STOCHASTIC INTEGRALS

$$X_t = \begin{matrix}\text{CONTINUOUS} \\ \text{PROCESS}\end{matrix} \begin{cases} S_t : & \text{STOCK PRICE} \\ M_t : & \text{MG} \\ W_t : & \text{BROWNIAN MOTION} \end{cases}$$

$\theta_t =$ PORTFOLIO: $\# X_t$ HELD AT t

DISCRETE TIME: $0 = t_0 < t_1 < \ldots < t_n = t$:

$$P/L_t = \sum_{i<n} \theta_{t_i} \underbrace{(X_{t_{i+1}} - X_{t_i})}_{\Delta X_{t_i}}$$

GRID BECOMES "DENSE": $\max_i \Delta t_i \to 0$

$$P/L_t \to \int_0^t \theta_u dX_u$$

INTEGRAL <u>DEFINED</u> AS LIMIT OF SUMS

PROPERTIES MOSTLY FROM SUMS:

$$\underbrace{\sum_{i<n}(a\theta_{t_i} + b\eta_{t_i})\Delta X_{t_i}}_{\downarrow \atop \int_0^t (a\theta_u + b\eta_u)dX_u} = \underbrace{a\sum_{i<n}\theta_{t_i}\Delta X_{t_i}}_{\downarrow \atop a\int_0^t \theta_u dX_u} + \underbrace{b\sum_{i<n}\eta t_i \Delta X_{t_i}}_{\downarrow \atop b\int_0^t \eta_u dX_u}$$

$\Rightarrow$ LINEARITY OK

TIME VARYING INTEGRAL:

$$\int_0^t \theta_u dX_u = \text{limit of}$$

$$\sum_{t_{i+1}\leq t} \theta_{t_i}(X_{t_{i+1}} - X_{t_i})$$

Limit in probability

MARTINGALE PROPERTY:

If $X_t = M_t = \text{MG}$:

$$U_t^{(n)} = \sum_{t_{i+1} \le t} \theta_{t_i} \Delta X_{t_i} : \quad \Delta U_{t_{i+1}}^{(n)} = \theta_{t_i} \Delta X_{t_i}$$

If on grid $t_0, t_1, \ldots$:

$$\begin{aligned} E(\Delta U_{t_{i+1}}^{(n)} \mid \mathcal{F}_{t_i}) &= E(\theta_{t_i} \Delta X_{t_i} \mid \mathcal{F}_{t_i}) \\ &= \theta_{t_i} E(\Delta X_{t_i} \mid \mathcal{F}_{t_i}) = 0 \end{aligned}$$

$\Rightarrow U_t^{(n)}$ is $\mathcal{F}_{t_i} -$ MG

Taking limits:

$$E(U_t \mid \mathcal{F}_s) = U_s$$

QUADRATIC VARIATION (Q. V.)

$$U_t^{(n)} = \sum_{t_{i+1} \le t} \theta_{t_i} \Delta X_{t_i}$$

so: $\Delta U_{t_i}^{(n)} = U_{t_{i+1}}^{(n)} - U_{t_i}^{(n)} = \theta_{t_i} \Delta X_{t_i}$

$$(\Delta U_{t_i}^{(n)})^2 = \theta_{t_i}^2 (\Delta X_{t_i})^2$$

Aggregate:

$$\underbrace{[U^{(n)}, U^{(n)}]_t}_{\downarrow\ [U,U]_t} = \sum_{t_{i+1} \le t} (\Delta U_{t_i}^{(n)})^2$$

$$= \sum_{t_{i+1} \le t} \theta_{t_i}^2 (\Delta X_{t_i})^2 = \underbrace{\sum_{t_{i+1} \le t} \theta_{t_i}^2 \Delta [X, X]_{t_i}}_{\downarrow\ \int_0^t \theta_u^2 d[X,X]_u}$$

$$[U, U]_t = \int_0^t \theta_u^2 d[X, X]_u$$

If $X_t = W_t = \text{B.M.}:\ d[X, X]_t = dt$

IT FOLLOWS THAT $[U, U]_t = \int_0^t \theta_u^2 du$

DIFFERENTIAL NOTATION

INTEGRAL:

