APPROXIMATE NORMALITY

BINOMIAL MODEL: $S_{n+1} = \begin{cases} uS_n \\ dS_n \end{cases}$

LOG SCALE (IID ADDITIVE INCREMENTS):

$$\log S_n = \log S_0 + X_1 + ... + X_n$$

WITH $X_i = \log(u)$ or $= \log(d)$

TWO TIME SCALES

CLOCK TIME: T – TIME PERIODS: n

$$t_0 = 0 \quad t_1 = \frac{T}{n} \quad t_2 = \frac{2T}{n} \quad t_3 = \frac{3T}{n} \quad \ldots \quad t_k = \frac{kT}{n}$$

T IS FIXED – n IS A MATTER OF CHOICE

RETURN ON RISK FREE ASSET

(in clock time)$e^{rT} = e^{\rho n}$(in time periods)

$$\text{in other words: } \rho = r\frac{T}{n} \tag{1}$$

r IS FIXED – ρ DEPENDS ON n

RISK NEUTRAL MEASURE PER STEP:

$$\pi_n(T) = \frac{u - e^{\rho}}{u - d} = \frac{u - e^{r\frac{T}{n}}}{u - d} \text{ and } \pi_n(H) = \frac{e^{\rho} - d}{u - d} \tag{2}$$

BEHAVIOR OF ADDITIVE INCREMENTS

MEAN:

$$E(X) = \log(u)\pi(H) + \log(d)\pi(T)$$

TOTAL MEAN:

$$\begin{aligned} E(\log(S_n) - \log(S_0)) &= E(X_1) + ... + E(X_n) \\ &= nE(X) \\ &= n(\log(u)\pi(H) + \log(d)\pi(T)) \end{aligned}$$

VARIANCE: $X = \log d + (\log u - \log d)I_{\{H\}}$, and so

$$\begin{aligned} \text{Var}\ (X) &= (\log u - \log d)^2\ \text{Var}\ (I_{\{H\}}) \\ &= (\log u - \log d)^2\pi(H)\pi(T) \end{aligned}$$

TOTAL VARIANCE:

$$\begin{aligned} \text{Var}\ (\log(S_n)) &= \text{Var}\ (X_1) + ...+ \text{Var}\ (X_n) \\ &= n\ \text{Var}\ (X_1) \\ &= n(\log u - \log d)^2\pi(H)\pi(T) \end{aligned}$$

WE WISH TO KEEP TOTAL MEAN, VARIANCE CONSTANT IN CLOCK TIME

$$\begin{aligned} \nu T &= E(\log S_n) \\ &= n(\log(u)\pi(H) + \log(d)\pi(T)) \qquad (3) \\ \sigma^2 T &= \text{Var}\,(\log(S_n)) \\ &= n(\log u - \log d)^2 \pi(H)\pi(T) \qquad (4) \end{aligned}$$

σ OR σ^2 IS VOLATILITY IN CLOCK TIME
NEED TO USE: $\nu \approx r - \frac{1}{2}\sigma^2$

EQUATIONS (1)-(4) *DEFINE* A BINOMIAL TREE (ρ, u, d, $\pi(H)$, $\pi(T)$) ON THE BASIS OF:

- VOLATILITY PER UNIT CLOCK TIME: σ^2
- INTEREST PER UNIT CLOCK TIME: r
- # OF UNITS OF CLOCK TIME: T
- # OF TIME PERIODS IN COMPUTATION: n

AN APPROXIMATION FOR THE CASE $r = \rho = 0$ (THE DISCOUNTED PROCESS)

UP AND DOWN STEPS:

$$u = 1 + \sqrt{\frac{\sigma^2 T}{n}} \text{ AND } d = 1 - \sqrt{\frac{\sigma^2 T}{n}}$$

RISK NEUTRAL PROBABILITIES:

$$\pi_n(T) = \frac{u - e^\rho}{u - d} = \frac{1}{2} \text{ AND } \pi_n(H) = \frac{e^\rho - d}{u - d} = \frac{1}{2}$$

