MathFinance 345/Stat 390. Autumn 2005.

Homework 6. Due Wed November 30. (Based on Lectures 6-8).

1. **The Black-Scholes-Merton formula.** Derive the B-S-M formula for the price of a call option according to p. 18-19 in Lecture Notes 6. (cf. Exercise 3.5 (p. 118) in Shreve II.)

2. **The Lookback Functional.**

(a) Let $h(x)$ be a functional that transforms a realization $(x_t)_{0\le t\le T}$ into a real value $h(x)$. Show that if there is a constant c so that for all x and y

$$|h(x) - h(y)| \le c \sup_{0\le t\le T} |x_t - y_t|,$$

then h is continuous in the sense of p. 25 of the notes.

(b) Deduce that $h(x) = \sup_{0\le t\le T} x_t$ is a continuous functional. [If this is too hard, solve the problem for $h(x) = \frac{1}{T}\int_0^T x_t dt$ instead.]

(c) Use the result in (b) along with the Central Limit Theorem to simulate (in R or Splus) the value of $\max_{0\le t\le 1} W_t$, where W_t is an (additive) Brownian motion. You should use a binomial tree and keep in mind that the increments should be additive.

(d) Compare the resulting histogram with that of the random variable $|N(0,1)|$ (which can be simulated using the R-function "rnorm"). Write down any conjectures you may have about the distribution of $\max_{0\le t\le 1} W_t$.

3. Shreve II, Exercise 3.7 (p. 119).

4. Shreve II, Exercise 4.13 (p. 197).

5. **The Black-Scholes-Merton formula again.** Let $s- > g(s)$ be a given function. Define the auxiliary function $(s, R, \Xi) \to B(s, R, \Xi)$, as follows:

$$B(s, R, \Xi) = \exp(-R)Eg(s\exp(R - \Xi/2 + \sqrt{\Xi}Z)), \tag{1}$$

where Z is standard normal.

(a) Show that

$$\frac{\partial B(s, R, \Xi)}{\partial \Xi} = \frac{1}{2}s^2\frac{\partial^2 B(s, R, \Xi)}{\partial s^2}$$

(b) Use (a), and any other needed relationship, to determine whether $e^{-rt}B(S_t, r(T-t), \sigma^2(T-t))$ is a martingale under the Black-Scholes model

$$dS_t = rS_t dt + \sigma S_t dW_t. \tag{1}$$

[Alternatively, solve the problem for $r = 0$, and argue why your solution is relevant to the case of general r.]

(c) Explain, using Ito's formula, that the probability measure described by equation (1) is risk neutral. If you assume that this measure is the only risk neutral one available, deduce from (b) the price at time 0 of the payoff $g(S_T)$ at time T.

(d) More generally, suppose that r_t and σ_t can be random, but that you know that $r_t \geq 0$, $\sigma_t \geq 0$

$$\int_0^T r_t dt \leq R^+ \text{ and } \int_0^T \sigma_t^2 dt \leq \Xi^+$$

S_t follows $dS_t = r_t S_t dt + \sigma_t S_t dW_t$. Determine if $e^{-rt} B(S_t, R^+ - \int_0^t r_u du, \Xi^+ - \int_0^t \sigma_u^2 du)$ is a martingale.