$$\Delta U^{(n)}_{t_i} = \theta_{t_i} \Delta X_{t_i} \text{ vs. } U_t = U_0 + \sum_{t_{i+1} \le t} \theta_{t_i} \Delta X_{t_i}$$

becomes

$$dU_t = \theta_t dX_t \text{ vs. } U_t = U_0 + \int_0^t \theta_s dX_s$$

QUADRATIC VARIATION:

$$(\Delta U^{(n)}_{t_i})^2 = \theta^2_{t_i} (\Delta X_{t_i})^2 \text{ vs. } [U^{(n)}, U^{(n)}]_t = \sum \theta^2_{t_i} \Delta [X, X_{t_i}]$$

” ”

$$\Delta [U^{(n)}, U^{(n)}]_{t_i} = \theta^2_{t_i} \Delta [X, X]_{t_i}$$

becomes:

$$(dU_t)^2 = \theta^2_t (dX_t)^2 \text{ vs. } [U, U]_t = \int_0^t \theta^2_u d[X, X]_u$$

” ”

$$d[U, U]_t = \theta^2_t d[X, X]_t$$

BROWNIAN MOTION:

$$(dW_t)^2 = dt \quad \text{AND} \quad d[U, U]_t = \theta^2_t dt$$

QUADRATIC COVARIATION:

$$U, Z: \quad [U, Z]_t = \text{limit of} \sum_{t_{i+1} \le t} \Delta U_{t_i} \Delta Z_{t_i}$$

CASE OF TWO INTEGRALS:

$$U_t = \int_0^t \theta_s dX_s, \quad Z_t = \int_0^t \eta_s dY_s$$

THEN:

$$[U, Z]_t = \int_0^t \theta_s \eta_s d[X, Y]_s$$

BECAUSE

$$\Delta U_{t_i} \Delta Z_{t_i} = \theta_{t_i} \eta_{t_i} \Delta X_{t_i} \Delta Y_{t_i}$$

or

$$\underbrace{dU_t dZ_t}_{d[U,Z]_t} = \theta_t \eta_t \underbrace{dX_t dY_t}_{d[X,Y]_t}$$

IF: $X_t = Y_t = W_t$ THE <u>SAME</u> B.M.:

$$d[U, Z]_t = \theta_t \eta_t dt$$

DETERMINISTIC INTEGRAND

IF θ_t IS NONRANDOM:

$$\int_0^t \theta_s dW_s = \text{ limit of } \sum_{t_{i+1} \le t} \theta_{t_i} \Delta W_{t_i}$$

$\sum_{t_{i+1} \le t} \theta_{t_i} \Delta W_{t_i}$:

• LINEAR COMBINATION OF NORMAL RANDOM VARIABLES IS A NORMAL RANDOM VARIABLE

• MEAN: $E \sum_{t_{i+1} \le t} \theta_{t_i} \Delta W_{t_i} = 0$

• VARIANCE:
$\text{Var } (\sum_{t_{i+1} \le t} \theta_{t_i} \Delta W_{t_i}) = \sum_{t_{i+1} \le t} \theta_{t_i}^2 \text{ Var } (\Delta W_{t_i}) = \sum_{t_{i+1} \le t} \theta_{t_i}^2 \Delta t_i$

IN THE LIMIT:

$\int_0^t \theta_s dW_s$:

• NORMAL RANDOM VARIABLE

• MEAN IS ZERO

• VARIANCE:
$\text{Var } (\int_0^t \theta_s dW_s) = E[\int_0^\cdot \theta_s dW_s, \int_0^\cdot \theta_s dW_s]_t = \int_0^t \theta_s^2 ds$

ITÔ's FORMULA

X_t: CONTINUOUS PROCESS (SOME RESTRICTIONS):
ξ: TWICE CONTINUOUSLY DIFFERENTIABLE

$$\xi(X_t) = \xi(X_0) + \int_0^t \xi'(X_u)dX_u + \frac{1}{2}\int_0^t \xi''(X_u)d[X,X]_t$$

