MILD INCOMPLETENESS:

- CONTINUITY
- 2 FACTORS, 1 TRADED

Case study: paper by Martin Schweizer (1992)

$B_t, \mathcal{E}_t$: independent Brownian motions

$(\mathcal{F}_t)$ generated by $(B_t, \mathcal{E}_t)$

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t$$
$$dF_t = m_t F_t dt + v_t F_t d\xi_t$$
$$\text{where } d\xi_t = \rho_t dB_t + \sqrt{1-\rho_t^2}\, d\mathcal{E}_t$$

Additional regularity conditions

$$F = \text{traded}, \quad S = \text{untraded}, \quad r = 0$$

Problem:

$$\min_{\theta \in \Theta} E\left[(\pi + L - G_T(\theta))^2\right]$$

$$\pi = \text{payoff}, \quad L = \text{fixed}, \quad G_t = \int_0^t \theta_u dF_u$$

¡E = actual probabilty measure!

QUADRATIC UTILITY FUNCTION:

$$
\begin{aligned}
&(\pi + L - G_T(\theta))^2 \\
&\qquad = [(G_T(\theta) - \pi) - L]^2 \\
&\qquad = (G_T(\theta) - \pi))^2 + L^2 - 2(G_T(\theta) - \pi)L \\
&\qquad = -\frac{1}{2L} u(G_T(\theta) - \pi) + L^2
\end{aligned}
$$

where

$$u(a) = x - cx^2 \quad c = \frac{1}{2L}$$

Hence

$$\arg\min_\theta E(\pi + L - G_T(\theta))^2 = \arg\max_\theta Eu(G_T(\theta) - \pi)$$

Overall minimum: minimize over L

Minimalization procedure:

Suppose θ_t^* is optimal delta, set

$$\theta_t = \theta_t^* + \delta\eta_t$$

$Z_t =$ tracking value for π: "price process"

$$
\begin{aligned}
&\frac{\partial}{\partial\delta} E\left[Z_t + L - G_t(\theta)\right]^2 \\
&\qquad = \frac{\partial}{\partial\delta} E\left[Z_t + L - G_t(\theta^*) - \delta G_t(\eta)\right]^2 \\
&\qquad = 2\delta E G_t(\eta)^2 - 2EG_t(\eta)[Z_t + L - G_t(\theta^*)]
\end{aligned}
$$

At $\delta = 0$: $EG_t(\eta)[Z_t + L - G_t(\theta^*)] = 0$, all t, η

First: more general approach: F traded, S untraded

- Choose a $\widetilde{P} \sim P$: F is $\widetilde{P}-$MG
- Take $V_t = \widetilde{E}(\pi \mid \mathcal{F}_t)$
- A hedge:

$$dV_t = \theta_t dF_t + dR_t$$

$$\theta_t = \frac{d[V,F]_t}{d[F,F]_t} \qquad R_t = \text{remainder}$$

One particular choice of $\widetilde{P}$:

the Minimal Martingale Measure $\widehat{P}$:

- F is $\widehat{P}-$MG
- if K is a $P-$MG, $[F,K]_t \equiv 0$, then K is a $\widehat{P}-$MG

$\widehat{P}$ = The risk neutral measure with the least change from P

Original system under P:

$B, \mathcal{E}$ independent BM's, generating $(\mathcal{F}_t)$

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma_t S_t dt \\ dF_t &= m_t F_t dt + v_t F_t d\xi_t \\ d\xi_t &= \rho_t dB_t + \sqrt{1-\rho_t^2}\, d\mathcal{E}_t \end{aligned}$$

Set also

$$dN_t = \sqrt{1-\rho_t^2}\, dB_t - \rho_t d\mathcal{E}_t$$

So

$$\begin{pmatrix} d\xi_t \\ dN_t \end{pmatrix} = A_t \begin{pmatrix} dB_t \\ d\mathcal{E}_t \end{pmatrix} \qquad A_t = \begin{pmatrix} \rho_t & \sqrt{1-\rho_t^2} \\ \sqrt{1-\rho_t^2} & -\rho_t \end{pmatrix}$$

Inverse system: A_t is orthonormal (each t, w)

$$A_t^{-1} = A_t^* = A_t$$

$$\begin{pmatrix} dB_t \\ d\mathcal{E}_t \end{pmatrix} = A_t^{-1} \begin{pmatrix} d\xi_t \\ dN_t \end{pmatrix} = A_t \begin{pmatrix} d\xi_t \\ dN_t \end{pmatrix}$$

$$dS_t = \mu_t S_t dt + \sigma_t S_t (\rho_t d\xi_t + \sqrt{1-\rho_t^2}\, dN_t)$$

