ONE PERIOD MODELS

$$t = \text{ TIME } = 0 \text{ or } 1$$

BASIC INSTRUMENTS:

* S_t: STOCK
* B_t: RISKLESS BOND

$$B_0 = 1 \qquad B_1 = 1 + r \text{ or } e^r$$

* FORWARD CONTRACT:
  AGREEMENT TO SWAP $$ FOR STOCK
  - AGREEMENT TIME: $t = 0$
  - AGREEMENT PRICE: F_0
  - SWAP TIME $t = 1$
  - SWAP: $$ F_0 FOR 1 STOCK S_1

ACTUAL INSTRUMENT: W_t:

$$W_0 = 0 \qquad W_1 = S_1 - F_0$$

ABITRAGE ARGUMENT: $F_0 = e^r S_0$

<u>If $F_0 < e^r S_0$:</u>

FORM PORTFOLIO AT $t = 0$: NET POSITION

SELL 1 STOCK	$-S_t$
BUY S_0 # BONDS	$S_0 B_t$
ENTER 1 FORWARD CONTR	W_t
TOTAL PORTFOLIO	$V_t = -S_t + S_0 B_t + W_t$

VALUE: $V_0 = -S_0 + S_0 + 0 = 0$

$$V_1 = -S_1 + e^r S_0 + S_1 - F_0 = e^r S_0 - F_0 > 0$$

V_t IS AN ARBITRAGE: MONEY FOR NOTHING

NOT SUPPOSED TO OCCUR

MORE GENERALLY...

* $K+1$ ASSETS, PRICE S_t^j, $j=1,\ldots,K$
  $S^0 = B$ IS BOND: $S_0^0 = 1$, $S_1^0 = e^r$

* PORTFOLIO $\Delta = (\Delta_{0,\ldots,}\Delta_K)$:

  HOLD Δ_j # OF SECURITY S_t^j

* VALUE OF PORTFOLIO:

$$V_t(\Delta) = \sum_{j=0}^{K} \Delta_j S_t^j$$

* ARBITRAGE: A PORTFOLIO FOR WHICH

  $V_0(\Delta) \leq 0$
  $V_1(\Delta) \geq 0$
  $V_1(\Delta) > 0$ IN SOME SCENARIO
  ($=$ WITH PROBABILITY > 0)

TWO STATE MODEL FOR STOCK

$$S_0 \begin{array}{l} \nearrow S_1 = uS_0 \quad \text{SCENARIO } H\text{: } S_1(H) \\ \searrow S_1 = dS_0 \quad \text{SCENARIO } T\text{: } S_1(T) \end{array}$$

Conditions to avoid arbitrage

Consider portfolio: buy $\frac{1}{B_0}$ # of bonds, $\frac{-1}{S_0}$ # of stocks at time 0. Properites:

$$V_0 = \frac{1}{B_0} B_0 - \frac{1}{S_0} S_0 = 0$$

$$V_1 = \frac{1}{B_0} B_1 - \frac{1}{S_0} S_1$$

$$= \begin{cases} e^r - u \text{ under scenario H} \\ e^r - d \text{ under scenario T} \end{cases}$$

If $e^r \geq u$: $\begin{array}{lll} V_1 \geq 0 & \text{under H} & \text{ARBITRAGE} \\ V_1 > 0 & \text{under T} & \text{NOT ALLOWED} \end{array}$

It follows that $u > e^r$ (unless P(T)=0: $e^r = u$ OK)

Similarly: portfolio $\frac{-1}{B_0}$ bonds, $\frac{1}{S_0}$ stocks => $e^r > d$

CONCLUSION: no arbitrage implies $u > e^r > d$.

CALL OPTIONS

$$V_1 = (S_1 - K)^+ \qquad V_0 = ???$$

$$S_0 \begin{cases} S_1 = uS_0 & \text{SCENARIO } H \\ S_1 = dS_0 & \text{SCENARIO } T \end{cases}$$

Suppose $uS_0 > K > dS_0$
otherwise $V_1 = S_1 - K$ $(dS_0 \geq K)$ or $V_1 = 0$ $(uS_0 \leq K)$

