HEDGING: THE EXACT CASE

SETUP: ON DISCOUNTED SCALE

$\widetilde{S}_t$: stock price $\quad \widetilde{\eta}$: payoff at $T \quad \widetilde{C}_t$: price at t

SELF FINANCING STRATEGY (SFS):

$$\widetilde{\eta} = \widetilde{C}_T$$

$$d\widetilde{C}_t = \theta_t d\widetilde{S}_t \quad \theta_t = \text{ “delta”}$$

QUADRATIC VARIATION $[\cdot,\cdot]_t$:

$$d[\widetilde{C},\widetilde{S}]_t = \theta_t d[\widetilde{S},\widetilde{S}]_t$$

OR:

$$\theta_t = \frac{d[\widetilde{C},\widetilde{S}]_t}{d[\widetilde{S},\widetilde{S}]_t} \qquad (\widetilde{1})$$

CONCLUSION: IF YOU KNOW HOW TO CALCULATE OPTIONS PRICES $\widetilde{C}_t$, AND IF SFS EXISTS, THEN YOU CAN CALCULATE θ_t.

WHAT DO YOU GET FROM $(\widetilde{1})$ IF SFS DOES NOT EXIST?
NEXT PAGE ON ORIGINAL SCALE

NUMERAIRE INVARIANCE: SFS means

$$\eta = C_T$$

$$dC_t = \theta_t^{(0)} d\Lambda_t + \theta_t^{(1)} dS_t \tag{1}$$

$$C_t = \theta_t^{(0)} \Lambda_t + \theta_t^{(1)} S_t \tag{2}$$

IF $\Lambda_t = \exp\left(\int_0^t r_u du\right) =$ MONEY MARKET BOND:

Λ_t IS dt TERM, SO

$$dC_t = \theta_t^{(1)} dS_t + dt \text{ TERM}$$

OR: $\qquad \theta_t^{(1)} = \dfrac{d[C,S]_t}{d[S,S]_t}$

$\theta_t^{(0)}$ IS GIVEN BY (2)

SINCE

$$C_t = e^{rt}\widetilde{C}_t, \quad \text{OR} \quad dC_t = e^{rt} d\widetilde{C}_t + dt \text{ TERM}$$

$$S_t = e^{rt}\widetilde{S}_t, \quad \text{OR} \quad dS_t = e^{rt} d\widetilde{S}_t + dt \quad \text{TERM}$$

ONE OBTAINS

$$d[C,S] = (e^{rt})^2 d[\widetilde{C},\widetilde{S}] \quad d[S,S] = (e^{rt})^2 d[S,S]_t$$

WHENCE

$$\theta_t^{(1)} = \frac{(e^{rt})^2 d[\widetilde{C},\widetilde{S}]_t}{(e^{rt})^2 d[\widetilde{S},\widetilde{S}]_t} = \theta_t \quad \text{FROM } (\widetilde{1})$$

AS BEFORE

BLACK-SCHOLES MODEL:

$$dS_t = rS_t dt + \sigma S_t dB_t$$

UNDER P^*

$\Lambda_t =$ MONEY MARKET BOND, EUROPEAN OPTIONS

$$C_t = f(S_t, T-t) = \text{B-S formula}$$

$$dC_t = f'_S(S_t, T-t)dS_t + dt \text{ terms}$$

so $d[C,S]_t = f'_S(S_t, T-t)d[S,S]_t$

OR $\theta_t = \dfrac{d[C,S]_t}{d[S,S]_t} = f'_S(S_t, T-t)$

= THE REGULAR B-S DELTA

SHOOTING SPARROWS WITH CANNON

AN ASIAN LIABILITY

$$\eta = \int_0^T g(Su)du$$

TAKE $r = 0$:

$$\begin{aligned} C_t &= E\left(\int_0^T g(S_u)du \mid \mathcal{F}_t\right) \\ &= E\left(\int_0^t g(S_u)du \mid \mathcal{F}_t\right) + E\left(\int_t^T g(S_u)du \mid \mathcal{F}_t\right) \\ &= \int_0^t g(S_u)du + \int_t^T f(S_t, u-t)du \end{aligned}$$

WHERE $f(s,t) =$ BS price at zero for payoff $g(S_t)$ at t.

