STOCHASTIC INTEGRALS

\[ X_t = \text{CONTINUOUS PROCESS} \]

\[
\begin{align*}
S_t : & \text{ STOCK PRICE} \\
M_t : & \text{ MG} \\
W_t : & \text{ BROWNIAN MOTION}
\end{align*}
\]

\[ \theta_t = \text{PORTFOLIO: \#} X_t \text{ HELD AT } t \]

DISCRETE TIME: \(0 = t_0 < t_1 < \ldots < t_n = t:\)

\[ P/L_t = \sum_{i<n} \theta_{t_i} \frac{(X_{t_{i+1}} - X_{t_i})}{\Delta X_{t_i}} \]

GRID BECOMES "DENSE": \(\max_i \Delta t_i \to 0\)

\[ P/L_t \to \int_0^t \theta_u dX_u \]

INTEGRAL DEFINED AS LIMIT OF SUMS
PROPERTIES MOSTLY FROM SUMS:

\[
\sum_{i<n} (a\theta_{t_i} + b\eta_{t_i}) \Delta X_{t_i} = a \sum_{i<n} \theta_{t_i} \Delta X_{t_i} + b \sum_{i<n} \eta_{t_i} \Delta X_{t_i}
\]

\[
\int_0^t (a\theta_u + b\eta_u) dX_u = a \int_0^t \theta_u dX_u + b \int_0^t \eta_u dX_u
\]

\[\Rightarrow \text{LINEARITY OK}\]

TIME VARYING INTEGRAL:

\[
\int_0^t \theta_u dX_u = \text{limit of} \sum_{t_i+1 \leq t} \theta_{t_i} (X_{t_{i+1}} - X_{t_i})
\]

Limit in probability
MARTINGALE PROPERTY:

If \( X_t = M_t = \text{MG} \):

\[
U_t^{(n)} = \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta X_{t_i} : \quad \Delta U_{t_{i+1}}^{(n)} = \theta_{t_i} \Delta X_{t_i}
\]

If on grid \( t_0, t_1, \ldots \):

\[
E(\Delta U_{t_{i+1}}^{(n)} \mid \mathcal{F}_{t_i}) = E(\theta_{t_i} \Delta X_{t_i} \mid \mathcal{F}_{t_i}) = \theta_{t_i} E(\Delta X_{t_i} \mid \mathcal{F}_{t_i}) = 0
\]

\( \Rightarrow U_t^{(n)} \) is \( \mathcal{F}_{t_i} - \text{MG} \)

Taking limits:

\[
E(U_t \mid \mathcal{F}_s) = U_s
\]
QUADRATIC VARIATION (Q. V.)

\[ U_t^{(n)} = \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta X_{t_i} \]

so: \[ \Delta U_{t_i}^{(n)} = U_{t_{i+1}}^{(n)} - U_{t_i}^{(n)} = \theta_{t_i} \Delta X_{t_i} \]

\[ (\Delta U_{t_i}^{(n)})^2 = \theta_{t_i}^2 (\Delta X_{t_i})^2 \]

Aggregate:

\[ [U^{(n)}, U^{(n)}]_t = \sum_{t_{i+1} \leq t} (\Delta U_{t_i}^{(n)})^2 \]

\[ = \sum_{t_{i+1} \leq t} \theta_{t_i}^2 (\Delta X_{t_i})^2 = \sum_{t_{i+1} \leq t} \theta_{t_i}^2 \Delta [X, X]_{t_i} \]

\[ \int_0^t \theta_u^2 d[X, X]_u \]

If \( X_t = W_t = \text{B.M.} \): \( d[X, X]_t = dt \)

IT FOLLOWS THAT \( [U, U]_t = \int_0^t \theta_u^2 du \)
DIFFERENTIAL NOTATION

INTEGRAL:

\[ \Delta U_{t_i}^{(n)} = \theta_{t_i} \Delta X_{t_i} \text{ vs. } U_t = U_0 + \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta X_{t_i} \]

becomes

\[ dU_t = \theta_t dX_t \text{ vs. } U_t = U_0 + \int_0^t \theta_s dX_s \]

