MILD INCOMPLETENESS:

- CONTINUITY
- 2 FACTORS, 1 TRADED

Case study: paper by Martin Schweizer (1992)

\( B_t, \mathcal{E}_t \): independent Brownian motions

\((\mathcal{F}_t)\) generated by \((B_t, \mathcal{E}_t)\)

\[
\begin{align*}
    dS_t &= \mu_t S_t \, dt + \sigma_t S_t \, dB_t \\
    dF_t &= m_t F_t \, dt + v_t F_t \, d\xi_t \\
    \text{where } d\xi_t &= \rho_t dB_t + \sqrt{1 - \rho_t^2} \, d\mathcal{E}_t
\end{align*}
\]

Additional regularity conditions

\( F = \text{traded}, \quad S = \text{untraded}, \quad r = 0 \)

Problem:

\[
\min_{\theta \in \Theta} E \left[ (\pi + L - G_T(\theta))^2 \right]
\]

\( \pi = \text{payoff}, \quad L = \text{fixed}, \quad G_t = \int_0^t \theta_u \, dF_u \)

\( \mathcal{E} = \text{actual probability measure!} \)
QUADRATIC UTILITY FUNCTION:

\[
\begin{align*}
& (\pi + L - G_T(\theta))^2 \\
& = \left[ (G_T(\theta) - \pi) - L \right]^2 \\
& = (G_T(\theta) - \pi)^2 + L^2 - 2(G_T(\theta) - \pi)L \\
& = -\frac{1}{2L} u(G_T(\theta) - \pi) + L^2
\end{align*}
\]

where

\[
u(a) = x - cx^2 \quad c = \frac{1}{2L}
\]

Hence

\[
\arg \min_{\theta} E(\pi + L - G_T(\theta))^2 = \arg \max_{\theta} E u(G_T(\theta) - \pi)
\]

Overall minimum: minimize over \( L \)

Minimalization procedure:
Suppose \( \theta^*_t \) is optimal delta, set

\[
\theta_t = \theta^*_t + \delta \eta_t
\]

\( Z_t = \) tracking value for \( \pi \): “price process”

\[
\frac{\partial}{\partial \delta} E \left[ Z_t + L - G_t(\theta) \right]^2
\]

\[
= \frac{\partial}{\partial \delta} E \left[ Z_t + L - G_t(\theta^*) - \delta G_t(\eta) \right]^2
\]

\[
= 2\delta EG_t(\eta)^2 - 2EG_t(\eta) [Z_t + L - G_t(\theta^*)]
\]

At \( \delta = 0 \): \( EG_t(\eta) [Z_t + L - G_t(\theta^*)] = 0 \), all \( t, \eta \)
First: more general approach: $F$ traded, $S$ untraded

- Choose a $\tilde{P} \sim P$: $F$ is $\tilde{P} - \text{MG}$
- Take $V_t = \tilde{E}(\pi | \mathcal{F}_t)$
- A hedge:
  \[
dV_t = \theta_t dF_t + dR_t
  \]
  \[
  \theta_t = \frac{d[V,F]_t}{d[F,F]_t} \quad R_t = \text{remainder}
  \]

One particular choice of $\tilde{P}$:

the Minimal Martingale Measure $\hat{P}$:

- $F$ is $\hat{P} - \text{MG}$
- if $K$ is a $P - \text{MG}$, $[F,K]_t \equiv 0$, then $K$ is a $\hat{P} - \text{MG}$

$\hat{P} = \text{The risk neutral measure with the least change from } P$
Original system under $P$:

$B, E$ independent BM's, generating $(\mathcal{F}_t)$

\[
\begin{align*}
    dS_t &= \mu_t S_t dt + \sigma_t S_t dt \\
    dF_t &= m_t F_t dt + v_t F_t d\xi_t \\
    d\xi_t &= \rho_t dB_t + \sqrt{1 - \rho_t^2} d\mathcal{E}_t
\end{align*}
\]

Set also

\[
dN_t = \sqrt{1 - \rho_t^2} dB_t - \rho_t d\mathcal{E}_t
\]

So

\[
\begin{pmatrix} d\xi_t \\ dN_t \end{pmatrix} = A_t \begin{pmatrix} dB_t \\ d\mathcal{E}_t \end{pmatrix} \quad A_t = \begin{pmatrix} \rho_t & \sqrt{1 - \rho_t^2} \\ \sqrt{1 - \rho_t^2} & -\rho_t \end{pmatrix}
\]