DIFFERENTIAL NOTATION:

$$d\xi(X_t) = \xi'(X_t)dX_t + \frac{1}{2}\xi''(X_t)d[X,X]_t$$

EX: $X_t = W_t =$ BROWNIAN MOTION:

$$d\xi(W_t) = \xi'(W_t)dW_t + \frac{1}{2}\xi''(W_t)dt$$

EX: $dX_t = \nu_t dt + \sigma_t dW_t$ ITÔ PROCESS

or: $X_t = X_0 + \int_0^t \nu_s ds + \int_0^t \sigma_s dW_s$

First question: ¿What is $d[X,X]_t$?

FIRST: CASE OF EXPLICIT INTEGRATION:

$W_t =$ B.M. ¿WHAT IS $\int_0^t W_s dW_s$?

$$U_t = W_t^2 = \zeta(W_t), \zeta(x) = x^2$$

$$\begin{aligned} dU_t &= \zeta'(W_t)dW_t + \frac{1}{2}\zeta''(W_t)dt \\ &= 2W_t dW_t + dt \end{aligned}$$

$$\text{so:} \quad W_t dW_t = \frac{1}{2}dU_t - \frac{1}{2}dt$$

$$\begin{aligned} >: \quad \int_0^t W_s dW_s &= \frac{1}{2}(U_t - U_0) - \frac{1}{2}\int_0^t ds \\ &= \frac{1}{2}W_t^2 - \frac{1}{2}t \end{aligned}$$

DIFFERENT FROM ORDINARY INTEGRAL:

If $X_t = g(t)$ g' exists, continuous, $g(t) = 0$

$$\begin{aligned} \int_0^t X_s dX_s &= \int_0^t g(s)g'(s)ds \\ &= \frac{1}{2}g(t)^2 \\ &= \frac{1}{2}X_t^2 \end{aligned}$$

ITÔ PROCESS:

$$X_t = X_0 + \underbrace{\int_0^t \nu_s ds}_{Z_t} + \underbrace{\int_0^t \sigma_s dW_s}_{U_t}$$

Grid:

$$\Delta X_{t_i} = \Delta Z_{t_i} + \Delta U_{t_i}$$

so:

$$(\Delta X_{t_i})^2 = (\Delta Z_{t_i})^2 + (\Delta U_{t_i})^2 + 2\Delta Z_{t_i} \Delta U_{t_i}$$

$$\sum (\Delta X_{t_i})^2 = \sum (\Delta Z_{t_i})^2 + \sum (\Delta U_{t_i})^2 + \sum 2\Delta Z_{t_i} \Delta U_{t_i}$$

$$\begin{aligned} |\Delta Z_{t_i}| &= |\int_{t_i}^{t_{i+1}} \nu_s ds| \le \int_{t_i}^{t_{i+1}} |\nu_s| ds \\ &\le \sup_s |\nu_s| (t_{i+1} - t_i) = \sup_s |\nu_s| \Delta t_i \end{aligned}$$

$$\begin{aligned} \sum (\Delta Z_{t_i})^2 &\le (\sup_s |\nu_s|)^2 \sum_i (\Delta t_i)^2 \\ &\le (\sup_s |\nu_s|)^2 \underbrace{\sup \Delta t_i}_{\to 0} \underbrace{\sum_i \Delta t_i}_{\simeq t} \to 0 \end{aligned}$$

$\subset$: $[Z, Z]_t = 0$ ALSO: $[Z, U]_t = 0$

ONLY: $[U, U]_t = \int_0^t \sigma_s^2 ds$

ITÔ PROCESS

$$X_t = X_0 + \underbrace{\int_0^t \nu_s d_s}_{Z_t} + \underbrace{\int_0^t \sigma_s dW_s}_{U_t}$$

$$\underset{(dZ_t)^2}{d[Z,Z]_t = 0} \qquad \underset{dZ_t dU_t}{d[Z,U]_t = 0} \qquad d[U,U]_t = \sigma_t^2 dt$$

USING DIFFERENTIALS:

ANY dt-TERM HAS ZERO Q.V.:

$$(dZ_t)^2 = \nu_t^2 (dt)^2 = 0 \text{ ETC}$$

COMBINING TERMS:

$$\begin{aligned}(dX_t)^2 &= (dZ_t + dU_t)^2 \\ &= (dZ_t)^2 + 2dZ_t dU_t + (dU_t)^2 \\ &= (dU_t)^2 = \sigma_t^2 dt\end{aligned}$$

RIGOROUS:

$$(\Delta X_{t_i})^2 = \Delta Z_{t_i}^2 + 2\Delta Z_{t_i} \Delta U_{t_i} + (\Delta U_{t_i})^2$$

SUM OVER t_i, TAKE LIMITS, GET SAME RESULT

INTEGRALS WITH RESPECT TO AN ITÔ PROCESS

$$X_t = X_0 + \int_0^t \nu_s ds + \int_0^t \sigma_s dW_s$$

CAN SHOW THAT:

$$\int_0^t \theta_s dX_s = \int_0^t \theta_s \nu_s ds + \int_0^t \theta_s \sigma_s dW_s$$

A NEW ITÔ PROCESS

BACK TO ITÔ'S FORMULA:

$$d\xi(X_t) = \xi'(X_t)dX_t + \frac{1}{2}\xi''(X_t)d[X,X]_t \qquad (*)$$

ITÔ PROCESS:

$$dX_t = \nu_t dt + \sigma_t dW_t$$

SO

$$d[X,X]_t = \sigma_t^2 dt$$

PLUG IN:

$$\begin{aligned} d\xi(X_t) &= \xi'(X_t)(\nu_t dt + \sigma_t dW_t) \\ &\quad + \frac{1}{2}\xi''(X_t)\sigma_t^2 dt \\ &= (\xi'(X_t)\nu_t + \frac{1}{2}\xi''(X_t)\sigma_t^2)dt \\ &\quad + \xi'(X_t)\sigma_t dW_t \end{aligned}$$

EASIER TO REMEMBER (*)...

"PROOF" OF ITÔ'S FORMULA:

$U_t = \xi(X_t)$:

$$\begin{aligned}\Delta U_{t_i} &= \xi(X_{t_{i+1}}) - \xi(X_{t_i}) \\ &= \xi(X_{t_i} + \Delta X_{t_i}) - \xi(X_{t_i}) \\ &= \zeta'(X_{t_i})\Delta X_{t_i} + \frac{1}{2}\zeta''(X_{t_i})\Delta X_{t_i}^2 \\ &\qquad + \frac{1}{3!}\zeta'''(X_{t_i})\Delta X_{t_i}^3 + \cdots\end{aligned}$$

sum up:

$$U_t - U_0 = \underbrace{\sum \zeta'(X_{t_i})\Delta X_{t_i}}_{\displaystyle\downarrow \atop \displaystyle\int_0^t \zeta'(X_s)dX_s} + \underbrace{\frac{1}{2}\sum \zeta''(X_{t_i})\Delta X_{t_i}^2}_{\displaystyle\downarrow \atop \displaystyle\frac{1}{2}\int_0^t \zeta''(X_s)d[X,X]_s}$$

OTHER "PROOF":

$$\begin{aligned}dU_t &= \zeta(X_t + dX_t) - \zeta(X_t) \\ &= \zeta'(X_t)dX_t + \frac{1}{2}\zeta''(X_t)\underbrace{(dX_t)^2}_{d[X,X]_t} + \cdots\end{aligned}$$

MULTIVARIATE FORMULA

$$
\begin{aligned}
U_t &= \zeta(X_t, Y_t) \\
dU_t &= \zeta'_x(X_t, Y_t)dX_t + \zeta'_y(X_t, Y_t)dY_t \\
&\quad + \frac{1}{2}\Big\{ \zeta''_{xx}(X_t, Y_t)d[X, X]_t \\
&\qquad + \zeta''_{yy}(X_t, Y_t)d[Y, Y]_t \\
&\qquad + 2\zeta''_{xy}(X_t, Y_t)d[X, Y]_t \Big\}
\end{aligned}
$$

etc.