WE SHOW THAT EQUATIONS (3)-(4) ARE APPROXIMATELY SATISFIED

WILL USE THIS APPROXIMATE BINOMIAL TREE

APPROXIMATION TO CONDITION (4):

$$\log(1+x) \;=\; x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$$

$$x = \sqrt{\frac{\sigma^2 T}{n}}: \quad \log(u) \;=\; \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2}\frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots$$

$$x = -\sqrt{\frac{\sigma^2 T}{n}}: \quad \log(d) \;=\; -\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2}\frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots$$

AND SO:

$$\begin{aligned}
\mathrm{Var}\,(\log(S_n)) \;&=\; n\left(\log u - \log d\right)^2 \pi(H)\pi(T) \\
&=\; n\frac{1}{4}\left(\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2}\frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots\right. \\
&\qquad \left. - \,(-\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2}\frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots)\right)^2 \\
&=\; n\frac{1}{4}\left(2\sqrt{\frac{\sigma^2 T}{n}} + \frac{1}{n\sqrt{n}} \times \ldots\right)^2 \\
&=\; \sigma^2 T + \frac{1}{n} \times \ldots
\end{aligned}$$

ABOUT EQUATION (3):

$$
\begin{aligned}
\log(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots \\
\log(u) &= \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2}\frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots \\
\log(d) &= -\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2}\frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots
\end{aligned}
$$

AND SO:

$$
\begin{aligned}
\nu T &= E(\log(S_n) - \log(S_0)) \\
&= n(\log(u)\pi(H) - \log(d)\pi(T)) \\
&= \frac{1}{2}n\left(\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2}\frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots\right. \\
&\quad \left. +(-\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2}\frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots)\right) \\
&= -\frac{1}{2}\sigma^2 T + \frac{1}{\sqrt{n}} \times \ldots
\end{aligned}
$$