Inverse system: $A_t$ is orthonormal (each $t, w$)

\[
A_t^{-1} = A_t^* = A_t
\]

\[
\begin{pmatrix} dB_t \\ d\mathcal{E}_t \end{pmatrix} = A_t^{-1} \begin{pmatrix} d\xi_t \\ dN_t \end{pmatrix} = A_t \begin{pmatrix} d\xi_t \\ dN_t \end{pmatrix}
\]

\[
dS_t = \mu_t S_t dt + \sigma_t S_t (\rho_t d\xi_t + \sqrt{1 - \rho_t^2} dN_t)
\]
ξ, N are independent Brownian motions:

\[
d[\xi, \xi]_t = \rho_t^2 d[B, B]_t + (1 - \rho_t^2) d[\mathcal{E}, \mathcal{E}]_t = dt
\]
\[
d[N, N]_t = (1 - \rho_t^2) d[B, B]_t + \rho_t^2 d[\mathcal{E}, \mathcal{E}]_t = dt
\]
\[
d[\xi, N]_t = \rho_t \sqrt{1 - \rho_t^2} d[B, B]_t - \sqrt{1 - \rho_t^2} \rho_t d[\mathcal{E}, \mathcal{E}]_t = 0
\]

Generating \((\mathcal{F}_t)\)

\[
\pi = c + \int_0^T f_u^{(1)} dB_u + \int_0^T f_u^{(2)} d\mathcal{E}_u
\]
then

\[
= c + \int_0^T \phi_u^{(1)} d\xi_u + \int_0^T \phi_u^{(2)} dN_u
\]

where

\[
\phi_u^{(1)} \rho_u + \phi_u^{(2)} \sqrt{1 - \phi_u^2} = f_u^{(1)}
\]
\[
\phi_u^{(1)} (1) \sqrt{1 - \rho_u^2} - \phi_u^{(2)} \rho_u = f_u^{(2)}
\]
or

\[
A_u \phi_u = f_u
\]

so

\[
\phi_u = A_u^{-1} f_u = A_u f_u
\]
\[
dF_t = m_t F_t \, dt + v_t F_t \, d\xi_t
\]
\[
dS_t = \mu_t S_t \, dt + \sigma_t S_t \left( \rho_t d\xi_t + \sqrt{1 - \rho_t^2} \, dN_t \right)
\]

For minimal martingale measure:

- need to drift — correct \( \xi_t \):
  \[
  dF_t = v_t F_t \left( m_t \frac{dt}{v_t} + d\xi_t \right)
  \]

\( \xi_t^* \) must be \( \hat{P} \)-martingale

- leave alone \( N \) as \( \hat{P} \)-martingale: \([\xi, N]_t \equiv 0\). If \( K \) is \( \hat{P} \)-martingale, \([\xi, K]_t \equiv 0\):

  \[
  K_t = c + \int_0^t \phi_u^{(1)} d\xi_u + \int_0^t \phi_u^{(2)} dN_u
  \]

  with \( 0 = [\xi, K]_t = \int_0^t \phi_u^{(1)} d[\xi, \xi]_u \)

  so \( K_t = c + \int_0^t \phi_u^{(2)} dN_u \)

  \[
  = \hat{P} - \text{martingale}
  \]

Hope \( (\mathcal{F}_t) \) is generated by \( \xi_t^*, N_t \)

OK since, by assumption, \( \frac{m_t}{v_t} \) nonrandom
Find conclusions for minimal martingale measure $\hat{P}$

$$
dF_t = v_tF_t d\xi^*_t$$

$$
dS_t = \mu_t S_t dt + \sigma_t S_t \left( \rho_t d\xi^*_t - \rho_t \frac{m_t}{v_t} dt + \sqrt{1 - \rho^2} dN_t \right)
= S_t \left( \mu_t - \sigma_t \rho_t \frac{m_t}{v_t} \right) dt + S_t \sigma_t \rho_t d\xi^*_t + S_t \sigma_t \sqrt{1 - \rho^2} dN_t
$$

$$
\hat{V}_t = \hat{E}(\pi \mid \mathcal{F}_t)
= \hat{\pi}_0 + \int_0^t \hat{\theta}_u dF_u + \int_0^t \nu_u dN_u
$$

$$
\frac{d\hat{P}}{dP} = \exp \left( - \int_0^T \frac{m_u}{v_u} d\xi_u - \frac{1}{2} \int_0^T \frac{m_u^2}{v_u^2} du \right)
$$
Solution of original problem