EXAMPLE: GEOMETRIC BROWNIAN MOTION

$$S_t = S_0 \exp\{\int_0^t \sigma_s dW_s + \int_0^t \left(r_s - \frac{1}{2}\sigma_s^2\right) ds\}$$

SET

- $X_t = \int_0^t \sigma_s dW_s + \int_0^t \left(r_s - \frac{1}{2}\sigma_s^2\right) ds$
- $S_t = f(X_t)$ $(f(x) = S_0 \exp\{x\})$

USE ITÔ'S FORMULA

$$dS_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X,X]_t$$

$f'(x) = f''(x) = f(x)$ AND $d[X,X]_t = \sigma_t^2 dt$, SO:

$$\begin{aligned} dS_t &= f(X_t)dX_t + \frac{1}{2}f(X_t)\sigma_t^2 dt \\ &= S_t dX_t + \frac{1}{2}S_t\sigma_t^2 dt \\ &= S_t\left(dX_t + \frac{1}{2}\sigma_t^2 dt\right) \\ &= S_t\left(\sigma_t dW_t + r_t dt\right) \\ &= S_t\sigma_t dW_t + S_t r_t dt \end{aligned}$$

DIFFERENTIAL REPRESENTATION OF S_t

VASICEK MODEL

$$dR_t = (\alpha - \beta R_t)dt + \sigma dW_t$$

• STEP 1: SET $U_t = R_t - \frac{\alpha}{\beta}$

EQUATION BECOMES:

$$dU_t = -\beta U_t dt + \sigma dW_t$$

• STEP 2: NOTE THAT (FROM ITO'S FORMULA)

$$\begin{aligned} d(\exp\{\beta t\}U_t) &= \exp\{\beta t\}dU_t + U_t d\exp\{\beta t\} \\ &= \exp\{\beta t\}dU_t + U_t\beta\exp\{\beta t\}dt \\ &= \exp\{\beta t\}\left(dU_t + U_t\beta dt\right) \\ &= \exp\{\beta t\}\sigma dW_t \end{aligned}$$

SO

$$\exp\{\beta t\}U_t = U_0 + \int_0^t \exp\{\beta s\}\sigma dW_s$$

OR

$$U_t = \exp\{-\beta t\}U_0 + \int_0^t \exp\{\beta(s-t)\}\sigma dW_s$$

IN OTHER WORDS: U_t IS NORMAL

- MEAN IS $\exp\{-\beta t\}U_0$
- VARIANCE IS

$$\begin{aligned}
&\int_0^t (\exp\{\beta(s-t)\}\sigma)^2\, ds \\
&\quad = \int_0^t \exp\{2\beta(s-t)\}\sigma^2 ds \\
&\quad = \left[\frac{1}{2\beta}\exp\{2\beta(s-t)\}\sigma^2\right]_{s=0}^{s=t} \\
&\quad = \frac{1}{2\beta}\left(1-\exp\{-2\beta t\}\right)\sigma^2
\end{aligned}$$

DEDUCE FOR $R_t = U_t + \frac{\alpha}{\beta}$ THAT

- R_t IS NORMAL
- $E(R_t) = \exp\{-\beta t\}U_0 + \frac{\alpha}{\beta}$
- Var (R_t) =Var (U_t)

LEVY'S THEOREM

IF M_t IS A CONTINUOUS (LOCAL) MARTINGALE, $M_0 = 0$, $[M, M]_t = t$ FOR ALL t, THEN M_t IS A CONTINUOUS BROWNIAN MOTION

PROOF: SET $f(x) = \exp\{hx\}$

ITO:

$$\begin{aligned} df(M_t) &= f'(M_t)dM_t + \frac{1}{2}f''(M_t)d[M, M]_t \\ &= f'(M_t)dM_t + \frac{1}{2}f''(M_t)dt. \end{aligned}$$

SINCE dM_t TERM IS MG, AND $f''(x) = h^2 f(x)$:

$$\begin{aligned} E(f(M_t)|\mathcal{F}_s) &= f(M_s) + \frac{1}{2}h^2 E\left(\int_s^t f(M_u)du|\mathcal{F}_s\right) \\ &= f(M_s) + \frac{1}{2}h^2 \int_s^t E(f(M_u)|\mathcal{F}_s)du \end{aligned}$$