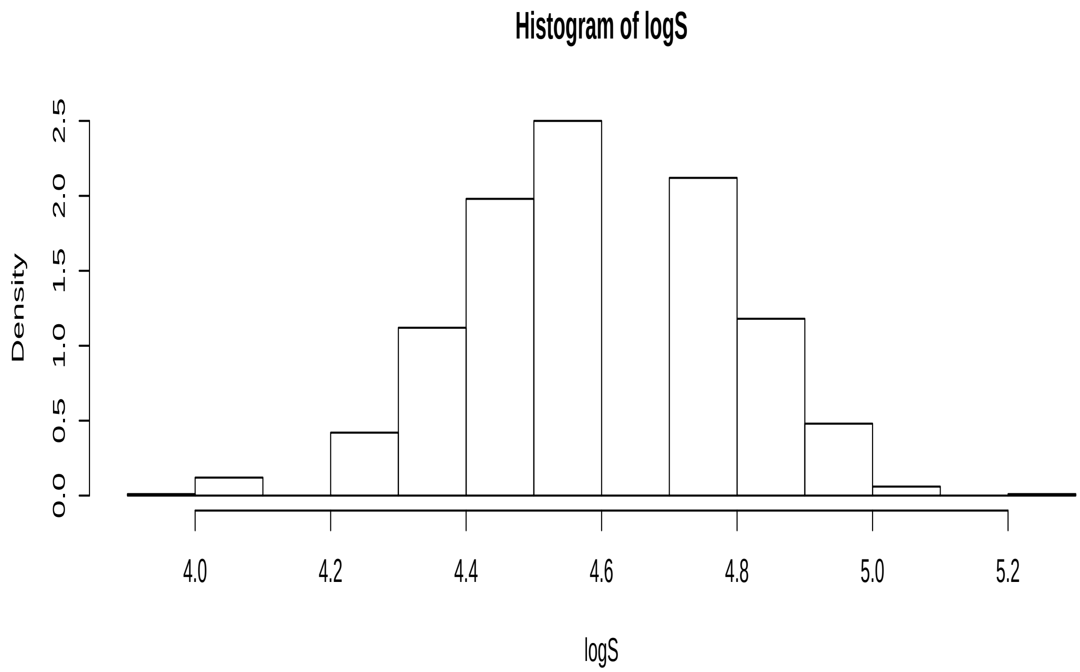
AS PREDICTED

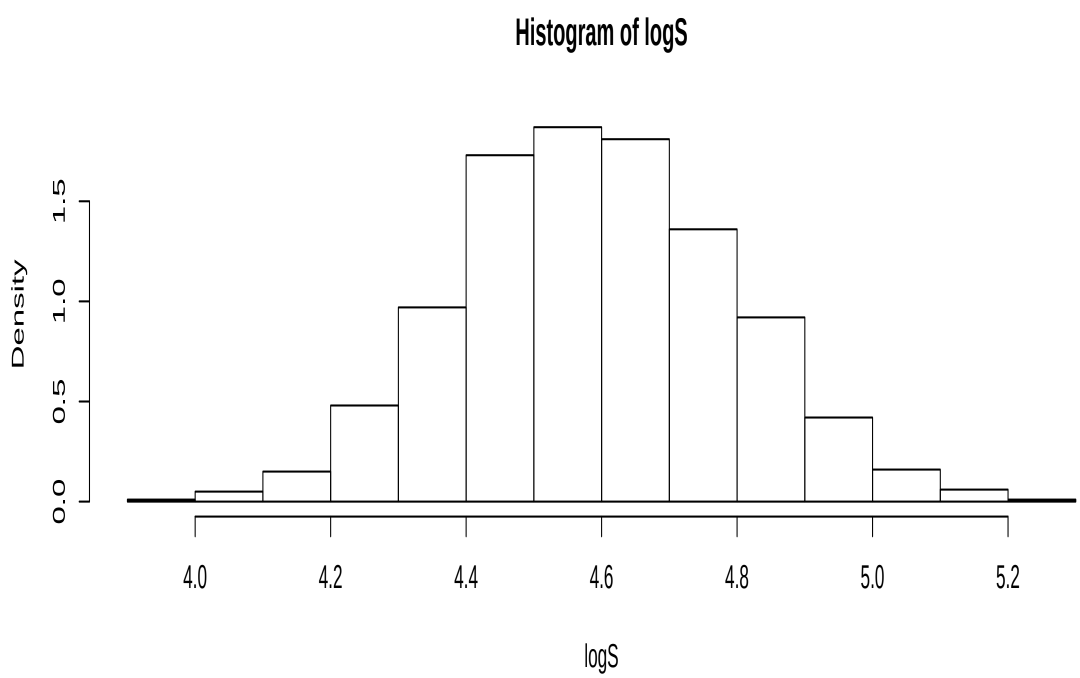
HOW MUCH DO OUR RESULTS DEPEND ON n?

TRYING THE MATTER OUT IN R

```
M <- 1000   # number of simulation steps
sigma <- .2 # clock time volatility
T <- 1      # clock time duration
S0 <- 100   # initial value
piH <- 1/2  # risk neutral probability
n <- 10     # steps
u <- 1 + sqrt(T*sigma^2/n) # up step
d <- 1 - sqrt(T*sigma^2/n) # down step
H<- rbinom(M,n,piH)        # simulation
logS <- log(S0) + log(u)*H + log(d)*(n-H)
par(mfrow=c(2,1)) # check this command out!
hist(logS,freq=F)
# try again with a larger number of steps
n <- 1000
# define u, d, H, logS as above, with new n
hist(logS,freq=F)
```

THE DISTRIBUTION OF $\log S_T$ STABILIZES





THE CENTRAL LIMIT PHENOMENON

THEOREM: SUPPOSE THAT

- X_i, $i = 1, ..., n$ ARE IID P_n

  (DISTRIBUTION CAN DEPEND ON n)
- $n \ \mathrm{Var}_n \ (X) \to \gamma^2$ AS $n \to \infty$

THEN

$$\sum_{i=1}^{n} X_i - nE_n(X) \xrightarrow{\mathcal{L}} N(0, \gamma^2)$$

IN WORDS:

$\sum_{i=1}^{n} X_i - nE_n(X)$ CONVERGES IN LAW TO $N(0, \gamma^2)$

THAT IS TO SAY:

THE DISTRIBUTION OF $\sum_{i=1}^{n} X_i - nE_n(X)$ IS APPROXIMATELY NORMAL $N(0, \gamma^2)$

DENSITY OF THE NORMAL DISTRIBUTION $N(\mu, \gamma^2)$

$$\frac{d}{dx} P(N(\mu, \gamma^2) \leq x) = \frac{1}{\gamma} \phi \left(\frac{x - \mu}{\gamma} \right)$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} x^2\}$$

IN OUR CASE

$$\log(S_T) - \log(S_0) = \sum_{i=1}^{n} X_i$$

$$\gamma^2 = \sigma^2 T$$

$$E(\log(S_T) - \log(S_0)) = nE_n(X) \approx -\frac{1}{2}\sigma^2 T$$

SO THAT

$$\log(S_T) - \left(\log(S_0) - \frac{1}{2}\sigma^2 T\right)$$

IS APPROXIMATELY NORMAL $N(0, \sigma^2 T)$

OR: $\log(S_T)$ IS APPROXIMATELY NORMAL

$$N(\log(S_0) - \frac{1}{2}\sigma^2 T, \sigma^2 T)$$

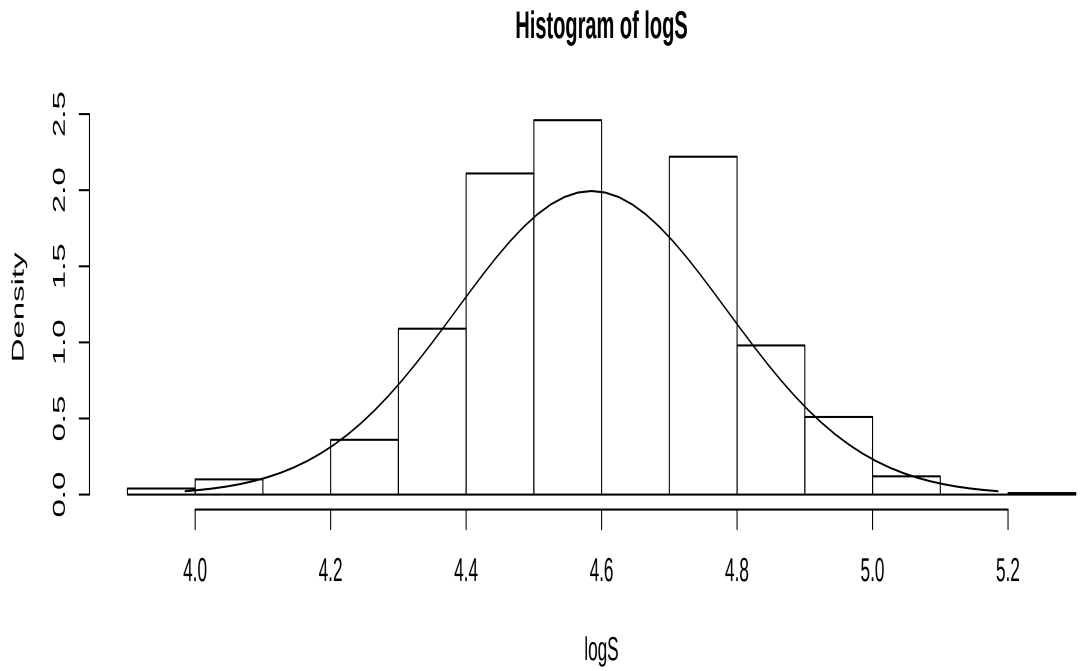
$$\text{Note: } Z \sim N(\mu, \gamma^2) <=> \quad Z \sim \mu \sim N(0, \gamma^2)$$

