APPROXIMATE NORMALITY

BINOMIAL MODEL: \[ S_{n+1} = \begin{cases} uS_n \\ dS_n \end{cases} \]

LOG SCALE (IID ADDITIVE INCREMENTS):
\[ \log S_n = \log S_0 + X_1 + \ldots + X_n \]
WITH \( X_i = \log(u) \) or \( = \log(d) \)

TWO TIME SCALES

CLOCK TIME: \( T \) – TIME PERIODS: \( n \)
\[ \begin{align*}
t_0 &= 0 \\
t_1 &= \frac{T}{n} \\
t_2 &= \frac{2T}{n} \\
t_3 &= \frac{3T}{n} \\
\vdots \\
t_k &= \frac{kT}{n}
\end{align*} \]

\( T \) IS FIXED – \( n \) IS A MATTER OF CHOICE

RETURN ON RISK FREE ASSET
(in clock time) \( e^{rT} = e^{\rho n} \) (in time periods)

in other words: \( \rho = r \frac{T}{n} \) (1)

\( r \) IS FIXED – \( \rho \) DEPENDS ON \( n \)

RISK NEUTRAL MEASURE PER STEP:
\[ \pi_n(T) = \frac{u - e^\rho}{u - d} = \frac{u - e^{r\frac{T}{n}}}{u - d} \]
and \( \pi_n(H) = \frac{e^\rho - d}{u - d} \) (2)
BEHAVIOR OF ADDITIVE INCREMENTS

MEAN:

\[ E(X) = \log(u) \pi(H) + \log(d) \pi(T) \]

TOTAL MEAN:

\[ E(\log(S_n) - \log(S_0)) = E(X_1) + \ldots + E(X_n) \]
\[ = n E(X) \]
\[ = n(\log(u) \pi(H) + \log(d) \pi(T)) \]

VARIANCE: \( X = \log d + (\log u - \log d) I_{\{H\}} \), and so

\[ \text{Var} \ (X) = (\log u - \log d)^2 \text{Var} \ (I_{\{H\}}) \]
\[ = (\log u - \log d)^2 \pi(H) \pi(T) \]

TOTAL VARIANCE:

\[ \text{Var} \ (\log(S_n)) = \text{Var} \ (X_1) + \ldots + \text{Var} \ (X_n) \]
\[ = n \text{Var} \ (X_1) \]
\[ = n(\log u - \log d)^2 \pi(H) \pi(T) \]
WE WISH TO KEEP TOTAL MEAN, VARIANCE CONSTANT IN CLOCK TIME

\[ \nu T = E(\log S_n) = n(\log(u)\pi(H) + \log(d)\pi(T)) \]  
\[ \sigma^2 T = \text{Var}(\log(S_n)) = n(\log u - \log d)^2 \pi(H)\pi(T) \]  

\[ \sigma \text{ OR } \sigma^2 \text{ IS VOLATILITY IN CLOCK TIME} \]

\text{NEED TO USE: } \nu \approx r - \frac{1}{2}\sigma^2

EQUATIONS (1)-(4) DEFINE A BINOMIAL TREE 
\((\rho, u, d, \pi(H), \pi(T))\) ON THE BASIS OF:

- VOLATILITY PER UNIT CLOCK TIME: \(\sigma^2\)
- INTEREST PER UNIT CLOCK TIME: \(r\)
- # OF UNITS OF CLOCK TIME: \(T\)
- # OF TIME PERIODS IN COMPUTATION: \(n\)
AN APPROXIMATION FOR THE CASE $r = \rho = 0$
(THE DISCOUNTED PROCESS)

UP AND DOWN STEPS:

$$u = 1 + \sqrt{\frac{\sigma^2 T}{n}} \text{ AND } d = 1 - \sqrt{\frac{\sigma^2 T}{n}}$$

RISK NEUTRAL PROBABILITIES:

$$\pi_n(T) = \frac{u - e^\rho}{u - d} = \frac{1}{2} \text{ AND } \pi_n(H) = \frac{e^\rho - d}{u - d} = \frac{1}{2}$$

WE SHOW THAT EQUATIONS (3)-(4) ARE APPROXIMATELY SATISFIED

WILL USE THIS APPROXIMATE BINOMIAL TREE
APPROXIMATION TO CONDITION (4):

\[ \log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots \]

\[ x = \sqrt{\frac{\sigma^2 T}{n}} : \log(u) = \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots \]

\[ x = -\sqrt{\frac{\sigma^2 T}{n}} : \log(d) = -\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots \]