QUADRATIC VARIATION:

\[ (\Delta U_{t_i}^{(n)})^2 = \theta_{t_i}^2 (\Delta X_{t_i})^2 \text{ vs. } [U^{(n)}, U^{(n)}]_t = \sum \theta_{t_i}^2 \Delta [X, X_{t_i}] \]

\[ \Delta [U^{(n)}, U^{(n)}]_{t_i} = \theta_{t_i}^2 \Delta [X, X]_{t_i} \]

becomes:

\[ (dU_t)^2 = \theta_t^2 (dX_t)^2 \text{ vs. } [U, U]_t = \int_0^t \theta_u^2 d[X, X]_u \]

\[ d[U, U]_t = \theta_t^2 d[X, X]_t \]

BROWNIAN MOTION:

\[ (dW_t)^2 = dt \text{ AND } d[U, U]_t = \theta_t^2 dt \]
QUADRATIC COVARIATION:

\[ U, Z : \quad [U, Z]_t = \lim_{t_{i+1} \leq t} \sum_{i} \Delta U_{t_i} \Delta Z_{t_i} \]

CASE OF TWO INTEGRALS:

\[ U_t = \int_0^t \theta_s dX_s, \quad Z_t = \int_0^t \eta_s dY_s \]

THEN:

\[ [U, Z]_t = \int_0^t \theta_s \eta_s d[X, Y]_s \]

BECAUSE

\[ \Delta U_{t_i} \Delta Z_{t_i} = \theta_{t_i} \eta_{t_i} \Delta X_{t_i} \Delta Y_{t_i} \]

or

\[ d[U, Z]_t = \theta_t \eta_t d[X, Y]_t \]

IF: \( X_t = Y_t = W_t \) THE SAME B.M.:

\[ d[U, Z]_t = \theta_t \eta_t dt \]
DETERMINISTIC INTEGRAND

IF $\theta_t$ IS NONRANDOM:

$$\int_0^t \theta_s dW_s = \text{limit of } \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta W_{t_i}$$

$\sum_{t_{i+1} \leq t} \theta_{t_i} \Delta W_{t_i}$:

- LINEAR COMBINATION OF NORMAL RANDOM VARIABLES IS A NORMAL RANDOM VARIABLE

- MEAN: $E \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta W_{t_i} = 0$

- VARIANCE:

  $$\text{Var} \left( \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta W_{t_i} \right) = \sum_{t_{i+1} \leq t} \theta_{t_i}^2 \text{Var} \left( \Delta W_{t_i} \right) = \sum_{t_{i+1} \leq t} \theta_{t_i}^2 \Delta t_i$$

IN THE LIMIT:

$$\int_0^t \theta_s dW_s$$:

- NORMAL RANDOM VARIABLE

- MEAN IS ZERO

- VARIANCE:

  $$\text{Var} \left( \int_0^t \theta_s dW_s \right) = E \left[ \int_0^t \theta_s dW_s, \int_0^t \theta_s dW_s \right]_t = \int_0^t \theta_s^2 ds$$
ITÔ’s FORMULA

\( X_t: \) CONTINUOUS PROCESS (SOME RESTRICTIONS):
\( \xi: \) TWICE CONTINUOUSLY DIFFERENTIABLE

\[ \xi(X_t) = \xi(X_0) + \int_0^t \xi'(X_u) dX_u + \frac{1}{2} \int_0^t \xi''(X_u) d[X, X]_t \]

DIFFERENTIAL NOTATION:

\[ d\xi(X_t) = \xi'(X_t) dX_t + \frac{1}{2} \xi''(X_t) d[X, X]_t \]