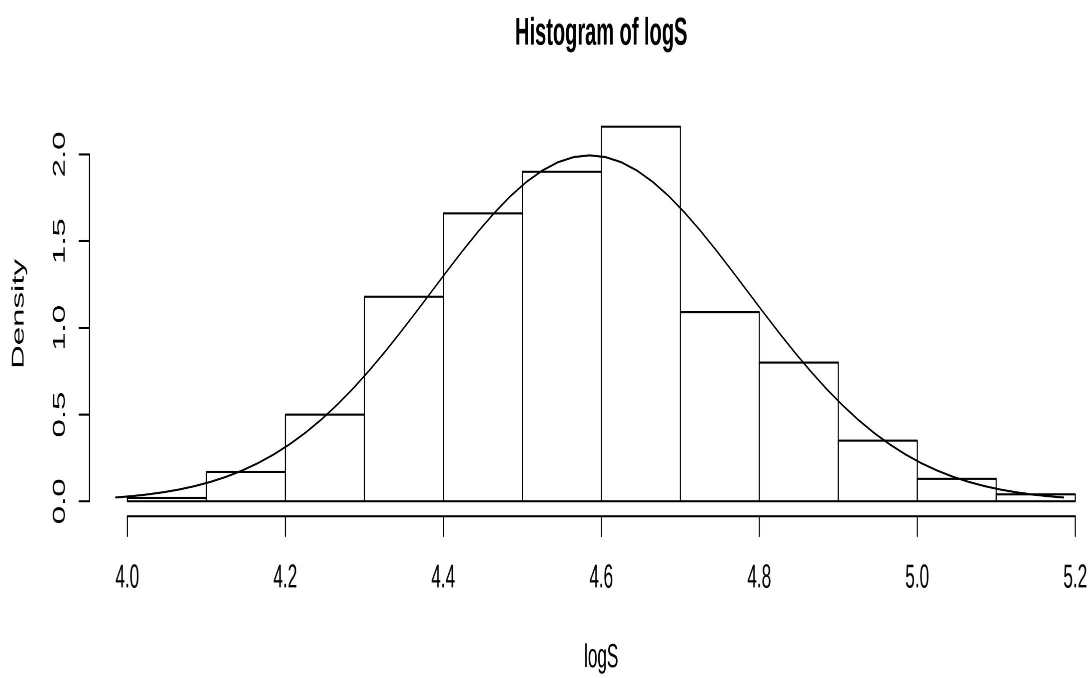
$$<=> \quad \frac{Z - \mu}{\gamma} \sim N(0, 1)$$

SUPERIMPOSING THE NORMAL CURVE ON THE HISTOGRAM

```
n <- 10     # steps
u <- 1 + sqrt(T*sigma^2/n) # up step
d <- 1 - sqrt(T*sigma^2/n) # down step
H<- rbinom(M,n,piH)        # simulation
logS <- log(S0) + log(u)*H + log(d)*(n-H)
par(mfrow=c(2,1)) # check this command out!
hist(logS,freq=F)
# compare to normal distribution
xpoints<-c(-30:30)/10
mu<-log(S0)-(sigma^2*T)/2
gamma<-sqrt(sigma^2*T)
xpoints<-c(-30:30)/10
xpoints<-mu+sigma*xpoints
density<-dnorm(xpoints,mean=mu,sd=gamma)
lines(xpoints,density)
# try again with a larger number of steps
n <- 1000
# define u, d, H, logS as above, with new n
hist(logS,freq=F)
# mu, gamma, xpoints stay the same
lines(xpoints,density)
```

NORMAL CURVE SUPERIMPOSED ON HISTOGRAMS





THE CLASSICAL CENTRAL LIMIT THEOREM

(A digression. Just so you know.)

SETUP:

$Y_1, ..., Y_n$ ARE IID, $E(Y) = 0$ AND $\text{Var}\,(Y) = \gamma^2$

THEN:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i \xrightarrow{\mathcal{L}} N(0, \gamma^2)$$

PROOF:

$$\text{TAKE } X_i = \frac{1}{\sqrt{n}} Y_i$$

IN EARLIER THEOREM

RESULT FOLLOWS

BEHAVIOR OF OPTIONS PRICES

STEP 1: CONTINUOUS FUNCTIONS

THEOREM: IF

- $Z_n \xrightarrow{\mathcal{L}} Z$ AS $n \longrightarrow \infty$
- $x \to h(x)$ IS A CONTINUOUS FUNCTION

