ONE PERIOD MODELS

\[ t = \text{TIME} = 0 \text{ or } 1 \]

BASIC INSTRUMENTS:

* \( S_t \): STOCK
* \( B_t \): RISKLESS BOND

\[ B_0 = 1 \quad B_1 = 1 + r \text{ or } e^r \]

* FORWARD CONTRACT:
  AGREEMENT TO SWAP $$ FOR STOCK
  - AGREEMENT TIME: \( t = 0 \)
  - AGREEMENT PRICE: \( F_0 \)
  - SWAP TIME \( t = 1 \)
  - SWAP: $$ \( F_0 \) FOR 1 STOCK \( S_1 \)

ACTUAL INSTRUMENT: \( W_t \):

\[ W_0 = 0 \quad W_1 = S_1 - F_0 \]
ABITRAGE ARGUMENT: $F_0 = e^r S_0$

If $F_0 < e^r S_0$:

FORM PORTFOLIO AT $t = 0$: NET POSITION

SELL 1 STOCK
BUY $S_0$ # BONDS
ENTER 1 FORWARD CONTR

TOTAL PORTFOLIO

$$V_t = -S_t + S_0 B_t + W_t$$

VALUE: $V_0 = -S_0 + S_0 + 0 = 0$

$$V_1 = -S_1 + e^r S_0 + S_1 - F_0 = e^r S_0 - F_0 > 0$$

$V_t$ IS AN ARBITRAGE: MONEY FOR NOTHING
NOT SUPPOSED TO OCCUR
MORE GENERALLY...

* $K + 1$ ASSETS, PRICE $S^j_t$, $j = 1, \ldots, K$  
  $S^0 = B$ IS BOND: $S^0_0 = 1$, $S^1_1 = e^r$

* PORTFOLIO $\Delta = (\Delta_0, \ldots, \Delta_K)$:

  HOLD $\Delta_j$ # OF SECURITY $S^j_t$

* VALUE OF PORTFOLIO:

  $$V_t(\Delta) = \sum_{j=0}^{K} \Delta_j S^j_t$$

* ARBITRAGE: A PORTFOLIO FOR WHICH

  $$V_0(\Delta) \leq 0$$
  $$V_1(\Delta) \geq 0$$
  $$V_1(\Delta) > 0 \text{ IN SOME SCENARIO}$$

  $$\left(= \text{ WITH PROBABILITY} > 0 \right)$$
TWO STATE MODEL FOR STOCK

\[ S_0 < \begin{cases} S_1 = uS_0 & \text{SCENARIO } H: \ S_1(H) \\ S_1 = dS_0 & \text{SCENARIO } T: \ S_1(T) \end{cases} \]

Conditions to avoid arbitrage

Consider portfolio: buy \( \frac{1}{B_0} \) # of bonds, \( -\frac{1}{S_0} \) # of stocks at time 0. Properties:

\[ V_0 = \frac{1}{B_0} B_0 - \frac{1}{S_0} S_0 = 0 \]

\[ V_1 = \frac{1}{B_0} B_1 - \frac{1}{S_0} S_1 = \begin{cases} e^r - u & \text{under scenario } H \\ e^r - d & \text{under scenario } T \end{cases} \]

If \( e^r \geq u \):

\[ V_1 \geq 0 \quad \text{under } H \quad \text{ARBITRAGE} \]

\[ V_1 > 0 \quad \text{under } T \quad \text{NOT ALLOWED} \]

It follows that \( u > e^r \) (unless \( P(T)=0: \ e^r = u \) OK)

Similarly: portfolio \( -\frac{1}{B_0} \) bonds, \( \frac{1}{S_0} \) stocks \( \Rightarrow e^r > d \)

CONCLUSION: no arbitrage implies \( u > e^r > d \).
CALL OPTIONS

\[ V_1 = (S_1 - K)^+ \quad V_0 = ??? \]

\[ S_0 \xleftarrow{\text{SCENARIO } H} S_1 = uS_0 \]
\[ S_1 = dS_0 \quad \text{SCENARIO } T \]

Suppose \( uS_0 > K > dS_0 \)
otherwise \( V_1 = S_1 - K \) (\( dS_0 \geq K \)) or \( V_1 = 0 \) (\( uS_0 \leq K \))

REPLICATING PORTFOLIO
Buy \( \Delta_0 \) bonds and \( \Delta_1 \) stocks at time 0

portfolio: \( V_t(\Delta) = \Delta_0 B_t + \Delta_1 S_t \)
replication: \( (S_1 - K)^+ = \Delta_0 B_1 + \Delta_1 S_1 \)