EX: \( X_t = W_t = \) BROWNIAN MOTION:

\[ d\xi(W_t) = \xi'(W_t) dW_t + \frac{1}{2} \xi''(W_t) dt \]

EX: \( dX_t = \nu_t dt + \sigma_t dW_t \) ITÔ PROCESS

or: \( X_t = X_0 + \int_0^t \nu_s ds + \int_0^t \sigma_s dW_s \)

First question: What is \( d[X, X]_t? \)
FIRST: CASE OF EXPLICIT INTEGRATION:

\( W_t = \text{B.M.} \) WHAT IS \( \int_0^t W_s dW_s \)?

\[
U_t = W_t^2 = \zeta(W_t), \zeta(x) = x^2
\]

\[
dU_t = \zeta'(W_t)dW_t + \frac{1}{2}\zeta''(W_t)dt
\]

\[
= 2W_t dW_t + dt
\]

so:

\[
W_t dW_t = \frac{1}{2}dU_t - \frac{1}{2}dt
\]

\[
\int_0^t W_s dW_s = \frac{1}{2}(U_t - U_0) - \frac{1}{2} \int_0^t ds
\]

\[
= \frac{1}{2}W_t^2 - \frac{1}{2}t
\]

DIFFERENT FROM ORDINARY INTEGRAL:

If \( X_t = g(t) \) \( g' \) exists, continuous, \( g(t) = 0 \)

\[
\int_0^t X_s dX_s = \int_0^t g(s)g'(s)ds
\]

\[
= \frac{1}{2}g(t)^2
\]

\[
= \frac{1}{2}X_t^2
\]
ITÔ PROCESS:

\[ X_t = X_0 + \int_0^t \nu_s ds + \int_0^t \sigma_s dW_s \]

Grid:

\[ \Delta X_{t_i} = \Delta Z_{t_i} + \Delta U_{t_i} \]

so:

\[ (\Delta X_{t_i})^2 = (\Delta Z_{t_i})^2 + (\Delta U_{t_i})^2 + 2\Delta Z_{t_i} \Delta U_{t_i} \]

\[ \sum (\Delta X_{t_i})^2 = \sum (\Delta Z_{t_i})^2 + \sum (\Delta U_{t_i})^2 + \sum 2\Delta Z_{t_i} \Delta U_{t_i} \]

\[ |\Delta Z_{t_i}| = |\int_{t_i}^{t_{i+1}} \nu_s ds| \leq \int_{t_i}^{t_{i+1}} |\nu_s| ds \]

\[ \leq \sup_s |\nu_s|(t_{i+1} - t_i) = \sup_s |\nu_s| \Delta t_i \]

\[ \sum (\Delta Z_{t_i})^2 \leq (\sup_s |\nu_s|)^2 \sum (\Delta t_i)^2 \]

\[ \leq (\sup_s |\nu_s|)^2 \sup \Delta t_i \sum_i \Delta t_i \rightarrow 0 \]

\[ \sim t \]

\[ [Z, Z]_t = 0 \quad \text{ALSO:} \quad [Z, U]_t = 0 \]

ONLY: \[ [U, U]_t = \int_0^t \sigma_s^2 ds \]
ITÔ PROCESS

\[ X_t = X_0 + \int_0^t \nu_s ds + \int_0^t \sigma_s dW_s \]

\[ d[Z, Z]_t = 0 \quad d[Z, U]_t = 0 \quad d[U, U]_t = \sigma_t^2 dt \]

USING DIFFERENTIALS:

ANY \( dt \)-TERM HAS ZERO Q.V.:

\[ (dZ_t)^2 = \nu_t^2 (dt)^2 = 0 \text{ ETC} \]

COMBINING TERMS:

\[ (dX_t)^2 = (dZ_t + dU_t)^2 \]
\[ = (dZ_t)^2 + 2dZ_t dU_t + (dU_t)^2 \]
\[ = (dU_t)^2 = \sigma_t^2 dt \]