THEN $h(Z_n) \xrightarrow{\mathcal{L}} h(Z)$ AS $n \longrightarrow \infty$

EXAMPLE

$$Z_n = \log(S_T^{(n)}) \xrightarrow{\mathcal{L}} Z = N\left(\log(S_0) - \frac{1}{2}\sigma^2 T, \sigma^2 T\right)$$

CONTINUOUS FUNCTION #1: $h(x) = e^x$:

$$S_T^{(n)} = \exp\{Z_n\} \xrightarrow{\mathcal{L}} S_T^{(\infty)} = \exp\{Z\}$$

CONTINUOUS FUNCTION #2: $h(x) = (x - e^{-rT}K)^+$:

$$V_T^{(n)} = (S_T^{(n)} - e^{-rT}K)^+ \xrightarrow{\mathcal{L}} (S_T^{(\infty)} - e^{-rT}K)^+$$

CHECK THIS IN R!

BEHAVIOR OF OPTIONS PRICES

STEP 2: THE DOMINATED CONVERGENCE THEOREM

SETUP:

- $(T_n, U_n) \xrightarrow{\mathcal{L}} (T, U)$ AS $n \longrightarrow \infty$
- $|T_n| \leq U_n$ a.s., FOR ALL n
- $E(U_n) \longrightarrow E(U)$ AS $n \longrightarrow \infty$

THEOREM:

UNDER THESE CONDITIONS:

$$E(T_n) \longrightarrow E(T) \text{ AS } n \longrightarrow \infty$$

- CHECK THAT THEOREM IN SHREVE IS SPECIAL CASE
- GENERAL THEOREM:
  - See Billingsley: *Probability and Measure*
  - Deduce using Skorokhod embedding
  - For final: need only to be able to use above Theorem

BEHAVIOR OF OPTIONS PRICES

STEP 3: COMBINE THEOREMS

TAKE: $T_n = (S_T^{(n)} - e^{-rT}K)^+$ AND $U_n = S_T^{(n)}$

WE KNOW:

- $(T_n, U_n) \xrightarrow{\mathcal{L}} (T, U)$ AS $n \longrightarrow \infty$
- $|T_n| \leq U_n$ a.s., FOR ALL n: $(S - e^{-rT}K)^+ \leq S$

WE NEED TO ESTABLISH

$$E(U_n) \longrightarrow E(U) \text{ AS } n \longrightarrow \infty \tag{5}$$

IF THIS IS THE CASE, WE CAN CONCLUDE THAT

$$\begin{aligned} \text{n step options price} &= E(S_T^{(n)} - e^{-rT}K)^+ \\ &\longrightarrow E(S_T^{(\infty)} - e^{-rT}K)^+ \end{aligned} \tag{6}$$

WHERE

$$S_T^{(\infty)} = \exp\{Z\}$$

AND

$$Z = N\left(\log(S_0) - \frac{1}{2}\sigma^2 T, \sigma^2 T\right)$$

COMPUTATION OF EXPECTED VALUES

$$\log S_T = \log S_0 - \frac{1}{2}\underbrace{\sigma^2 T}_{\gamma^2} + \sqrt{\sigma^2 T}N(0,1)$$

$$\begin{aligned} E[f(S_T)] &= E[f(\exp\{\log S_0 - \frac{1}{2}\sigma^2 T + \sqrt{\sigma^2 T}N(0,1)\})] \\ &= E[f(S_0 \exp\{-\frac{1}{2}\sigma^2 T + \sqrt{\sigma^2 T}N(0,1)\})] \\ &= \int_{-\infty}^{+\infty} f(S_0 \exp\{-\frac{1}{2}\sigma^2 T + \sqrt{\sigma^2 T}z\})]\phi(z)dz \end{aligned} \tag{7}$$

$$\text{where } \phi(z) = \frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}z^2\}$$