Set $g(t) = E(\exp\{h(M_t - M_s)\}|\mathcal{F}_s)$:

$$g(t) = 1 + \frac{1}{2}h^2 \int_s^t g(u)du$$

SOLUTION:

$$g(t) = \exp\{\frac{1}{2}h^2(t-s)\}$$

IN OTHER WORDS:

$$E(\exp\{h(M_t - M_s)\}|\mathcal{F}_s) = \exp\{\frac{1}{2}h^2(t-s)\}$$

CHARACTERISTIC FUNCTION ARGUMENT GIVES:

- $M_t - M_s$ IS INDEPENDENT OF $\mathcal{F}_s$
- $M_t - M_s$ IS $N(0, t-s)$

ITÔ PROCESSES

$$X_t = X_0 + \underbrace{\int_0^t a_s ds}_{dt \text{ term}} + \underbrace{\int_0^t b_s dW_s}_{dW_t \text{ term}}$$

Decomposition unique:

- dW_t term is martingale
- dt term is drift

(Doob-Meyer decomposition)

UNDER RISK NEUTRAL MEASURE P^*

Discounted securities only have dW term:

$$d\widetilde{S}_t = \underbrace{\mu_t \widetilde{S}_t dt}_{=0} + \sigma_t \widetilde{S}_t dW_t$$

UNDISCOUNTED SECURITIES UNDER P^*:

$$d\widetilde{S}_t = \sigma_t \widetilde{S}_t dW_t$$

1) Numeraire $= B_t = \exp(\int_0^t r_u du)$
properties: $dB_t = r_t B_t dt$, $[B,B]_t = 0$, $[B,\widetilde{S}]_t = 0$

$$\begin{aligned} S_t &= \widetilde{S}_t B_t \Rightarrow \\ dS_t &= B_t d\widetilde{S}_t + \widetilde{S}_t dB_t \\ &= \underbrace{B_t \sigma_t \widetilde{S}_t dW_t} + \underbrace{\widetilde{S}_t B_t r_t} dt \\ &= \sigma_t S_t dW_t + r_t S_t dt \end{aligned}$$

2) Other numeraire:
$\Lambda_t \neq B_t$, Λ_t has $dW_t^{(2)}$ term

$$d\langle W, W^{(2)} \rangle_t = \rho_t dt$$

$d\widetilde{S}_t \Lambda_t =$ full use of Itô's formula

Not same P^*!!!

UNDISCOUNTED SECURITIES UNDER P^*:

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t$$

LOG SCALE: ITO'S FORMULA

$$\begin{aligned}
d\log(S_t) &= \frac{1}{S_t} dS_t + \frac{1}{2}(-\frac{1}{S_t^2}) d[S,S]_t \\
&= \frac{1}{S_t}(r_t S_t dt + \sigma_t S_t dW_t) + \frac{1}{2}(-\frac{1}{S_t^2})\sigma_t^2 S_t dt \\
&= (r_t - \frac{1}{2}\sigma_t^2) dt + \sigma_t dW_t
\end{aligned}$$

OPTIONS PRICES: PDE'S

PAYOFF: $f(S_T)$

DISCOUNTED PAYOFF: $e^{-rT}f(S_T) = \widetilde{f}(\widetilde{S}_T)$

$$\widetilde{S}_T = e^{-rT}S_T \quad \widetilde{f}(\widetilde{s}) = e^{-rT}f(e^{rT}\widetilde{s})$$

CALL: $f(s) = (s-K)^+$ $\widetilde{f}(\widetilde{s}) = (\widetilde{s} - e^{-rT}K)^+$

CANDIDATE PRICE: DISCOUNTED:

$$\widetilde{C}(\widetilde{S}_t, t) \text{ satisfies: } \widetilde{C}(\widetilde{S}, T) = \widetilde{f}(\widetilde{S})$$

