# $P$ AND $P^*$

$(B_t, S_t^{(t)}, \ldots, S_t^{(p)})$: securities

$P$: actual probability

$P^*$: risk neutral probability

Realtionship: mutual absolute continuity $P \sim P^*$

<u>For example</u>:

$P: \; dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$

$P^*: \; dS_t = \mu_t^* S_t dt + \sigma_t^* S_t dW_t^*$

Money market bond numeraire:

$$\mu_t^* = r_t$$

$Q: \; \sigma_t^* = ??$ $\qquad A: \; {\sigma_t^*}^2 = \sigma_t^2.$ Why?

# CONTINUOUS CASE: $\sigma_t$ AND $\sigma_t^*$

$$P: \quad dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$

$$P^*: \quad dS_t = \mu_t^* S_t dt + \sigma_t^* S_t dW_t^*$$

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$$P: \quad d\log S_t = \ldots dt + \sigma_t dW_t$$

$$\text{hence:} \quad \underbrace{(d\log S_t)^2}_{d[\log S, \log S]_t} = \sigma_t^2 (dW_t)^2 = \sigma_t^2 dt$$

$$P^*: \quad \text{same argument}: \quad d[\log S, \log S]_t = {\sigma_t^*}^2 dt$$

Process same under $P$, $P^*$:

$${\sigma_t^*}^2 dt = d[\log S, \log S]_t = \sigma_t^2 dt$$

$$\text{hence:} \quad {\sigma_t^*}^2 = \sigma_t^2$$

In particular: if $\sigma$ is constant:

$$\sigma^* = \sigma$$

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If numeraire = Money Market Bond:

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t^*$$

$$d\widetilde{S}_t = \sigma_t \widetilde{S}_t dW_t^*$$

# CONTINUOUS CASE: THE MARKET PRICE OF RISK

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$
$$=$$
$$dS_t = \mu_t^* S_t dt + \sigma_t S_t dW_t^*$$

$$\mu_t S_t dt + \sigma_t S_t dW_t = \mu_t^* S_t dt + \sigma_t S_t dW_t^*$$

$$\text{hence:} \quad \mu_t dt + \sigma_t dW_t = \mu_t^* dt + \sigma_t dW_t^*$$

$$\text{hence:} \quad \underbrace{\frac{\mu_t - \mu_t^*}{\sigma_t}}_{\lambda_t} dt + dW_t = dW_t^*$$

Change from $P$ to $P^*$:

Market price of risk

$$\underbrace{dW_t^*}_{P^* - BM} = \underbrace{dW_t}_{P - BM} + \lambda_t dt$$

# OTHER NUMERAIRE

$$\left.\begin{aligned} d\log S_t &= \ldots dt + \sigma_t dW_t \\ d\log \Lambda_t &= \ldots dt + \gamma_t dV_t \end{aligned}\right] P, P^*$$

$$\widetilde{S}_t = \frac{S_t}{\Lambda_t} \Rightarrow \log \widetilde{S}_t = \log S_t - \log \Lambda_t$$

$$\Rightarrow$$

$$\begin{aligned} (d\log \widetilde{S}_t)^2 &= (d\log S_t)^2 + (d\log \Lambda_t)^2 - 2(d\log S_t)(d\log \Lambda_t) \\ &= \sigma_t^2 (dW_t)^2 + \gamma_t^2 (dV_t)^2 - 2\sigma_t\gamma_t \underbrace{(dW_t)(dV_t)}_{\rho_t dt} \\ &= (\sigma_t^2 + \gamma_t^2 - 2\sigma_t\gamma_t\rho_t)dt \end{aligned}$$

$$\rho_t = \frac{1}{dt}\, d[W, V]_t = \text{correlation } (dW, dV)$$

(can be random)

$$\begin{aligned} \widetilde{\sigma}_t^2 &= \sigma_t^2 + \gamma_t^2 - 2\sigma_t\gamma_t\rho_t \quad P, P^* \\ &\neq \sigma_t^2 \text{ ex } \gamma_t = 0 : \text{ MONEY MARKET BOND} \end{aligned}$$