RIGOROUS:

\[ (\Delta X_{t_i})^2 = \Delta Z_{t_i}^2 + 2\Delta Z_{t_i} \Delta U_{t_i} + (\Delta U_{t_i})^2 \]

SUM OVER \( t_i \), TAKE LIMITS, GET SAME RESULT
INTEGRALS WITH RESPECT TO AN ITÔ PROCESS

\[ X_t = X_0 + \int_0^t \nu_s ds + \int_0^t \sigma_s dW_s \]

CAN SHOW THAT:

\[ \int_0^t \theta_s dX_s = \int_0^t \theta_s \nu_s ds + \int_0^t \theta_s \sigma_s dW_s \]

A NEW ITÔ PROCESS
BACK TO ITÔ’S FORMULA:

\[ d\xi(X_t) = \xi'(X_t)dX_t + \frac{1}{2}\xi''(X_t)d[X, X]_t \]  

ITÔ PROCESS:

\[ dX_t = \nu_t dt + \sigma_t dW_t \]

SO

\[ d[X, X]_t = \sigma_t^2 dt \]

PLUG IN:

\[
\begin{align*}
    d\xi(X_t) &= \xi'(X_t)(\nu_t dt + \sigma_t dW_t) \\
    &= \xi'(X_t)\nu_t dt + \frac{1}{2}\xi''(X_t)\sigma_t^2 dt \\
    &= (\xi'(X_t)\nu_t + \frac{1}{2}\xi''(X_t)\sigma_t^2)dt \\
    &+ \xi'(X_t)\sigma_t dW_t
\end{align*}
\]

EASIER TO REMEMBER (*)...
“PROOF” OF ITÔ’S FORMULA:

\[ U_t = \xi(X_t) : \]

\[ \Delta U_{t_i} = \xi(X_{t_i+1}) - \xi(X_{t_i}) \]
\[ = \xi(X_{t_i} + \Delta X_{t_i}) - \xi(X_{t_i}) \]
\[ = \zeta'(X_{t_i}) \Delta X_{t_i} + \frac{1}{2} \zeta''(X_{t_i}) \Delta X_{t_i}^2 \]
\[ + \frac{1}{3!} \zeta'''(X_{t_i}) \Delta X_{t_i}^3 + \cdots \]

sum up:

\[ U_t - U_0 = \sum \zeta'(X_{t_i}) \Delta X_{t_i} + \frac{1}{2} \sum \zeta''(X_{t_i}) \Delta X_{t_i}^2 \]
\[ \downarrow \]
\[ \int_0^t \zeta'(X_s) dX_s \quad + \quad \frac{1}{2} \int_0^t \zeta''(X_s) d[X, X]_s \]

OTHER “PROOF”:

\[ dU_t = \zeta(X_t + dX_t) - \zeta(X_t) \]
\[ = \zeta'(X_t) dX_t + \frac{1}{2} \zeta''(X_t) (dX_t)^2 + \cdots \]
\[ d[X, X]_t \]
MULTIVARIATE FORMULA

\[ U_t = \zeta(X_t, Y_t) \]
\[ dU_t = \zeta'_x(X_t, Y_t)dX_t + \zeta'_y(X_t, Y_t)dY_t \]
\[ + \frac{1}{2} \left\{ \zeta''_{xx}(X_t, Y_t)d[X, X]_t \right. \]
\[ - \zeta''_{yy}(X_t, Y_t)d[Y, Y]_t \]
\[ + 2\zeta''_{xy}(X_t, Y_t)d[X, Y]_t \right\} \]

etc.
EXAMPLE: GEOMETRIC BROWNIAN MOTION

\[ S_t = S_0 \exp \left\{ \int_0^t \sigma_s dW_s + \int_0^t \left( r_s - \frac{1}{2} \sigma_s^2 \right) ds \right\} \]