IN PARTICULAR: $f(s) = s$:

$$
\begin{aligned}
E[U] &= E[S_T] \\
&= \int_{-\infty}^{+\infty} S_0 \exp\{-\frac{1}{2}\sigma^2 T + \sqrt{\sigma^2 T} z\}\phi(z) dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S_0 \exp\{-\frac{1}{2}\sigma^2 T + \sqrt{\sigma^2 T} z - \frac{1}{2} z^2\} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S_0 \exp\{-\frac{1}{2}(z - \sqrt{\sigma^2 T})^2\} dz \\
&= S_0 \int_{-\infty}^{+\infty} \phi(z - \sqrt{\sigma^2 T}) dz \\
&= S_0 \int_{-\infty}^{+\infty} \phi(u) du \quad (u = z - \sqrt{\sigma^2 T}) \\
&= S_0
\end{aligned}
$$

IT FOLLOWS THAT EQUATION (5) IS SATISFIED

THE BLACK-SCHOLES-MERTON FORMULA

• THE OPTIONS PRICE FOR LARGE n IS

$$E(\tilde{S}_T^{(\infty)} - e^{-rT}K)^+$$

• CAN COMPUTE IT EXPLICITELY USING EQUATION (7)

• THIS IS THE B-S-M FORMULA FOR THE PRICE OF A CALL OPTION

• YOU DON'T NEED TO USE A TREE IN THIS CASE

CONTINUOUS MARTINGALES

TWO CONTINUITIES:

- TIME ITSELF:

$$M_t, \qquad 0 \leq t \leq T \qquad (\text{or } 0 \leq t < \infty)$$

- PROCESS PATH:

$$t \to M_t = M_t(\omega) \quad \text{CONTINUOUS FUNCTION OF TIME}$$

(ADDITIVE) BROWNIAN MOTION $W_t,\ 0 \le t \le T$

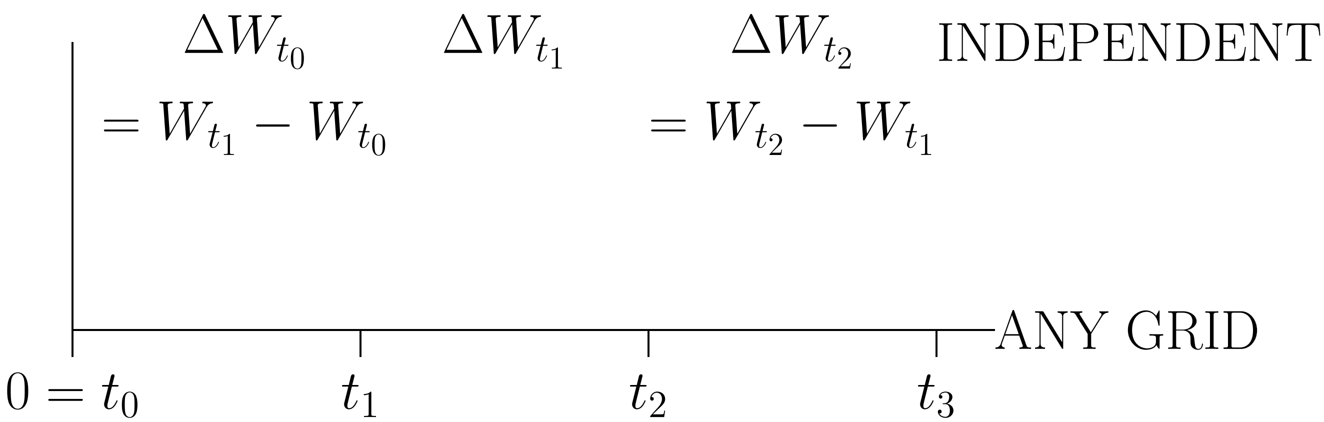
(1) $W_0 = 0$

(2) $t \to W_t(\omega)$ IS CONTINUOUS for each ω

(3) HAS INDEPENDENT INCREMENTS

(4) $W_{t+s} - W_s \sim N(0, t)$

PICTURE OF (3):



ADDITIVE PROPERTY (4):

$\Delta W_{t_0} \sim N(0, t_1)$, $\Delta W_{t_1} \sim N(0, t_2 - t_1)$

$$\text{DELETE } t_1 : W_{t_2} - W_{t_0} \;=\; \underbrace{\begin{array}{c}\Delta W_{t_0} \quad + \quad \Delta W_{t_1} \\ N(0, t_1) + N(0, t_2 - t_1)\end{array}}_{\text{BY INDEP: } N(0, t_2)}$$