ξ, N are independent Brownian motions:

$$
\begin{aligned}
d[\xi,\xi]_t &= \rho_t^2 d[B,B]_t + (1-\rho_t^2) d[\mathcal{E},\mathcal{E}]_t = dt \\
d[N,N]_t &= (1-\rho_t^2) d[B,B]_t + \rho_t^2 d[\mathcal{E},\mathcal{E}]_t = dt \\
d[\xi,N]_t &= \rho_t \sqrt{1-\rho_t^2}\, d[B,B]_t - \sqrt{1-\rho_t^2}\, \rho_t d[\mathcal{E},\mathcal{E}]_t = 0
\end{aligned}
$$

Generating $(\mathcal{F}_t)$

$$\pi = c + \int_0^T f_u^{(1)} dB_u + \int_0^T f_u^{(2)} d\mathcal{E}_u$$

then

$$= c + \int_0^T \phi_u^{(1)} d\xi_u + \int_0^T \phi_u^{(2)} dN_u$$

where

$$
\begin{aligned}
\phi_u^{(1)} \rho_u + \phi_u^{(2)} \sqrt{1-\rho_u^2} &= f_u^{(1)} \\
\phi_u^{(1)} (1) \sqrt{1-\rho_u^2} - \phi_u^{(2)} \rho_u &= f_u^{(2)}
\end{aligned}
$$

or

$$A_u \phi_u = f_u$$

so

$$\phi_u = A_u^{-1} f_u = A_u f_u$$

$$dF_t = m_t F_t dt + v_t F_t d\xi_t$$
$$dS_t = \mu_t S_t dt + \sigma_t S_t \left(\rho_t d\xi_t + \sqrt{1-\rho_t^2}\, dN_t \right)$$

For minimal martingale measure:

- need to drift — correct ξ_t:
$$dF_t = v_t F_t \underbrace{\left(\frac{m_t}{v_t} dt + d\xi_t \right)}_{d\xi_t^*}$$
  ξ_t^* must be $\widehat{P}$-martingale
- leave alone N as $\widehat{P}$-martingale: $[\xi, N]_t \equiv 0$. If K is $\widehat{P}$-martingale, $[\xi, K]_t \equiv 0$:
$$K_t = c + \int_0^t \phi_u^{(1)} d\xi_u + \int_0^t \phi_u^{(2)} dN_u$$
$$\text{with } 0 = [\xi, K]_t = \int_0^t \phi_u^{(1)} d[\xi, \xi]_u$$
$$\text{so } K_t = c + \int_0^t \phi_u^{(2)} dN_u$$
$$= \widehat{P} - \text{ martingale}$$

Hope $(\mathcal{F}_t)$ is generated by ξ_t^*, N_t

OK since, by assumption, $\frac{m_t}{v_t}$ nonrandom

Find conclusions for minimal martingale measure $\widehat{P}$

$$dF_t = v_t F_t d\xi_t^*$$

$$dS_t = \mu_t S_t dt + \sigma_t S_t \left(\rho_t d\xi_t^* - \rho_t \frac{m_t}{v_t} dt + \sqrt{1 - \rho\xi}\, dN_t \right)$$

$$= S_t \left(\mu_t - \sigma_t \rho_t \frac{m_t}{v_t} \right) dt + S_t \sigma_t \rho_t d\xi_t^* + S_t \sigma_t \sqrt{1 - \rho_t^2}\, dN_t$$

$$\widehat{V}_t = \widehat{E}(\pi \mid \mathcal{F}_t)$$

$$= \widehat{\pi}_0 + \int_0^t \widehat{\theta}_u dF_u + \int_0^t \nu_u dN_u$$

$$\frac{d\widehat{P}}{dP} = \exp\left(- \int_0^T \frac{m_u}{v_u} d\xi_u - \frac{1}{2} \int_0^T \frac{m_u^2}{v_u^2}\, du \right)$$

Solution of original problem

Set

$$\Phi(G_t) = \widehat{\theta}_t + \frac{m_t}{v_t^2 F_t}(\widehat{V}_t + 2 - G_t) \tag{3.1}$$