SET

- \( X_t = \int_0^t \sigma_s dW_s + \int_0^t \left( r_s - \frac{1}{2} \sigma_s^2 \right) ds \)
- \( S_t = f(X_t) \) \((f(x) = S_0 \exp \{ x \})\)

USE ITÔ’S FORMULA

\[ dS_t = f'(X_t)dX_t + \frac{1}{2} f''(X_t) d[X, X]_t \]

\( f'(x) = f''(x) = f(x) \) AND \( d[X, X]_t = \sigma_t^2 dt \), SO:

\[ dS_t = f(X_t)dX_t + \frac{1}{2} f(X_t) \sigma_t^2 dt \]

\[ = S_t dX_t + \frac{1}{2} S_t \sigma_t^2 dt \]

\[ = S_t \left( dX_t + \frac{1}{2} \sigma_t^2 dt \right) \]

\[ = S_t \left( \sigma_t dW_t + r_t dt \right) \]

\[ = S_t \sigma_t dW_t + S_t r_t dt \]

DIFFERENTIAL REPRESENTATION OF \( S_t \)
VASICEK MODEL

\[ dR_t = (\alpha - \beta R_t)dt + \sigma dW_t \]

- **STEP 1:** SET \( U_t = R_t - \frac{\alpha}{\beta} \)

EQUATION BECOMES:

\[ dU_t = -\beta U_t dt + \sigma dW_t \]

- **STEP 2:** NOTE THAT (FROM ITO’S FORMULA)

\[
\begin{align*}
    d(\exp{\{\beta t\} U_t}) &= \exp{\{\beta t\}} dU_t + U_t d\exp{\{\beta t\}} \\
    &= \exp{\{\beta t\}} dU_t + U_t \beta \exp{\{\beta t\}} dt \\
    &= \exp{\{\beta t\}} (dU_t + U_t \beta dt) \\
    &= \exp{\{\beta t\}} \sigma dW_t
\end{align*}
\]

SO

\[ \exp{\{\beta t\}} U_t = U_0 + \int_0^t \exp{\{\beta s\}} \sigma dW_s \]

OR

\[ U_t = \exp{-\beta t} U_0 + \int_0^t \exp{\beta(s - t)} \sigma dW_s \]
IN OTHER WORDS: $U_t$ IS NORMAL

- MEAN IS $\exp\{-\beta t\}U_0$
- VARIANCE IS

$$\int_0^t (\exp\{\beta(s-t)\}\sigma)^2 \, ds$$

$$= \int_0^t \exp\{2\beta(s-t)\}\sigma^2 \, ds$$

$$= \left[ \frac{1}{2\beta} \exp\{2\beta(s-t)\}\sigma^2 \right]_{s=0}^{s=t}$$

$$= \frac{1}{2\beta} (1 - \exp\{-2\beta t\}) \sigma^2$$

DEDUCE FOR $R_t = U_t + \frac{\alpha}{\beta}$ THAT

- $R_t$ IS NORMAL
- $E(R_t) = \exp\{-\beta t\}U_0 + \frac{\alpha}{\beta}$
- $\text{Var} (R_t) = \text{Var} (U_t)$
LEVY’S THEOREM

IF \( M_t \) IS A CONTINUOUS (LOCAL) MARTINGALE, \( M_0 = 0, [M,M]_t = t \) FOR ALL \( t \), THEN \( M_t \) IS A CONTINUOUS BROWNIAN MOTION

PROOF: SET \( f(x) = \exp\{hx\} \)

ITO:

\[
df(M_t) = f'(M_t) dM_t + \frac{1}{2} f''(M_t) d[M,M]_t
\]

\[
= f'(M_t) dM_t + \frac{1}{2} f''(M_t) dt.
\]