SET $h(s,t) = \int_0^t f(s,u)du$:

$$C_t = \int_0^t g(S_u)du + h(S_t, T-t)$$

THEN $dC_t = h'_S(S_t, T-t) + dt -$ TERMS

$$\theta_t = \frac{d[C,S]}{d[S,S]_t} = h'_S(S_t, T-t)$$

MORE INTERESTING: A LOOKBACK OPTION

$$\eta = (M_T - K)^+ \quad \text{WHERE} \quad M_t = \max_{0 \le u \le t} S_u$$

$$C_t = e^{-r(T-t)} E^*(\eta \mid \mathcal{F}_t) = \text{ PRICE AT } t$$

IN LECTURE 1, WE LEARNED TO CALCULATE C_0 FOR FIXED r, σ^2. SET

$$C_0 = f(s_0, K, T)$$

FIRST OBTAIN C_t

SET: $K_t = \max(K, M_t)$

NOTE: $(M_T - K) = (M_T - K_t) + (K_t - K)$

$M_T \ge K_t \ge K$ WHEN $M_T \ge K$:

$$(M_T - K)^+ = (M_T - K_t)^+ + (K_t - K)^+$$

SO

$$\begin{aligned} C_t &= e^{-r(T-t)} E^* \left[(M_T - K_t)^+ \mid \mathcal{F}_t \right] \\ &\quad + e^{-r(T-t)} E^+ \left[(K_t - K)^+ \mid \mathcal{F}_t \right] \\ &= f(S_t, K_t, T-t) + e^{-r(T-t)} (K_t - K)^+ \end{aligned}$$

LOOKBACK (CONTINUED)

$$C_t = f(S_t, K_t, T-t) + e^{-r(T-t)}(K_t - K)^+$$

FINDING THE HEDGE

K_t IS NONDECREASING (BUT NOT A dt TERM):

$$dC_t = f'_S(S_t, K_t, T-t)dS_t \\ + \text{NONDECREASING (INCL } dt\text{) TERMS}$$

SO: $d[C,S]_t = f'_S(S_t, K_t, T-t)d[S,S]_t$

FROM EITHER ABOVE EQUATION:

$$\theta_t = f'_s(S_t, K_t, T-T)$$

THE DELTA IS STILL DERIVATIVE W.R.T. S, BUT HARDER TO SEE DIRECTLY.

ALSO: $K_t =$ FUNCTION OF PATH OF S_t.

MORE COMPLEX LOOKBACKS

PAYOFF $g(M_T)$ at T

RECALL FROM LECTURE 1:

$$\begin{aligned} V_0 &= f(s_0, T; g) \\ &= e^{-rT} E[g(M_T)] \\ &= e^{-rT} \iint\limits_{b \geq a} g(S_0 \exp(\sigma b)) \\ &\qquad \exp\left(\nu a - \frac{1}{2}\nu^2 T\right) f_{X,M^X}(a,b) da db \end{aligned}$$

where $\nu = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$

and

$$f_{X,M^X}(a,b) = \begin{cases} 2\,\dfrac{(2b-a)}{\sqrt{2\pi T^3}}\,\exp\left(-\dfrac{(2b-a)^2}{2T}\right) & \text{if } b \geq a \\ 0 & \text{otherwise} \end{cases}$$

PRICE AT TIME t

Let $m = M_t$, $s = S_t$, $N = \max_{t \le u \le T} S_u$

$$M_T = \begin{cases} m \text{ on set } N < m \\ N \text{ on set } N \ge m \end{cases}$$

$$g(M_T) = g(m)I(N < m) + g(N)I(N \ge m)$$

$$\begin{aligned} &E^*(g(M_T) \mid \mathcal{F}_t) \\ &\quad = g(m)P^*(N < m \mid S_t = s) + E^*[g(N)I(N \ge m) \mid S_t = s] \\ &\quad = g(m)P^*(M_{T-t} < m \mid S_0 = s) + \\ &\quad\quad E^*[g(M_{T-t})I(M_{T-t} \ge m) \mid S_0 = s] \\ &\quad = e^{r(T-t)}[g(m)f(s, T-t; I(\cdot < m)) \\ &\quad\quad + f(s, T-t, g(\cdot)I(\cdot \ge m))] \end{aligned}$$