REPLICATING PORTFOLIO

Buy Δ_0 bonds and Δ_1 stocks at time 0

$$\text{portfolio: } V_t(\Delta) = \Delta_0 B_t + \Delta_1 S_t$$

$$\text{replication: } (S_1 - K)^+ = \Delta_0 B_1 + \Delta_1 S_1$$

FINDING THE Δs: 2 EQUATIONS, 2 UNKNOWNS:

$$uS_0 - K = \Delta_0 e^r + \Delta_1 uS_0 \text{ SCENARIO } H$$

$$0 = \Delta_0 e^r + \Delta_1 dS_0 \text{ SCENARIO } T$$

OR: $\Delta_1 = \frac{uS_0 - K}{uS_0 - dS_0}$ and $\Delta_0 = -e^{-r}\Delta_1 dS_0$

PRICE FOR THIS OPTION:

$$\begin{aligned} V_0 &= \Delta_0 B_0 + \Delta_1 S_0 \\ &= \Delta_1 e^{-r}(-dS_0 + e^r S_0) \end{aligned}$$

ARGUMENT DEPENDS ON

• bond, stock can be bought or sold in any quantity
• bond, stock can be short sold (in the case of bond: this means that borrowing rate is same as lending rate)
• no bid-ask spread
• binomial model

Binomial model is oversimplification
"Brownian motion" is close to binomial model
Increasingly realistic models as the course progresses

ARGUMENT DOES NOT DEPEND ON

• Assumption of no arbitrage, except that $u > e^r > d$

MORE GENERAL DERIVATIVE SECURITIES IN THE ONE PERIOD BINOMIAL MODEL

payoff $V_1(H)$ or $V_1(T)$
or $V_1 = f(S_1)$

$$\text{where } f(s) = \begin{cases} (s-K)^+ \text{ call option} \\ (K-s)^+ \text{ put option} \\ \text{etc} \end{cases}$$

REPLICATING PORTFOLIO

Buy Δ_0 bonds and Δ_1 stocks at time 0

$$\text{portfolio: } V_t(\Delta) = \Delta_0 B_t + \Delta_1 S_t$$

$$\text{replication: } f(S_1) = \Delta_0 B_1 + \Delta_1 S_1$$

FINDING THE Δs: 2 EQUATIONS, 2 UNKNOWNS:

$$f(uS_0) = \Delta_0 e^r + \Delta_1 uS_0 \text{ SCENARIO } H$$

$$f(dS_0) = \Delta_0 e^r + \Delta_1 dS_0 \text{ SCENARIO } T$$

OR: $\Delta_1 = \frac{f(uS_0)-f(dS_0)}{uS_0-dS_0}$ and $\Delta_0 = e^{-r}\frac{uf(dS_0)-df(uS_0)}{u-d}$

PRICE FOR THIS OPTION:

$$V_0 = \Delta_0 B_0 + \Delta_1 S_0$$

DISCOUNTING

Discounted stock: $S_t^* = S_t/B_t$
Discounted bond: $B_t^* = B_t/B_t = 1$
Discounted portfolio value: $V_t^* = V_t/B_t$

NUMERAIRE INVARIANCE

Portfolio in original numeraire: $V_t(\Delta) = \Delta_0 B_t + \Delta_1 S_t$

Portfolio in discounted numeraire:

$$\begin{aligned} V_t^*(\Delta) &= \frac{\Delta_0 B_t + \Delta_1 S_t}{B_t} \\ &= \Delta_0 B_t^* + \Delta_1 S_t^* \end{aligned}$$

The number Δ_0, Δ_1 of bonds, stocks is the same in original and discounted numeraire

Exit interest. This is often convenient

TWO EQUATIONS, TWO UNKNOWNS ON DISCOUNTED SCALE

$$V_1^*(H) = \Delta_1 + \Delta_2 u S_0^* \text{ SCENARIO } H$$
$$V_1^*(T) = \Delta_1 + \Delta_2 d S_0^* \text{ SCENARIO } T$$

PROBABILISTIC INTERPRETATION

Let $\pi(H)$, $\pi(T)$ be two numbers

From $V_t^* = \Delta_0 B_t^* + \Delta_1 S_t^*$:

$$\begin{aligned}
&\pi(H)V_1^*(H) + \pi(T)V_1^*(T) \\
&\quad = \pi(H)\left(\Delta_0 B_1^*(H) + \Delta_1 S_1^*(H)\right) \\
&\qquad + \pi(T)\left(\Delta_0 B_1^*(T) + \Delta_1 S_1^*(T)\right) \\
&\quad = \Delta_0\left(\pi(H)B_1^*(H) + \pi(T)B_1^*(T)\right) \\
&\qquad + \Delta_1\left(\pi(H)S_1^*(H) + \pi(T)S_1^*(T)\right) \\
&\quad = \Delta_0 B_0^* + \Delta_1 S_0^* = V_0^*
\end{aligned} \qquad (*)$$