SINCE \( dM_t \) TERM IS MG, AND \( f''(x) = h^2 f(x) \):

\[
E(f(M_t)|\mathcal{F}_s) = f(M_s) + \frac{1}{2} h^2 E \left( \int_s^t f(M_u) du | \mathcal{F}_s \right)
\]

\[
= f(M_s) + \frac{1}{2} h^2 \int_s^t E(f(M_u) | \mathcal{F}_s) du
\]

Set \( g(t) = E(\exp\{h(M_t - M_s)\} | \mathcal{F}_s) \):

\[
g(t) = 1 + \frac{1}{2} h^2 \int_s^t g(u) du
\]
SOLUTION:

\[ g(t) = \exp\left\{ \frac{1}{2}h^2(t - s) \right\} \]

IN OTHER WORDS:

\[ E(\exp\{h(M_t - M_s)\}|\mathcal{F}_s) = \exp\left\{ \frac{1}{2}h^2(t - s) \right\} \]

CHARACTERISTIC FUNCTION ARGUMENT GIVES:

- \( M_t - M_s \) IS INDEPENDENT OF \( \mathcal{F}_s \)
- \( M_t - M_s \) IS \( N(0, t - s) \)
**ITO PROCESSES**

\[ X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dW_s \]

\[ \underbrace{dt \text{ term}}_{\text{drift}} \quad \underbrace{dW_t \text{ term}}_{\text{martingale}} \]

Decomposition unique:

- \( dW_t \) term is martingale
- \( dt \) term is drift

(Doob-Meyer decomposition)

**UNDER RISK NEUTRAL MEASURE** \( P^* \)

Discounted securities only have \( dW \) term:

\[ d\tilde{S}_t = \mu_t \tilde{S}_t dt + \sigma_t \tilde{S}_t dW_t \]

\[ = 0 \]
UNDISCOUNTED SECURITIES UNDER $P^*$:

$$d\tilde{S}_t = \sigma_t \tilde{S}_t dW_t$$

1) Numeraire $= B_t = \exp(\int_0^t r_u du)$

properties: $dB_t = r_t B_t dt$, $[B, B]_t = 0$, $[B, \tilde{S}]_t = 0$

$$S_t = \tilde{S}_t B_t \Rightarrow$$

$$dS_t = B_t d\tilde{S}_t + \tilde{S}_t dB_t$$

$$= B_t \sigma_t \tilde{S}_t dW_t + \tilde{S}_t B_t r_t dt$$

$$= \sigma_t S_t dW_t + r_t S_t dt$$

2) Other numeraire:

$\Lambda_t \neq B_t$, $\Lambda_t$ has $dW_t^{(2)}$ term

$$d\langle W, W^{(2)} \rangle_t = \rho_t dt$$

$$d\tilde{S}_t \Lambda_t = \text{full use of Itô’s formula}$$

Not same $P^*$!!!
UNDISCOUNTED SECURITIES UNDER $P^*$:

\[ dS_t = r_t S_t dt + \sigma_t S_t dW_t \]

LOG SCALE: ITO’S FORMULA

\[
d \log(S_t) = \frac{1}{S_t} dS_t + \frac{1}{2} (-\frac{1}{S^2_t}) d[S, S]_t \\
= \frac{1}{S_t} (r_t S_t dt + \sigma_t S_t dW_t) + \frac{1}{2} (-\frac{1}{S^2_t}) \sigma^2_t S_t dt \\
= (r_t - \frac{1}{2} \sigma^2_t) dt + \sigma_t dW_t
\]
OPTIONS PRICES: PDE’S

PAYOFF: \( f(S_T) \)

DISCOUNTED PAYOFF: \( e^{-rT} f(S_T) = \tilde{f}(\tilde{S}_T) \)

\[ \tilde{S}_T = e^{-rT} S_T \quad \tilde{f}(\tilde{s}) = e^{-rT} f(e^{rT}\tilde{s}) \]

CALL: \( f(s) = (s - K)^+ \quad \tilde{f}(\tilde{s}) = (\tilde{s} - e^{-rT} K)^+ \)