Set

\[ \Phi(G_t) = \hat{\theta}_t + \frac{m_t}{v_t^2 F_t} (\hat{V}_t + 2 - G_t) \quad (3.1) \]

If

\[ dG_t^* = \Phi(G_t^*) dF_t \quad G_0^* = 0 \quad (3.2) \]

then

\[ \theta_t^* = \Phi(G_t^*) \quad (*) \]

solves

\[ \arg \min_{\theta} E[(\pi + L - G_T(\theta))^2] \]

\[ (*) \Rightarrow \theta_t^* dF_t = \Phi(G_t^*) dF_t \]

\[ = dG_t^* \quad \text{by (3.2)} \]
Set

\[ D_t = \hat{V}_t + L - G_t^* \]

\[ dD_t = \hat{\theta}_t dF_t + \nu_t dN_t - \Phi(G_t^*)dF_t \]

= \( \hat{\theta}_t dF_t + \nu_t dN_t - \left( \hat{\theta}_t + \frac{m_t}{v_t^2 F_t} D_t \right) dF_t \)

= -\( \frac{m_t}{v_t^2 F_t} D_t dF_t + \nu_t dN_t \)

\[ D_t dG_t(\eta) = D_t \eta_t dF_t \]

= \( D_t \eta_t m_t F_t dt + dMG_t \)

\[ d[D, G(\eta)]_t = -\frac{m_t}{v_t^2 F_t} D_t \eta_t d[F, F]_t \]

= -\( \frac{m_t}{v_t^2 F_t} D_t \eta_t v_t^2 F_t^2 dt \)

so

\[ D_t dG_t(\eta) + d[D, G(\eta)]_t = dMG_t \]

Hence

\[ d(D_t G_t(\eta)) = G_t(\eta) dD_t + D_t dG_t(\eta) + d[D, G(\eta)]_t \]

= \( G_t(\eta) dD_t + dMG_t \)

= -\( G_t(\eta) \frac{m_t}{v_t^2 F_t} D_t F_t m_t dt + dMG_t \)

= -\( \frac{m_t^2}{v_t^2} G_t(\eta) D_t dt + dMG_t \)
\[ D_t = \hat{V}_t + L - G_t^* \]

\[ d[D_tG_t(\eta)] = -\frac{m_t^2}{v_t^2} G_t(\eta)D_t dt + dMG_t \]

so

\[ ED_tG_t(\eta) = -\int_0^t \frac{m_t^2}{v_t^2} E G_t(\eta)D_t dt \]

or

\[ H'(t) = -\frac{m_t^2}{v_t^2} H(t) \]

where

\[ H(t) = ED_tG_t(\eta) \]

Since \( H(0) = 0 \):

\[ ED_TG_T(\eta) = H(T) = 0 \]

Hence \( G_t^* \) is optimal.
Solution of a Linear Stochastic Differential Equation

We shall in the following ignore all regularity conditions.

1. Suppose that \( G_t \) satisfies

\[
dG_t = (A_t - B_t G_t) dF_t.
\]

To solve this, set \( H_t = G_t \exp(X_t) \). Use Itô’s formula to get

\[
dH_t = \exp(X_t)(dG_t + G_t dX_t + \frac{1}{2} G_t d\langle X, X \rangle_t + d\langle G, X \rangle_t)
\]

\[
= \exp(X_t)((A_t - B_t G_t) dF_t + G_t dX_t + \frac{1}{2} G_t d\langle X, X \rangle_t
\]

\[
+ (A_t - B_t G_t) d\langle F, X \rangle_t),
\]

where the second line is obtained by replacing all \( dG_t \) terms by \( (A_t - B_t G_t) dF_t \). It is now clear that \( X_t \) must be on the form

\[
X_t = \int_0^t \left( a_s d\langle F, F \rangle_s + b_s dF_s \right).
\]

We plug this expression into (2) to get

\[
dH_t = \exp(X_t)\{(A_t - B_t G_t + b_t G_t) dF_t
\]

\[
+ [a_t G_t + \frac{1}{2} b_t^2 G_t + (A_t - B_t G_t) b_t] d\langle F, F \rangle_t \}.
\]
If we set
\[ b_t = B_t \text{ and } a_t = \frac{1}{2}B_t^2, \]
equation (4) reduces to
\[ dH_t = \exp(X_t)(A_t dF_t + A_t B_t d\langle F, F \rangle_t). \]
In other words,
\[ G_t = \exp(-X_t)\{G_0 + \int_0^t \exp(X_s)(A_s dF_s + A_s B_s d\langle F, F \rangle_s)\}. \]