If

$$dG_t^* = \Phi(G_t^*)dF_t \quad G_0^* = 0 \tag{3.2}$$

then

$$\theta_t^* = \Phi(G_t^*) \tag{$*$}$$

solves

$$\arg\min_\theta E[(\pi + L - G_T(\theta))^2]$$

$$\begin{aligned}(*) \Rightarrow \theta_t^* dF_t &= \Phi(G_t^*)dF_t \\ &= dG_t^* \quad \text{by (3.2)}\end{aligned}$$

Set

$$
\begin{aligned}
D_t &= \widehat{V}_t + L - G_t^* \\
dD_t &= \widehat{\theta}_t dF_t + \nu_t dN_t - \Phi(G_t^*) dF_t \\
&= \widehat{\theta}_t dF_t + \nu_t dN_t - \left(\widehat{\theta}_t + \frac{m_t}{v_t^2 F_t} D_t \right) dF_t \\
&= -\frac{m_t}{v_t^2 F_t} D_t dF_t + \nu_t dN_t \\
D_t dG_t(\eta) &= D_t \eta_t dF_t \\
&= D_t \eta_t m_t F_t dt + dMG_t \\
d[D, G(\eta)]_t &= -\frac{m_t}{v_t^2 F_t} D_t \eta_t d[F, F]_t \\
&= -\frac{m_t}{v_t^2 F_t} D_t \eta_t v_t^2 F_t^2 dt
\end{aligned}
$$

so

$$
D_t dG_t(\eta) + d[D, G(\eta)]_t = dMG_t
$$

Hence

$$
\begin{aligned}
d(D_t G_t(\eta)) &= G_t(\eta) dD_t + D_t dG_t(\eta) + d[D, G(\eta)]_t \\
&= G_t(\eta) dD_t + dMG_t \\
&= -G_t(\eta) \frac{m_t}{v_t^2 F_t} D_t F_t m_t dt + dMG_t \\
&= -\frac{m_t^2}{v_t^2} G_t(\eta) D_t dt + dMG_t
\end{aligned}
$$

$$D_t = \widehat{V}_t + L - G_t^*$$

$$d[D_t G_t(\eta)] = -\frac{m_t^2}{v_t^2} G_t(\eta) D_t dt + dMG_t$$

so

$$ED_t G_t(\eta) = -\int_0^t \frac{m_t^2}{v_t^2} EG_t(\eta) D_t dt$$

or

$$H'(t) = -\frac{m_t^2}{v_t^2} H(t)$$

where

$$H(t) = ED_t G_t(\eta)$$

Since $H(0) = 0$:

$$ED_T G_T(\eta) = H(T) = 0$$

Hence G_t^* is optimal.

Solution of a Linear Stochastic Differential Equation

We shall in the following ignore all regularity conditions.
– Suppose that G_t satisfies

$$dG_t = (A_t - B_t G_t)dF_t.$$

To solve this, set $H_t = G_t \exp(X_t)$. Use Itô's formula to get

$$\begin{aligned} dH_t &= \exp(X_t)(dG_t + G_t dX_t + \frac{1}{2}G_t d\langle X, X\rangle_t + d\langle G, X\rangle_t) \\ &= \exp(X_t)((A_t - B_t G_t)dF_t + G_t dX_t + \frac{1}{2}G_t d\langle X, X\rangle_t \\ &\qquad + (A_t - B_t G_t)d\langle F, X\rangle_t), \end{aligned}$$

where the second line is obtained by replacing all dG_t terms by $(A_t - B_t G_t)dF_t$. It is now clear that X_t must be on the form

$$X_t = \int_0^t (a_s d\langle F, F\rangle_s + b_s dF_s).$$

We plug this expression into (2) to get

$$\begin{aligned} dH_t = \exp(X_t)\{&(A_t - B_t G_t + b_t G_t)dF_t \\ &+ [a_t G_t + \frac{1}{2}b_t^2 G_t + (A_t - B_t G_t)b_t]d\langle F, F\rangle_t\}. \end{aligned}$$

If we set

$$b_t = B_t \text{ and } a_t = \frac{1}{2} B_t^2,$$

equation (4) reduces to

$$dH_t = \exp(X_t)(A_t dF_t + A_t B_t d\langle F, F\rangle_t).$$

In other words,

$$G_t = \exp(-X_t)\{G_0 + \int_0^t \exp(X_s)(A_s dF_s + A_s B_s d\langle F, F\rangle_s)\}.$$