FINDING THE \( \Delta \)s: 2 EQUATIONS, 2 Unknowns:

\[ uS_0 - K = \Delta_0 e^r + \Delta_1 uS_0 \quad \text{SCENARIO } H \]
\[ 0 = \Delta_0 e^r + \Delta_1 dS_0 \quad \text{SCENARIO } T \]

OR: \( \Delta_1 = \frac{uS_0 - K}{uS_0 - dS_0} \) and \( \Delta_0 = -e^{-r} \Delta_1 dS_0 \)

PRICE FOR THIS OPTION:

\[ V_0 = \Delta_0 B_0 + \Delta_1 S_0 \]
\[ = \Delta_1 e^{-r} (-dS_0 + e^r S_0) \]
ARGUMENT DEPENDS ON

• bond, stock can be bought or sold in any quantity
• bond, stock can be short sold (in the case of bond: this means that borrowing rate is same as lending rate)
• no bid-ask spread
• binomial model

Binomial model is oversimplification
“Brownian motion” is close to binomial model
Increasingly realistic models as the course progresses

ARGUMENT DOES NOT DEPEND ON

• Assumption of no arbitrage, except that $u > e^r > d$
MORE GENERAL DERIVATIVE SECURITIES
IN THE ONE PERIOD BINOMIAL MODEL

payoff $V_1(H)$ or $V_1(T)$
or $V_1 = f(S_1)$

where $f(s) = \begin{cases} 
(s - K)^+ & \text{call option} \\
(K - s)^+ & \text{put option} \\
\text{etc} & 
\end{cases}$

REPLICATING PORTFOLIO
Buy $\Delta_0$ bonds and $\Delta_1$ stocks at time 0

portfolio: $V_t(\Delta) = \Delta_0 B_t + \Delta_1 S_t$
replication: $f(S_1) = \Delta_0 B_1 + \Delta_1 S_1$

FINDING THE $\Delta$s: 2 EQUATIONS, 2 unknowns:

$f(uS_0) = \Delta_0 e^r + \Delta_1 uS_0$ SCENARIO $H$
$f(dS_0) = \Delta_0 e^r + \Delta_1 dS_0$ SCENARIO $T$

OR: $\Delta_1 = \frac{f(uS_0) - f(dS_0)}{uS_0 - dS_0}$ and $\Delta_0 = e^{-r} \frac{uf(dS_0) - df(uS_0)}{u - d}$

PRICE FOR THIS OPTION:

$V_0 = \Delta_0 B_0 + \Delta_1 S_0$
DISCOUNTING

Discounted stock: \( S_t^* = S_t / B_t \)
Discounted bond: \( B_t^* = B_t / B_t = 1 \)
Discounted portfolio value: \( V_t^* = V_t / B_t \)

NUMERAIRE INVARIANCE

Portfolio in original numeraire: \( V_t(\Delta) = \Delta_0 B_t + \Delta_1 S_t \)

Portfolio in discounted numeraire:

\[
V_t^*(\Delta) = \frac{\Delta_0 B_t + \Delta_1 S_t}{B_t} = \Delta_0 B_t^* + \Delta_1 S_t^*
\]

The number \( \Delta_0, \Delta_1 \) of bonds, stocks is the same in original and discounted numeraire

Exit interest. This is often convenient

TWO EQUATIONS, TWO UNKNOWNS
ON DISCOUNTED SCALE

\[
V_1^*(H) = \Delta_1 + \Delta_2 u S_0^* \quad \text{SCENARIO H}
\]
\[
V_1^*(T) = \Delta_1 + \Delta_2 d S_0^* \quad \text{SCENARIO T}
\]
PROBABILISTIC INTERPRETATION

Let \( \pi(H) \), \( \pi(T) \) be two numbers

From \( V_t^* = \Delta_0 B_t^* + \Delta_1 S_t^* \):

\[
\pi(H)V_1^*(H) + \pi(T)V_1^*(T) = \pi(H) (\Delta_0 B_1^*(H) + \Delta_1 S_1^*(H)) + \pi(T) (\Delta_0 B_1^*(T) + \Delta_1 S_1^*(T))
\]

\[
= \Delta_0 (\pi(H)B_1^*(H) + \pi(T)B_1^*(T)) + \Delta_1 (\pi(H)S_1^*(H) + \pi(T)S_1^*(T))
\]

\[
= \Delta_0 B_0^* + \Delta_1 S_0^* = V_0^*
\]

provided

\[
\left\{ \begin{array}{l}
\pi(H)B_1^*(H) + \pi(T)B_1^*(T) = B_0^* \quad (**) \\
\pi(H)S_1^*(H) + \pi(T)S_1^*(T) = S_0^* \quad (***)
\end{array} \right.
\]