AND SO:

\[ \text{Var} \left( \log(S_n) \right) = n \left( \log u - \log d \right)^2 \pi(H)\pi(T) \]

\[ = n \frac{1}{4} \left( \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots \right)^2 \]

\[ - \left( -\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots \right)^2 \]

\[ = n \frac{1}{4} \left( 2\sqrt{\frac{\sigma^2 T}{n}} + \frac{1}{n\sqrt{n}} \times \ldots \right)^2 \]

\[ = \sigma^2 T + \frac{1}{n} \times \ldots \]
ABOUT EQUATION (3):

\[
\begin{align*}
\log(1 + x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots \\
\log(u) &= \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots \\
\log(d) &= -\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots 
\end{align*}
\]

AND SO:

\[
\begin{align*}
\nu T &= E(\log(S_n) - \log(S_0)) \\
&= n(\log(u)\pi(H) - \log(d)\pi(T)) \\
&= \frac{1}{2} n \left( \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots \\
& \quad + (-\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots) \right) \\
&= -\frac{1}{2} \sigma^2 T + \frac{1}{\sqrt{n}} \times \ldots 
\end{align*}
\]

AS PREDICTED
HOW MUCH DO OUR RESULTS DEPEND ON $n$?

TRYING THE MATTER OUT IN R

```r
M <- 1000     # number of simulation steps
sigma <- .2   # clock time volatility
T <- 1        # clock time duration
S0 <- 100     # initial value
piH <- 1/2    # risk neutral probability
n <- 10       # steps
u <- 1 + sqrt(T*sigma^2/n) # up step
d <- 1 - sqrt(T*sigma^2/n) # down step
H <- rbinom(M,n,piH) # simulation
logS <- log(S0) + log(u)*H + log(d)*(n-H)
par(mfrow=c(2,1)) # check this command out!
hist(logS,freq=F)
# try again with a larger number of steps
n <- 1000
# define u, d, H, logS as above, with new n
hist(logS,freq=F)
```
THE DISTRIBUTION OF $\log S_T$ STABILIZES
THE CENTRAL LIMIT PHENOMENON

**THEOREM:** SUPPOSE THAT

- \(X_i, i = 1, \ldots, n\) ARE IID \(P_n\)
  
  (DISTRIBUTION CAN DEPEND ON \(n\))

- \(n \text{ Var}_n(X) \to \gamma^2\) AS \(n \to \infty\)

THEN

\[
\sum_{i=1}^{n} X_i - nE_n(X) \xrightarrow{\text{L}} N(0, \gamma^2)
\]

IN WORDS:

\(\sum_{i=1}^{n} X_i - nE_n(X)\) CONVERGES IN LAW TO \(N(0, \gamma^2)\)

THAT IS TO SAY:

THE DISTRIBUTION OF \(\sum_{i=1}^{n} X_i - nE_n(X)\) IS APPROXIMATELY NORMAL \(N(0, \gamma^2)\)

DENSITY OF THE NORMAL DISTRIBUTION \(N(\mu, \gamma^2)\)

\[
\frac{d}{dx} P(N(\mu, \gamma^2) \leq x) = \frac{1}{\gamma} \phi \left( \frac{x - \mu}{\gamma} \right)
\]

\[
\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} x^2\right\}
\]
IN OUR CASE

\[ \log(S_T) - \log(S_0) = \sum_{i=1}^{n} X_i \]

\[ \gamma^2 = \sigma^2 T \]

\[ E(\log(S_T) - \log(S_0)) = nE_n(X) \approx -\frac{1}{2} \sigma^2 T \]

SO THAT

\[ \log(S_T) - \left( \log(S_0) - \frac{1}{2} \sigma^2 T \right) \]

IS APPROXIMATELY NORMAL \( N(0, \sigma^2 T) \)

OR: \( \log(S_T) \) IS APPROXIMATELY NORMAL

\[ N(\log(S_0) - \frac{1}{2} \sigma^2 T, \sigma^2 T) \]