CANDIDATE PRICE: DISCOUNTED:

\( \tilde{C}(\tilde{S}_t, t) \) satisfies: \( \tilde{C}(\tilde{S}, T) = \tilde{f}(\tilde{S}) \)

AND (UNDER \( P^* \)):

\[ d\tilde{C}(\tilde{S}_t, t) \]

Hedge \( \left\{ \begin{array}{l}
\quad = \tilde{C}'_s(\tilde{S}_t, t)d\tilde{S}_t \\
\text{MG term} \quad d\tilde{S} = \sigma_t \tilde{S}_t dW_t
\end{array} \right. \)

BS \( \left\{ \begin{array}{l}
\quad + \tilde{C}'_t(\tilde{S}_t, t)dt \\
\quad + \frac{1}{2} \tilde{C}''_{ss}(\tilde{S}_t, t) d[\tilde{S}, \tilde{S}]_t
\end{array} \right. \)

PDE \( = 0 \)

\( \left\{ \begin{array}{l}
\quad + \frac{1}{2} \sigma_t^2 \tilde{S}_t^2 dt
\end{array} \right. \)
2 APPROACHES

THE BS PDE:

\[ \begin{align*}
\text{Solve } & \quad \tilde{C}_t(\tilde{s}, t) + \frac{1}{2} \tilde{C}_{ss}(\tilde{s}, t) \sigma^2 \tilde{s}^2 = 0 \\
\text{(*) } & \quad \tilde{C}(\tilde{s}, T) = \tilde{f}(\tilde{s})
\end{align*} \]

THE MARTINGALE APPROACH:

Set

\[ \tilde{C}(\tilde{s}, t) = E^* [\tilde{f}(\tilde{S}_T) | \tilde{S}_t = \tilde{s}] \]

Markov: \[ \tilde{C}(\tilde{S}_t, t) = E^* [\tilde{f}(\tilde{S}_T) | \mathcal{F}_t] = \text{price under } P^* \]

This \( \tilde{C} \) either

1) Market is complete:
   \( \tilde{C} \) automatically satisfies (*)

2) Otherwise: check if \( \tilde{C} \) satisfies (*):
   yes: solution OK
   no: try something else
REVERSAL OF DISCOUNTING

Numeraire: $B_t = \exp\{rt\}$

$$C(S_t, t) = B_t \tilde{C}(\tilde{S}_t, t)$$

$$= B_t \tilde{C}\left(\frac{S_t}{B_t}, t\right)$$

$$= e^{rt} \tilde{C}(e^{-rt} S_t, t)$$

Hence: $C(s, t) = e^{rt} \tilde{C}(e^{-rt} s, t)$

$$= e^{rt} E^*[\tilde{f}(\tilde{S}_T) \mid \tilde{S}_t = e^{-rt} s]$$

$$= e^{r(T-t)} E^*[f(S_T) \mid S_t = s]$$

since

$$\tilde{f}(\tilde{s}) = e^{-rT} f(e^{rT} \tilde{s})$$
COMPUTATION OF EXPECTED VALUES

\[
\log \tilde{S}_T = \log \tilde{S}_t + \nu (T-t) + \sigma \left( W_T - W_t \right)
\]

\[
\nu = -\frac{\sigma^2}{2}
\]

and so:

\[
\tilde{S}_T = \tilde{S}_t \exp(\nu (T-t) + \sigma \sqrt{T-t} Z)
\]

\[
\tilde{C}(s, t) = E[\tilde{f}(\tilde{S}_T) \mid \tilde{S}_t = \tilde{s}]
\]

\[
= E[\tilde{f}(\tilde{s} \exp(\nu (T-t) + \sigma \sqrt{T-t} Z)]
\]

\[
= \int_{-\infty}^{+\infty} \tilde{f}(\tilde{s} \exp(\nu (T-t) + \sigma \sqrt{T-t} z) \phi(z) dz
\]

where \( \phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2) \)