# OTHER NUMERAIRE

$$d\log S_t = \ldots dt + \sigma_t dW_t$$

$$d\log \Lambda_t = \ldots dt + \gamma_t dV_t$$

so $\underbrace{d\log \widetilde{S}_t}_{= \ldots dt + \widetilde{\sigma}_t d\widetilde{W}_t \,\big]\ \text{previous page}} = \ldots dt + \sigma_t + \sigma_t dW_t - \gamma_t dV_t$

or $d\widetilde{S}_t = \underbrace{\ldots \widetilde{S}_t dt}_{= 0 \text{ if } P^* \text{ for numeraire } \Lambda} + \widetilde{\sigma}_t \widetilde{S}_t d\widetilde{W}_t$

$$\widetilde{\sigma}_t d\widetilde{W}_t = \sigma_t dW_t - \gamma_t dV_t$$

or

$$d\widetilde{W}_t = \frac{\sigma_t}{\widetilde{\sigma}_t} dW_t - \frac{\gamma_t}{\widetilde{\sigma}_t} dV_t$$

Yet another Brownian motion

# $P^*$ DEPENDS ON NUMERAIRE

$\Lambda_t, B_t$: NUMERAIRES, SUPPOSE $P^*$ SAME

$$M_t = \frac{\Lambda_t}{B_t} = P^* - \mathrm{MG}$$

$$\frac{1}{M_t} = \frac{B_t}{\Lambda_t} = P^* - \mathrm{MG}$$

$$\underbrace{d\,\frac{1}{M_t}}_{d\mathrm{MG}} = \underbrace{-\frac{1}{M_t^2}\,dM_t}_{d\mathrm{MG}} + \underbrace{\frac{1}{M_t^3}\,d[M,M]_t}_{d\mathrm{MG}??}$$

NEED $\underbrace{d[M,M]_t}_{dt \text{ term}} = d\mathrm{MG}$

UNIQUENESS OF DOOB-MEYER $\Rightarrow$

$$[M,M]_t = 0$$

$\Rightarrow M_t$ constant if continuous

$\Rightarrow \Lambda, B$ are the same

# THE RADON-NIKODYM THEOREM

**Theorem**. *Let $P, Q$ be probabilities, $Q \ll P$. Then $\exists$r.v. $\frac{dQ}{dP}$:*

$$\textit{if } E_Q|X| < \infty, \textit{ then } E_Q X = E_P\left(X\frac{dQ}{dP}\right).$$

*$\frac{dQ}{dP}$ is unique $P-a.s.$*

*Proof and elaborations*: Billingsley, Section 32.

*Uniqueness*: Suppose both $Y, Z$ satisfy conditions on $\frac{dQ}{dP}$. Set

$$A = \{\omega : Y(\omega) > Z(\omega)\}.$$

Then

$$Q(A) = E_P Y I_A > E_P Z I_A = Q(A)$$

unless $P(A) = 0$.

The finite case: $\mathcal{F} = \sigma(\mathcal{P})$:

$$\frac{dQ}{dP}(\omega) = \frac{Q(A)}{P(A)}$$

when $\omega \in A \in \mathcal{P}$

# CONTINUOUS DISTRIBUTIONS

$$P(Z \in A) = \int_A f(z)dz \qquad Q(Z \in A) = \int_A g(z)dz.$$

Then:

$$\frac{dQ}{dP}(\omega) = \frac{g(Z)}{f(Z)}.$$

*Proof*:

$$\begin{aligned} E_P \frac{g(Z)}{f(Z)} I(Z \in A) &= \int \frac{g(z)}{f(z)} I(z \in A) f(z) dz \\ &= \int I(z \in A) g(z) dz \\ &= Q(A). \end{aligned}$$

Generally: $Z_1, \ldots, Z_p$

$$P(A) = \int_A f(z_1, \ldots, z_p) dz_1 \ldots dz_p$$

$$Q(A) = \int_A g(z_1, \ldots, z_p) dz_1 \ldots dz_p$$

$$\frac{dQ}{dP}(\omega) = \frac{g(Z_1, \ldots, Z_p)}{f(Z_1, \ldots, Z_p)}.$$

# NORMAL PROCESSES WITH DRIFT

$$P_\theta : \quad X_{t+1} = X_t + \theta_t + \sigma \epsilon_{t+1}$$

$$\epsilon_{t+1} \sim N(0,1), \ \amalg\ \mathcal{F}_t.$$

$$f_\theta(x_{t+1} \mid \mathcal{F}_t) = \frac{1}{\sqrt{2\pi}\ \sigma} \exp\left\{ -\frac{(x_{t+1} - X_t - \theta_t)^2}{2\sigma^2} \right\}$$

$$f_\theta(x_1, \ldots, x_t) = (2\pi\sigma^2)^{-\frac{t}{2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{u=0}^{t-1} (x_{u+1} - x_u - \theta_u)^2 \right\}$$

$$\frac{dP_\theta}{dP_0} = \frac{f_\theta(X_1, \ldots, X_t)}{f_0(X_1, \ldots, X_t)}$$

