGEABLE}} + \underbrace{\left( \frac{1}{2} \tilde{C}''_{SS} S_t^2 - \tilde{C}'_{\Xi} \right) \sigma_t^2 dt}_{= 0}$$

BECAUSE, FOR ALL $s, \Xi$,

$$C'_\Xi(s, 0, \Xi) = \frac{1}{2} C''_{SS}(s, 0, \Xi)s^2$$

BY DIFFERENTIATION OF (1)