$(3) + (4) \Longrightarrow W_t$ IS A MARTINGALE

$$
\begin{aligned}
E(W_{t+s} \mid \mathcal{F}_s) &= E(W_{t+s} - W_s + W_s \mid \mathcal{F}_s) \\
&= E(W_{t+s} - W_s \mid \mathcal{F}_s) + W_s \\
&= \underbrace{E(W_{t+s} - W_s)}_{=0 \quad \text{since } W_{t+s} - W_s \sim N(0,t)} + W_s \quad \text{(independence)} \\
&= W_s
\end{aligned}
$$

THE BLACK-SCHOLES MODEL: MULTIPLICATIVE BROWNIAN MOTION

$$\tilde{S}_t = \tilde{S}_0 \times \exp(\sigma W_t - \frac{1}{2}\sigma^2 t)$$

EVOLUTION:

$$\left.\tilde{S}_t = \tilde{S}_0 \times \exp(\sigma W_u - \tfrac{1}{2}\sigma^2 u)\right\} \tilde{S}_u$$

independent
multiplicative
increment

$$\times \exp(\sigma(W_t - W_u) - \tfrac{1}{2}\sigma^2(t-u))$$

$$= \tilde{S}_u \times \exp(\underbrace{\sigma N(0, t-u)}_{\sigma\sqrt{t-u}\ N(0,1)} - \tfrac{1}{2}\sigma^2(t-u))$$

$$= \tilde{S}_u \times \exp(\alpha Z - \tfrac{1}{2}\alpha^2) \quad \alpha^2 = \sigma^2(t-u) \quad Z \sim N(0,1)$$

MARTINGALE:

$$\begin{aligned} E(\tilde{S}_t \mid \mathcal{F}_u) &= \tilde{S}_u E(\exp(\sigma Z - \frac{1}{2}\sigma^2) \mid \mathcal{F}_u) \\ &= \tilde{S}_u E(\exp(\sigma Z - \frac{1}{2}\sigma^2) \text{ BY INDEPENDENCE} \\ &= \tilde{S}_u \times 1 \quad \text{(NORMAL)} \\ &= \tilde{S}_u \end{aligned}$$

CLT FOR THE WHOLE PROCESS

$$t_0 = 0 \quad t_1 = \frac{\sigma^2 T}{n} \quad t_2 = \frac{2\sigma^2 T}{n} \quad t_3 = \frac{3\sigma^2 T}{n} \qquad\qquad t_k = \frac{k\sigma^2 T}{n}$$

STOCK PRICE PROCESS

$$\log(\tilde{S}_t^{(n)}) - \log(S_0) = \sum_{t_i \le t} X_i, \quad 0 \le t \le T$$

CONVERGENCE: AS $n \to \infty$:

$$\log(\tilde{S}_t^{(n)}) \xrightarrow{\mathcal{L}} \log(S_t) = \log(S_0) + \sigma W_t - \frac{1}{2}\sigma^2 t$$

GEOMETRIC BROWNIAN MOTION

APPLICATION TO OPTIONS

CONTINUOUS FUNCTIONALS

- $x = \{x_t, 0 \leq t \leq T\}$ A REALIZATION OF THE PROCESS
- $x \to h(x)$ TAKES REAL VALUES
- $x \to h(x)$ IS CONTINUOUS:

$$\sup_{0 \leq t \leq T} |x_t^{(n)} - x_t| \to 0 \;\; => \;\; h(x^{(n)}) \to h(x_t)$$

FOR h CONTINUOUS:

$$h(\log(\tilde{S}_t^{(n)})) \xrightarrow{\mathcal{L}} h(\log(\tilde{S}_t))$$

OR

$$h(\tilde{S}_t^{(n)}) \xrightarrow{\mathcal{L}} h(\tilde{S}_t)$$

EXAMPLE OF MEANINGFUL LIMIT:

$$h(x) = \sup_{0 \leq t \leq T} x_t$$

LOOKBACK OPTIONS