\( B_t^* = 1: (***) \iff \pi(H) + \pi(T) = 1 \)

\( \pi \) is a probability measure, provided \( \pi(H), \pi(T) \geq 0 \)

This is the case since, by solving (**)-(***):

\[
\pi(T) = \frac{u - e^r}{u - d} \quad \text{and} \quad \pi(H) = \frac{e^r - d}{u - d}
\]

If \( E_\pi \) is expectation under \( \pi \):

\( (E_\pi X = X(H)\pi(H) + X(T)\pi(T)) \)

\[
(*) \iff E_\pi V_1^* = V_0^*
\]

\( (***) \iff E_\pi S_1^* = S_0^* \)

\( \pi \) IS A “RISK NEUTRAL” PROBABILITY MEASURE
A RISK NEUTRAL PROBABILITY DISTRIBUTION:

$$\pi : \quad S_j^0 = e^{-r} E_\pi S_j^1 = E_\pi S_j^1^* \text{ for all } j$$

FUNDAMENTAL THEOREM OF ARBITRAGE PRICING:

THERE EXISTS A RISK NEUTRAL MEASURE IF AND ONLY IF ARBITRAGE DOES NOT OCCUR.

EVALUATING PRICES USING $\pi$:

$$V_0 = \sum_{j=1}^{K} \Delta_j S_j^0$$

$$= e^{-r} \sum_{j=1}^{K} \Delta_j E_\pi S_j^1$$

$$= e^{-r} E_\pi V_1$$

BUT WHAT IS $\pi$?

BRUNO DE FINETTI (1937): FORESIGHT: ITS LOGICAL LAWS, ITS SUBJECTIVE SOURCES:

"PROBABILITY DOES NOT EXIST."
HORSE RACING
(from Baxter and Rennie: Financial Calculus. An introduction to derivative pricing.)

TWO HORSES: $H1, H2$

<table>
<thead>
<tr>
<th>ACTUAL CHANCE OF WINNING</th>
<th>BETS PLACED ON HORSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H1$ 25%</td>
<td>$5000</td>
</tr>
<tr>
<td>$H2$ 75%</td>
<td>$10000</td>
</tr>
<tr>
<td>TOTAL FOR BOOKIE</td>
<td>$15000</td>
</tr>
</tbody>
</table>

PRICE OF BETS

ACTUAL PROBABILITIES: $\bar{\pi}_1 = \frac{1}{4} \bar{\pi}_2 = \frac{3}{4}$

BETTING $1$ ON HORSE $H1$: WIN $4$
BETTING $1$ ON HORSE $H2$: WIN $\frac{4}{3}$

OUTCOME FOR BOOKIE:

WINNING HORSE  $$
\begin{align*}
H1 & : 15000 - 4 \times 5000 = -5000 \\
H2 & : 15000 - \frac{4}{3} \times 10000 = 1666
\end{align*}

A RISKY BUSINESS
HORSE RACING:

RISK NEUTRAL PROBABILITY

<table>
<thead>
<tr>
<th>ACTUAL CHANCE OF WINNING</th>
<th>BETS PLACED ON HORSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H1$ IRRELEVANT</td>
<td>$5000$</td>
</tr>
<tr>
<td>$H2$</td>
<td>$10000$</td>
</tr>
</tbody>
</table>

PRICE OF BETS:

$1$ ON $H1$ GIVES $\frac{1}{\pi_1}$$ IF WIN
$1$ ON $H2$ GIVES $\frac{1}{\pi_2}$$ IF WIN

OUTCOME FOR BOOKIE:

<table>
<thead>
<tr>
<th>WINNER</th>
<th>$$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H1$</td>
<td>$15000 - \frac{1}{\pi_1} \times 5000$</td>
</tr>
<tr>
<td>$H2$</td>
<td>$15000 - \frac{1}{\pi_2} \times 10000$</td>
</tr>
</tbody>
</table>

OUTCOME = 0 IF $\bar{\pi}_1 = \frac{1}{3}$, $\bar{\pi}_2 = \frac{2}{3}$ A SAFE BUSINESS.

JUST LIKE SELLING OPTIONS...
COMPLETE MARKETS

EQUIVALENT:

1) ALL RANDOM VARIABLES $V_1$ CAN BE REPRESENTED

$$V_1 = \Delta_0 B_1 + \Delta_1 S^1_1 + \cdots + \Delta_K S^K_1$$

2) $\pi$ IS UNIQUE

IN CALL OPTION — TWO STATE MODEL: DID NOT NEED TO VERIFY EXISTENCE OF PORTFOLIO