AND (UNDER P^*):

$$d\widetilde{C}(\widetilde{S}_t, t)$$

$$\text{Hedge}\left\{ \quad = \widetilde{C}'_s(\widetilde{S}_t, t)d\widetilde{S}_t \quad \right\}\begin{matrix}\text{MG term} \\ d\widetilde{S} = \sigma_t \widetilde{S}_t dW_t\end{matrix}$$

$$\begin{matrix}\text{BS} \\ \text{PDE} \\ = 0\end{matrix}\left\{ \begin{matrix} +\widetilde{C}'_t(\widetilde{S}_t, t)dt \\ +\frac{1}{2}\widetilde{C}''_{ss}(\widetilde{S}_t, t)\underbrace{d[\widetilde{S}, \widetilde{S}]_t}_{\sigma_t^2 \widetilde{S}_t^2 dt} \end{matrix} \right\} dt - \text{terms}$$

2 APPROACHES

THE BS PDE:

$$\begin{array}{c}\text{Solve}\\(*)\end{array}\begin{cases}\widetilde{C}'_t(\widetilde{s},t)+\dfrac{1}{2}\widetilde{C}''_{ss}(\widetilde{s},t)\sigma^2\widetilde{s}^2=0\\ \widetilde{C}(\widetilde{s},T)=\widetilde{f}(\widetilde{s})\end{cases}$$

THE MARTINGALE APPROACH:

Set

$$\widetilde{C}(\widetilde{s},t)=E^*[\widetilde{f}(\widetilde{S}_T)\mid\widetilde{S}_t=\widetilde{s}]$$

Markov: $\widetilde{C}(\widetilde{S}_t,t)=E^*[\widetilde{f}(\widetilde{S}_T)\mid\mathcal{F}_t]=$ price under P^*

This $\widetilde{C}$ either

1) Market is complete:
   $\widetilde{C}$ automatically satisfies (*)
2) Otherwise: check if $\widetilde{C}$ satisfies (*):
   yes: solution OK
   no: try something else

REVERSAL OF DISCOUNTING

Numeraire: $B_t = \exp\{rt\}$

$$\begin{aligned} C(S_t, t) &= B_t \widetilde{C}(\widetilde{S}_t, t) \\ &= B_t \widetilde{C}\left(\frac{S_t}{B_t}, t\right) \\ &= e^{rt} \widetilde{C}(e^{-rt} S_t, t) \\ \text{Hence: } C(s, t) &= e^{rt} \widetilde{C}(e^{-rt} s, t) \\ &= e^{rt} E^*[\widetilde{f}(\widetilde{S}_T) \mid \widetilde{S}_t = e^{-rt} s] \\ &= e^{r(T-t)} E^*[f(S_T) \mid S_t = s] \end{aligned}$$

since

$$\widetilde{f}(\widetilde{s}) = e^{-rT} f(e^{rT} \widetilde{s})$$

COMPUTATION OF EXPECTED VALUES

$$\log \widetilde{S}_T = \log \widetilde{S}_t + \underbrace{\nu(T-t)}_{\nu = -\frac{\sigma^2}{2}} + \sigma \underbrace{(W_T - W_t)}_{\substack{\sqrt{T-t}\, Z \\ Z \sim N(0,1)}}$$

$$\text{and so : } \widetilde{S}_T = \widetilde{S}_t \exp(\nu(T-t) + \sigma\sqrt{T-t}\, Z)$$

$$\begin{aligned} \widetilde{C}(s,t) &= E[\widetilde{f}(\widetilde{S}_T) \mid \widetilde{S}_t = \widetilde{s}] \\ &= E[\widetilde{f}(\widetilde{s} \exp(\nu(T-t) + \sigma\sqrt{T-t}\, Z)] \\ &= \int_{-\infty}^{+\infty} \widetilde{f}(\widetilde{s} \exp(\nu(T-t) + \sigma\sqrt{T-t}\, z)\phi(z)dz \end{aligned}$$

$$\text{where } \phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2)$$

For non-discounted S_t: $\nu = r\ \frac{1}{2}\sigma^2$

MORE GENERAL

"EUROPEAN CONTINGENT CLAIMS" (ECC)