$$= \exp\left\{ -\frac{1}{2\sigma^2} \sum_{u=0}^{t-1} (X_{u+1} - X_u - \theta_u)^2 + \frac{1}{2\sigma^2} \sum_{u=0}^{t-1} (X_{u+1} - X_u)^2 \right\}$$

$$= \exp\left\{ \frac{1}{\sigma^2} \sum_{u=0}^{t-1} \theta_u (X_{u+1} - X_u) - \frac{1}{2\sigma^2} \sum_{u=0}^{t-1} \theta_u^2 \right\}$$

$$= \exp\left\{ Y_t - \frac{1}{2} [Y, Y]_t \right\}$$

where: $Y_t = \frac{1}{\sigma^2} \int_0^t \theta_u dX_u$ and $\theta_u = \theta_t$ for $t < u \leq t+1$

# GIRSANOV'S THEOREM

$$\underbrace{dX_t}_{P-BM} = \underbrace{dW_t}_{Q-BM} + \theta_t dt$$

$$\frac{dQ}{dP} = \exp(M_T - \frac{1}{2}[M,M]_T)$$

$$\text{where } M_t = \int_0^t \theta_u dX_u = P - \text{MG}$$

$$\begin{aligned}
\frac{dP}{dQ} &= \exp(-M_T + \frac{1}{2}[M,M]_T) \\
&= \exp(-\int_0^T \theta_u [dW_u + \frac{1}{2}\theta_u du] + \frac{1}{2}\int_0^T \theta_u^2 du) \\
&= \exp(-\int_0^T \theta_u dW_u - \frac{1}{2}\int_0^T \theta_u^2 du) \\
&= \exp(\widetilde{M}_T - \frac{1}{2}[\widetilde{M},\widetilde{M}]_T) \text{ where } \widetilde{M}_t = -\int_0^t \theta_u dW_u
\end{aligned}$$

$$Q - \text{MG as function of } T$$

# SMALL INCREMENTS, $\theta$ CONSTANT

$$P_\theta : \quad X_{t+\Delta} = X_t + \theta\Delta + \sigma(B_{t+\Delta} - B_t)$$

$\theta$ replaced by $\theta\Delta$ $\qquad$ $\sigma$ replaced by $\sigma\sqrt{\Delta}$

$$\begin{aligned} \frac{dP_\theta}{dP_0} &= \exp\left\{ \frac{\theta\Delta}{\sigma^2\Delta}\ X_{N\Delta} - \frac{1}{2}\frac{\theta^2\Delta^2}{\sigma^2\Delta}\ N \right\} \\ &= \exp\left\{ \frac{\theta}{\sigma^2}\ X_T - \frac{1}{2}\frac{\theta^2}{\sigma^2}\ T \right\} \qquad T = \Delta N \end{aligned}$$

If $T$ fixed, $N \to \infty$, $\Delta \to 0$:

$$P_\theta : \quad X_t = \theta t + \sigma B_t$$

# BROWNIAN MOTION WITH DRIFT

Form of $\frac{dP_\theta}{dP_0}$: simplest case of Girsanov's Theorem.

# APPLICATION: EVALUATION OF THE EURO-RUSSIAN OPTION

Recall: $V_0 = e^{-rT} E^*[\text{payoff}]$

$$V_0 = e^{-rT} E^* f(\max_{0 \le t \le T} S_t)$$
$$= e^{-rT} E^* f\left(\exp(\max_{0 \le t \le T} \log S_t)\right)$$
$$= e^{-rT} E^* f\left(\exp(\max_{0 \le t \le T} \log S_t)\right)$$
$$= e^{-rT} E^* f\left(\exp\left(\log S_0 + \sigma \max_{0 \le t \le T} (\nu t + B_t^*)\right)\right)$$
$$\sigma\nu = r - \frac{1}{2}\sigma^2$$
$$= e^{-rT} E_Q f\left(\exp(\log S_0 + \sigma \max_{0 \le t \le T} X_t)\right) \frac{dP^*}{dQ}$$

$$P^* : \quad X_t = \nu t + B_t^*$$

$$Q : \quad X_t \text{ is Brownian motion.}$$

$$\frac{dP^*}{dQ} = \exp\{\nu X_T - \frac{1}{2}\nu^2 T\}.$$

Problem reduced to one involving maxima of Brownian motion.