Set

$$h(m, s, T-t) = g(m) f(s, T-t, I(\cdot < m)) + f(s, T-t, g(\cdot) I(\cdot \geq$$

$$= e^{-r(T-t)} \Big\{ g(m) \int\int_{b \geq a} I(S_0 \exp(\sigma b) < m) \exp\left(va - \frac{1}{2} v^2 T\right) f_{X,M}(a, b) da db + \int\int_{b \geq a} g(S_0 \exp(\sigma b)) I(S_0 \exp(\sigma b) \geq m) \exp\left(va - \frac{1}{2} v^2 T\right) f_{X,M}(a, b) da db \Big\}$$

Then

$$V_t = h(M_t, S_t, T-t)$$

$$dV_t = h'_S dS_t + \text{terms with } d \text{ (increasing quantities)}$$

Again

$$\text{delta} = f'_S(M_t, S_t, T-t) \qquad (A)$$

GENERAL PRINCIPLE

If $V_t = h(M_t, S_t, T-t)$ and M_t is any quantity without quadratic variation, then (A) holds

WHAT IF YOU DO NOT KNOW FORM OF V_t?

MONTE CARLO SIMULATION

FOR EXAMPLE, SUPPOSE

(i) you know that $V_t = f(M_t, S_t, T - t)$ where $M_t =$ maximum, or any other quantity without quadratic variation

(ii) you don't know the form of f

(iii) you know how to simulate (V_T, M_t, S_t), but not V_t

PROCEDURE: SIMULATE n COPIES $(V_T^{(i)}, M_t^{(i)}, S_t^{(i)})$ $i = 1, \ldots, n$ UNDER P^*

Since $f(m, s, T - t) = E^*(V_T | M_t = m, S_t = s)$, use estimate

$\widehat{f}(m, s, T - t) =$ NONPARAMETRIC REGRESSION

of $V_T^{(i)}$ on $M_t^{(i)}, S_t^{(i)}$

KERNEL REGRESSION
NADARAYA-WATSON ESTIMATOR

$$\begin{aligned} K(m,s) &= \text{“KERNEL”} \\ &= \text{any density in } m, s \\ \widehat{f}(m,s,T-t) &= \frac{\sum_i V_T^{(i)} K(M_t^{(i)} - m, S_t^{(i)} - s)}{\sum_i K(M_t^{(i)} - m, S_t^{(i)} - s)} \\ \text{DELTA} &= \widehat{f}'_S(m,s,T-t) \end{aligned}$$

FOR EXAMPLE,

$$K(m,s) = \frac{1}{h_1 h_2} \phi\left(\frac{m}{h_1}\right) \phi\left(\frac{s}{h_2}\right)$$

FOR $n \to \infty$, OPTIMAL $h_1, h_2 \to 0$

- MUCH THEORY AVAILABLE
- OTHER ESTIMATORS AVAILABLE: Local linear regression, LOcally WEighted Scatter plot Smoothing (LOWESS)
- NEED TO REDUCE # ARGUMENTS TO MINIMUM

HOW DO YOU KNOW YOU HAVE THE RIGHT ARGUMENTS?

$\widehat{f}'_S(M_t, S_t, T-t) =$ CORRECT DELTA FOR PAYOFF $g(M_T)$

$\widehat{f}'_S(S_t, T-t) =$ WRONG DELTA FOR PAYOFF $g(M_T)$

TO TEST THIS:

- OBTAIN $\widehat{f}'(m, s, T-t)$ FOR SEVERAL t (REUSE SAMPLE FOR DIFFERENT t, OR REDUCE DIMENSION)
- DO NEW SIMULATION TO SEE DISTRIBUTION OF

$$g(M_T^{(new,i)}) - \sum_{t_j} \widehat{f}'_S(M_{t_j}^{(new,i)} S_{t_j}^{(new,i)} T-t_j) \Delta S_{t_j}^{(new,i)}$$

IF ≈ 0: YOU'RE OK

$\widehat{f}(s, T-t)$ FAILS TEST FOR $g(M_T)$