$$\eta = \text{PAYOFF} \qquad \text{AT } \underbrace{\text{TIME } T}_{\text{FIXED}}$$

$$\text{LOOKBACK: } \eta = (\max_{0 \le t \le T} S_t - K)^+$$

$$\text{ASIAN: } \eta = \left(\frac{1}{T} \int_0^T S_u du - K \right)^+$$

$$\text{BARRIER } \eta = \begin{cases} (S_T - K)^+ \\ \text{UNLESS} \\ \min_{0 \le t \le T} S_t \le X \end{cases}$$

ONLY THE IMAGINATION IS THE LIMIT...

OPTION PRICES: GENERAL SCHEME

SELF FINANCING STRATEGIES:

$$\eta = C_T$$

$$dC_t = \theta_t^{(0)} dB_t + \sum_{i=1}^{K} \theta_t^{(i)} dS_t^{(i)}$$

$$C_t = \theta_t^{(0)} B_t + \sum_{i=1}^{K} \theta_t^{(i)} S_t^{(i)}$$

Same as (by numeraire invariance):

$$\left.\begin{aligned} \widetilde{\eta} = \tfrac{1}{B_T}\eta \ &: \quad \widetilde{\eta} = \widetilde{C}_T \\ d\widetilde{C}_t &= \textstyle\sum_{i=1}^{K} \theta_t^{(i)} d\widetilde{S}_t^{(i)} \end{aligned}\right] \text{Key requirement}$$

$$\left.\widetilde{C}_t = \theta_t^{(0)} + \sum_{i=1}^{K} \theta_t^{(i)} \widetilde{S}_t^{(i)} \right] \text{Defines } \theta_t^{(0)}$$

PROOF: ITÔ'S FORMULA

ON THE DISCOUNTED SCALE

$$\left.\begin{aligned}\widetilde{\eta} &= \widetilde{C}_T \\ d\widetilde{C}_t &= \sum_{i=1}^{K} \theta_t^{(i)} d\widetilde{S}_t^{(i)}\end{aligned}\right]$$

UNDER P^*: SAME AS

$$\widetilde{\eta} = c + \sum_{i=1}^{K} \int_0^T \theta_t^{(i)} d\widetilde{S}_t^{(i)} \qquad (*)$$

BY TAKING

$$\widetilde{C}_t = E^*(\widetilde{\eta} \mid \mathcal{F}_t)$$

If $B_0 = 1$: $c = \widetilde{C}_0 =$ PRICE AT 0

(*): "MARTINGALE REPRESENTATION THEOREM"

WHEN DOES THE REPRESENTATION THEOREM HOLD?

THEOREM: *IF* $W^{(1)}, \ldots, W^{(K)}$ *INDEPENDENT B.M.'S*

$$\mathcal{F}_t = \mathcal{F}_t^{W^{(1)},\ldots,W^{(K)}}$$

$$\widetilde{\eta} \text{ @ } \mathcal{F}_T, \quad E^*|\widetilde{\eta}| < \infty :$$

$$\widetilde{\eta} = c + \sum_{i=1}^{P} \int_0^T f_t dW_t^{(i)}$$

Brownian motions are like binomial trees

FROM BROWNIAN MOTION TO STOCK PRICE

$$d\widetilde{S}_t = \sigma \widetilde{S}_t dW_t$$

or:

$$\log \widetilde{S}_t = \log \widetilde{S}_0 - \frac{1}{2}\sigma^2 t + \sigma \widetilde{W}_t$$

$$\widetilde{\eta} \text{ @ } \mathcal{F}_T^{\widetilde{S}} \Leftrightarrow \widetilde{\eta} \text{ @ } \mathcal{F}_T^{W}$$

$$\begin{aligned} \text{Get: } \widetilde{\eta} &= c + \int_0^T f_t dW_t \\ &= c + \int_0^T \underbrace{\frac{f_t}{\sigma \widetilde{S}_t}}_{\theta_t} \, d\widetilde{S}_t \end{aligned}$$

More complex if σ_t or r_t random...