For non-discounted \( S_t \): \( \nu = r \frac{1}{2} \sigma^2 \)
MORE GENERAL

“EUROPEAN CONTINGENT CLAIMS” (ECC)

\[ \eta = \text{PAYOFF AT } \underbrace{\text{TIME } T}_{\text{FIXED}} \]

LOOKBACK: \[ \eta = (\max_{0 \leq t \leq T} S_t - K)^+ \]

ASIAN: \[ \eta = \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \]

BARRIER \[ \eta = \begin{cases} (S_T - K)^+ \\ \text{UNLESS} \\ \min_{0 \leq t \leq T} S_t \leq X \end{cases} \]

ONLY THE IMAGINATION IS THE LIMIT...
OPTION PRICES:
GENERAL SCHEME

SELF FINANCING STRATEGIES:

\[ \eta = C_T \]

\[ dC_t = \theta_t^{(0)} dB_t + \sum_{i=1}^{K} \theta_t^{(i)} dS_t^{(i)} \]

\[ C_t = \theta_t^{(0)} B_t + \sum_{i=1}^{K} \theta_t^{(i)} S_t^{(i)} \]

Same as (by numeraire invariance):

\[ \tilde{\eta} = \frac{1}{B_T} \eta : \tilde{\eta} = \tilde{C}_T \]

\[ d\tilde{C}_t = \sum_{i=1}^{K} \theta_t^{(i)} d\tilde{S}_t^{(i)} \]

\[ \tilde{C}_t = \theta_t^{(0)} + \sum_{i=1}^{K} \theta_t^{(i)} \tilde{S}_t^{(i)} \]

PROOF: ITÔ’S FORMULA
ON THE DISCOUNTED SCALE

\[ \tilde{\eta} = \tilde{C}_T \]
\[ d\tilde{C}_t = \sum_{i=1}^{K} \theta_t^{(i)} d\tilde{S}_t^{(i)} \]

UNDER \( P^* \): SAME AS

\[ \tilde{\eta} = c + \sum_{i=1}^{K} \int_0^T \theta_t^{(i)} d\tilde{S}_t^{(i)} \] (*)

BY TAKING

\[ \tilde{C}_t = E^*(\tilde{\eta} \mid \mathcal{F}_t) \]

If \( B_0 = 1 \): \( c = \tilde{C}_0 = \text{PRICE AT 0} \)

(*): “MARTINGALE REPRESENTATION THEOREM”
WHEN DOES THE REPRESENTATION THEOREM HOLD?

**Theorem:** IF \( W^{(1)}, \ldots, W^{(K)} \) INDEPENDENT B.M.’S

\[
\mathcal{F}_t = \mathcal{F}^{W^{(1)}, \ldots, W^{(K)}}_t
\]

\[\tilde{\eta} \in \mathcal{F}_T, \quad E^*|\tilde{\eta}| < \infty: \]

\[\tilde{\eta} = c + \sum_{i=1}^{P} \int_{0}^{T} f_t dW_t^{(i)}\]

Brownian motions are like binomial trees
FROM BROWNIAN MOTION TO STOCK PRICE

\[ d\tilde{S}_t = \sigma\tilde{S}_tdW_t \]

or:

\[ \log\tilde{S}_t = \log\tilde{S}_0 - \frac{1}{2}\sigma^2t + \sigma\tilde{W}_t \]

\[ \tilde{\eta} @ \mathcal{F}\tilde{S}_T \Leftrightarrow \tilde{\eta} @ \mathcal{F}^W_T \]

Get: \[ \tilde{\eta} = c + \int_0^T f_t dW_t \]

\[ = c + \int_0^T \frac{f_t}{\sigma\tilde{S}_t} \tilde{S}_t \theta_t d\tilde{S}_t \]

More complex if \( \sigma_t \) or \( r_t \) random...