$$M_T = \max_{0 \le t \le T} X_t$$

$$V_0 = e^{-rT} E_Q f\left(S_0 \exp(\sigma M_T)\right) \exp\left(\nu X_T - \frac{1}{2}\nu^2 T\right)$$

$$f_{X,M}(a,b) = \frac{2(2b-a)}{\sqrt{2\pi T^3}} \exp\left\{-\frac{(2b-a)^2}{2T}\right\}$$

$b \ge a, 0$

$$V_0 = e^{-rT} \int\int f(S_0 \exp(\sigma b)) \exp\left(\nu a - \frac{1}{2}\nu^2 T\right) f_{X,M}(a,b)\, da db$$

# ADDITIVE AND MULTIPLICATIVE MARTINGALES

## $M_t$ IS A MARTINGALE

$$Z_t = \exp(\underbrace{M_t - \frac{1}{2}[M,M]_t}_{X_t}) = f(X_t)$$

Itô:

$$\begin{aligned} dZ_t &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X,X]_t \\ &= \exp(X_t)\underbrace{\left[dX_t + \frac{1}{2}d[X,X]_t\right]}_{dM_t} \\ &= Z_t dM_t = \text{MG} \end{aligned}$$

Since

$$\overbrace{= dM_t - \frac{1}{2}d[M,M]_t}^{dX_t} + \overbrace{\frac{1}{2}d[M,M]_t}^{+\frac{1}{2}d[X,X]_t} \qquad = dM_t$$

In particular:

$$EZ_T = EZ_0 = E1 = 1$$

# THE DISTRIBUTION OF HITTING TIMES

$$P^* : \quad X_t = \nu t + \sigma W_t \text{ and } Q : \quad X_t = \sigma W_t^*$$

$$\left.\frac{dP^*}{dQ}\right|_t = \exp\left\{\frac{\nu}{\sigma^2} X_t - \frac{1}{2}\frac{\nu^2}{\sigma^2} t\right\}.$$

Martingale property:

$$\left.\frac{dP^*}{dQ}\right|_\tau = \exp\left\{\frac{\nu}{\sigma^2} X_\tau - \frac{1}{2}\frac{\nu^2}{\sigma^2} \tau\right\}.$$

A special case

$$\tau = \inf \{t : \quad X_t = b\}.$$

$$\begin{aligned} P^*(\tau \leq u) &= E^* I_{\{\tau \leq u\}} \\ &= E_Q I_{\{\tau \leq u\}} \exp\left\{\frac{\nu}{\sigma^2} X_{\tau\wedge u} - \frac{1}{2}\frac{\nu^2}{\sigma^2}(\tau \wedge u)\right\} \\ &= E_Q I\{\tau \leq u\} \exp\left\{\frac{\nu}{\sigma^2} b - \frac{1}{2}\frac{\nu^2}{\sigma^2}\tau\right\} \\ &= \int_0^u \exp\left(\frac{\nu}{\sigma^2} b - \frac{1}{2}\frac{\nu^2}{\sigma^2} t\right) f_{Q,\tau}(t)\, dt \end{aligned}$$

$$P^*(\tau \le u) = \int_0^u \exp\left(\frac{\nu}{\sigma^2} b - \frac{1}{2}\frac{\nu^2}{\sigma^2} t\right) f_{Q,\tau}(t)\, dt$$

or:

$$f_{P^*,\tau}(t) = \exp\left(\frac{\nu}{\sigma^2} b - \frac{1}{2}\frac{\nu^2}{\sigma^2} t\right) f_{Q,\tau}(t).$$

From distribution of maximum:

$$f_{Q,\tau}(t) = \frac{|b|}{\sqrt{2\pi\sigma^2 t^3}}\ \exp\left\{-\frac{b^2}{2\sigma^2 t}\right\}, \qquad t \ge 0$$

EXAMPLE: DOWN AND IN OPTIONS

$$\eta = \begin{cases} (S_T - K)^+ & \text{of } \min\limits_{0 \le t \le T} S_t \le K' \\ 0 & \text{otherwise} \end{cases}$$

$$X_t = \log S_t - \log S_0 \qquad b = \log K' - \log S_0$$

$$\begin{aligned} \text{price} &= E^* e^{-rT} \eta \\ &= E^* E^*[e^{-rT}\eta \mid \mathcal{F}_\tau] I_{(\tau \le T)} \\ &= E^* e^{-r\tau} E^* \left[e^{-r(T-\tau)}\eta \mid \mathcal{F}_\tau\right] I_{(\tau \le T)} \\ &= E^* e^{-r\tau} BS(T-\tau) I_{(\tau \le T)} \\ &= \int_0^T e^{-rt} BS(T-t) f_{P^*,\tau}(t)\, dt \end{aligned}$$