Note: \( Z \sim N(\mu, \gamma^2) \) \( \iff \]

\[ Z \sim \mu \sim N(0, \gamma^2) \]

\[ \iff \quad \frac{Z - \mu}{\gamma} \sim N(0, 1) \]
SUPERIMPOSING THE NORMAL CURVE ON THE HISTOGRAM

\[ n \leftarrow 10 \quad \# \text{steps} \]
\[ u \leftarrow 1 + \sqrt{T \cdot \text{sigma}^2/n} \quad \# \text{up step} \]
\[ d \leftarrow 1 - \sqrt{T \cdot \text{sigma}^2/n} \quad \# \text{down step} \]
\[ H \leftarrow \text{rbinom(M,n,piH)} \quad \# \text{simulation} \]
\[ \logS \leftarrow \log(S_0) + \log(u) \cdot H + \log(d) \cdot (n-H) \]
\[ \text{par(mfrow=c(2,1))} \quad \# \text{check this command out!} \]
\[ \text{hist(logS,freq=F)} \]
\[ \# \text{compare to normal distribution} \]
\[ \text{xpoints} \leftarrow \text{c(-30:30)/10} \]
\[ \mu \leftarrow \log(S_0) - (\text{sigma}^2 \cdot T)/2 \]
\[ \gamma \leftarrow \sqrt{\text{sigma}^2 \cdot T} \]
\[ \text{xpoints} \leftarrow \text{c(-30:30)/10} \]
\[ \text{xpoints} \leftarrow \mu + \text{sigma} \cdot \text{xpoints} \]
\[ \text{density} \leftarrow \text{dnorm(xpoints,mean=mu,sd=gamma)} \]
\[ \text{lines(xpoints,density)} \]
\[ \# \text{try again with a larger number of steps} \]
\[ n \leftarrow 1000 \]
\[ \# \text{define u, d, H, logS as above, with new n} \]
\[ \text{hist(logS,freq=F)} \]
\[ \# \mu, \gamma, \text{xpoints} \text{ stay the same} \]
\[ \text{lines(xpoints,density)} \]
NORMAL CURVE SUPERIMPOSED ON HISTOGRAMS
THE CLASSICAL CENTRAL LIMIT THEOREM

(A digression. Just so you know.)

SETUP:

$Y_1, \ldots, Y_n$ ARE IID, $E(Y) = 0$ AND $\text{Var}(Y) = \gamma^2$

THEN:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i \overset{\mathcal{L}}{\longrightarrow} N(0, \gamma^2)$$

PROOF:

TAKE $X_i = \frac{1}{\sqrt{n}} Y_i$

IN EARLIER THEOREM

RESULT FOLLOWS
BEHAVIOR OF OPTIONS PRICES

STEP 1: CONTINUOUS FUNCTIONS

THEOREM: IF

• $Z_n \xrightarrow{L} Z$ AS $n \to \infty$
• $x \to h(x)$ IS A CONTINUOUS FUNCTION

THEN $h(Z_n) \xrightarrow{L} h(Z)$ AS $n \to \infty$

EXAMPLE

$Z_n = \log(S_T^{(n)}) \xrightarrow{L} Z = N \left( \log(S_0) - \frac{1}{2} \sigma^2 T, \sigma^2 T \right)$

CONTINUOUS FUNCTION #1: $h(x) = e^x$:

$S_T^{(n)} = \exp\{Z_n\} \xrightarrow{L} S_T^{(\infty)} = \exp\{Z\}$

CONTINUOUS FUNCTION #2: $h(x) = (x - e^{-rT} K)^+$:

$V_T^{(n)} = (S_T^{(n)} - e^{-rT} K)^+ \xrightarrow{L} (S_T^{(\infty)} - e^{-rT} K)^+$

CHECK THIS IN R!
BEHAVIOR OF OPTIONS PRICES

STEP 2: THE DOMINATED CONVERGENCE THEOREM

SETUP:

• \((T_n, U_n) \xrightarrow{\mathcal{L}} (T, U)\) AS \(n \to \infty\)
• \(|T_n| \leq U_n\) a.s., FOR ALL \(n\)
• \(E(U_n) \to E(U)\) AS \(n \to \infty\)

THEOREM:

UNDER THESE CONDITIONS:

\[ E(T_n) \to E(T) \text{ AS } n \to \infty \]

• CHECK THAT THEOREM IN SHREVE IS SPECIAL CASE

• GENERAL THEOREM:

  • See Billingsley: *Probability and Measure*
  • Deduce using Skorokhod embedding
  • For final: need only to be able to use above Theorem
BEHAVIOR OF OPTIONS PRICES

STEP 3: COMBINE THEOREMS

TAKE: $T_n = (S_T^{(n)} - e^{-rT}K)^+$ AND $U_n = S_T^{(n)}$

WE KNOW:
• $(T_n, U_n) \xrightarrow{\mathcal{L}} (T, U)$ AS $n \to \infty$
• $|T_n| \leq U_n$ a.s., FOR ALL $n$: $(S - e^{-rT}K)^+ \leq S$

WE NEED TO ESTABLISH

$$E(U_n) \to E(U) \text{ AS } n \to \infty$$ (5)

IF THIS IS THE CASE, WE CAN CONCLUDE THAT

$$\text{n step options price} = E(S_T^{(n)} - e^{-rT}K)^+ \to E(S_T^{(\infty)} - e^{-rT}K)^+$$ (6)

WHERE

$$S_T^{(\infty)} = \exp\{Z\}$$

AND

$$Z = N \left( \log(S_0) - \frac{1}{2} \sigma^2 T, \sigma^2 T \right)$$
COMPUTATION OF EXPECTED VALUES

\[ \log S_T = \log S_0 - \frac{1}{2} \sqrt{\frac{\sigma^2 T}{\gamma^2}} + \sqrt{\sigma^2 T} N(0, 1) \]

\[ E[f(S_T)] = E[f(\exp\{\log S_0 - \frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} N(0, 1)\})] \]

\[ = E[f(S_0 \exp\{-\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} z\})] \]

\[ = \int_{-\infty}^{+\infty} f(S_0 \exp\{-\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} z\}) \phi(z) dz \]

where \( \phi(z) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} z^2\} \)
IN PARTICULAR: \( f(s) = s \):

\[
E[U] = E[S_T] = \int_{-\infty}^{+\infty} S_0 \exp\left\{-\frac{1}{2}\sigma^2 T + \sqrt{\sigma^2 T} z\right\} \phi(z) dz
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S_0 \exp\left\{-\frac{1}{2}\sigma^2 T + \sqrt{\sigma^2 T} z - \frac{1}{2}z^2\right\} dz
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S_0 \exp\left\{-\frac{1}{2}(z - \sqrt{\sigma^2 T})^2\right\} dz
\]

\[
= S_0 \int_{-\infty}^{+\infty} \phi(z - \sqrt{\sigma^2 T}) dz
\]

\[
= S_0 \int_{-\infty}^{+\infty} \phi(u) du \quad (u = z - \sqrt{\sigma^2 T})
\]

\[
= S_0
\]

IT FOLLOWS THAT EQUATION (5) IS SATISFIED
THE BLACK-SCHOLES-MERTON FORMULA

• THE OPTIONS PRICE FOR LARGE $n$ IS

$$E(\tilde{S}^{(\infty)}_T - e^{-rT} K)^+$$

• CAN COMPUTE IT EXPLICITELY USING EQUATION (7)

• THIS IS THE B-S-M FORMULA FOR THE PRICE OF A CALL OPTION

• YOU DON’T NEED TO USE A TREE IN THIS CASE
CONTINUOUS MARTINGALES

TWO CONTINUITIES:

• TIME ITSELF:

\[ M_t, \quad 0 \leq t \leq T \quad \text{(or } 0 \leq t < \infty) \]

• PROCESS PATH:

\[ t \rightarrow M_t = M_t(\omega) \quad \text{CONTINUOUS FUNCTION} \]

\[ \text{OF TIME} \]
(ADDITIONAL) BROWNIAN MOTION $W_t$, $0 \leq t \leq T$

1. $W_0 = 0$
2. $t \rightarrow W_t(\omega)$ IS CONTINUOUS for each $\omega$
3. HAS INDEPENDENT INCREMENTS
4. $W_{t+s} - W_s \sim N(0, t)$

PICTURE OF (3):

\[
\begin{array}{cccc}
\Delta W_{t_0} & \Delta W_{t_1} & \Delta W_{t_2} & \text{INDEPENDENT} \\
= W_{t_1} - W_{t_0} & = W_{t_2} - W_{t_1} \\
0 = t_0 & t_1 & t_2 & t_3 \\
\end{array}
\]