provided $\begin{cases} \pi(H)B_1^*(H) + \pi(T)B_1^*(T) = B_0^* & (**) \\ \pi(H)S_1^*(H) + \pi(T)S_1^*(T) = S_0^* & (***) \end{cases}$

$$B_t^* = 1\text{: } (**) <=> \pi(H) + \pi(T) = 1$$

π is a probability measure, provided $\pi(H), \pi(T) \geq 0$

This is the case since, by solving (**)-(***):

$$\pi(T) = \frac{u - e^r}{u - d} \text{ and } \pi(H) = \frac{e^r - d}{u - d}$$

If E_π is expectation under π:
$(E_\pi X = X(H)\pi(H) + X(T)\pi(T))$

$$(*) <=> E_\pi V_1^* = V_0^*$$

$$(***) <=> E_\pi S_1^* = S_0^*$$

π IS A "RISK NEUTRAL" PROBABILITY MEASURE

A RISK NEUTRAL PROBABILITY DISTRIBUTION:

$$\pi : \quad S_0^j = e^{-r} E_\pi S_1^j = E_\pi S_1^{j*} \text{ for all } j$$

FUNDAMENTAL THEOREM OF ARBITRAGE PRICING:

THERE EXISTS A RISK NEUTRAL MEASURE IF AND ONLY IF ARBITRAGE DOES NOT OCCUR.

EVALUATING PRICES USING π:

$$\begin{aligned} V_0 &= \sum_{j=1}^{K} \Delta_j S_0^j \\ &= e^{-r} \sum_{j=1}^{K} \Delta_j E_\pi S_1^j \\ &= e^{-r} E_\pi V_1 \end{aligned}$$

BUT WHAT IS π?

BRUNO DE FINETTI (1937): <u>FORESIGHT: ITS LOGICAL LAWS, ITS SUBJECTIVE SOURCES</u>:

"PROBABILITY DOES NOT EXIST."

HORSE RACING
(from Baxter and Rennie: Financial Calculus. An introduction to derivative pricing.)

TWO HORSES: $H1$, $H2$

	ACTUAL CHANCE OF WINNING	BETS PLACED ON HORSE
$H1$	25%	\$ 5000
$H2$	75%	\$10000
TOTAL FOR BOOKIE		\$15000

PRICE OF BETS

ACTUAL PROBABILITIES: $\bar{\pi}_1 = \frac{1}{4} \bar{\pi}_2 = \frac{3}{4}$

$$\text{BETTING \$1 ON HORSE } H1: \quad \text{WIN \$4}$$
$$\text{BETTING \$1 ON HORSE } H2: \quad \text{WIN \$}\,\tfrac{4}{3}$$

OUTCOME FOR BOOKIE:

WINNING HORSE	\$\$
$H1$	$15000 - 4 \times 5000 = -5000$
$H2$	$15000 - \frac{4}{3} \times 10000 = 1666$

A RISKY BUSINESS

HORSE RACING:

RISK NEUTRAL PROBABILITY

	ACTUAL CHANCE OF WINNING	BETS PLACED ON HORSE
$H1$	IRRELEVANT	\$ 5000
$H2$		\$10000

PRICE OF BETS:

$$\$\ 1 \text{ ON } H1 \text{ GIVES } \frac{1}{\pi_1}\$ \text{ IF WIN}$$
$$\$\ 1 \text{ ON } H2 \text{ GIVES } \frac{1}{\pi_2}\$ \text{ IF WIN}$$

OUTCOME FOR BOOKIE:

WINNER	\$\$
$H1$	$15000 - \frac{1}{\pi_1} \times 5000$
$H2$	$15000 - \frac{1}{\pi_2} \times 10000$

OUTCOME $=$ 0 IF $\bar{\pi}_1 = \frac{1}{3}$, $\bar{\pi}_2 = \frac{2}{3}$ A SAFE BUSINESS.

JUST LIKE SELLING OPTIONS...

COMPLETE MARKETS

EQUIVALENT:

1) ALL RANDOM VARIABLES V_1 CAN BE REPRESENTED

$$V_1 = \Delta_0 B_1 + \Delta_1 S_1^1 + \cdots + \Delta_K S_1^K$$

2) π IS UNIQUE

IN CALL OPTION — TWO STATE MODEL: DID NOT NEED TO VERIFY EXISTENCE OF PORTFOLIO