ANY GRID

ADDITIVE PROPERTY (4):

$\Delta W_{t_0} \sim N(0, t_1)$, $\Delta W_{t_1} \sim N(0, t_2 - t_1)$

DELETE $t_1$: $W_{t_2} - W_{t_0} = \Delta W_{t_0} + \Delta W_{t_1}$

$N(0, t_1) + N(0, t_2 - t_1)$

BY INDEP: $N(0, t_2)$
\[(3) + (4) \implies W_t \text{ is a martingale}\]

\[
E(W_{t+s} \mid \mathcal{F}_s) = E(W_{t+s} - W_s + W_s \mid \mathcal{F}_s)
\]
\[
= E(W_{t+s} - W_s \mid \mathcal{F}_s) + W_s
\]
\[
= E(W_{t+s} - W_s) + W_s \quad \text{(independence)}
\]
\[
= 0 \quad \text{since } W_{t+s} - W_s \sim N(0, t)
\]
\[
= W_s
\]
THE BLACK-SCHOLES MODEL: MULTIPLICATIVE BROWNIAN MOTION

\[ \tilde{S}_t = \tilde{S}_0 \times \exp(\sigma W_t - \frac{1}{2} \sigma^2 t) \]

EVOLUTION:

\[
\tilde{S}_t = \tilde{S}_0 \times \exp(\sigma W_u - \frac{1}{2} \sigma^2 u) \quad \tilde{S}_u \quad \text{independent multiplicative increment}
\]

\[
\times \exp(\sigma(W_t - W_u) - \frac{1}{2} \sigma^2 (t - u))
\]

\[
= \tilde{S}_u \times \exp(\sigma N(0, t - u) - \frac{1}{2} \sigma^2 (t - u))
\]

\[
= \tilde{S}_u \times \exp(\alpha Z - \frac{1}{2} \alpha^2) \quad \alpha^2 = \sigma^2 (t - u) \quad Z \sim N(0, 1)
\]

MARTINGALE:

\[
E(\tilde{S}_t \mid \mathcal{F}_u) = \tilde{S}_u E(\exp(\sigma Z - \frac{1}{2} \sigma^2) \mid \mathcal{F}_u)
\]

\[
= \tilde{S}_u E(\exp(\sigma Z - \frac{1}{2} \sigma^2)) \text{ BY INDEPENDENCE}
\]

\[
= \tilde{S}_u \times 1 \quad (\text{NORMAL})
\]

\[
= \tilde{S}_u
\]
CLT FOR THE WHOLE PROCESS

\[ t_0 = 0 \quad t_1 = \frac{\sigma^2 T}{n} \quad t_2 = \frac{2\sigma^2 T}{n} \quad t_3 = \frac{3\sigma^2 T}{n} \quad t_k = \frac{k\sigma^2 T}{n} \]

STOCK PRICE PROCESS

\[ \log(\tilde{S}_t^{(n)}) - \log(S_0) = \sum_{t_i \leq t} X_i, \quad 0 \leq t \leq T \]

CONVERGENCE: AS \( n \to \infty \):

\[ \log(\tilde{S}_t^{(n)}) \xrightarrow{\mathcal{L}} \log(S_t) = \log(S_0) + \sigma W_t - \frac{1}{2} \sigma^2 t \]

GEOMETRIC BROWNIAN MOTION
APPLICATION TO OPTIONS

CONTINUOUS FUNCTIONALS

• $x = \{x_t, 0 \leq t \leq T\}$ A REALIZATION OF THE PROCESS

• $x \rightarrow h(x)$ TAKES REAL VALUES

• $x \rightarrow h(x)$ IS CONTINUOUS:

$$\sup_{0 \leq t \leq T} |x_t^{(n)} - x_t| \rightarrow 0 \implies h(x^{(n)}) \rightarrow h(x_t)$$

FOR $h$ CONTINUOUS:

$$h(\log(\tilde{S}_t^{(n)})) \xrightarrow{L} h(\log(\tilde{S}_t))$$

OR

$$h(\tilde{S}_t^{(n)}) \xrightarrow{L} h(\tilde{S}_t)$$

EXAMPLE OF MEANINGFUL LIMIT:

$$h(x) = \sup_{0 \leq t \leq T} x_t$$

LOOKBACK OPTIONS