APPROXIMATE NORMALITY

BINOMIAL MODEL: \( S_{n+1} = \begin{cases} uS_n \\ dS_n \end{cases} \)

LOG SCALE (IID ADDITIVE INCREMENTS):
\[
\log S_n = \log S_0 + X_1 + \ldots + X_n
\]
WITH \( X_i = \log(u) \) or \( = \log(d) \)

TWO TIME SCALES

CLOCK TIME: \( T \) – TIME PERIODS: \( n \)
\[
t_0 = 0 \quad t_1 = \frac{T}{n} \quad t_2 = \frac{2T}{n} \quad t_3 = \frac{3T}{n} \quad \ldots \quad t_k = \frac{kT}{n}
\]
\( T \) IS FIXED – \( n \) IS A MATTER OF CHOICE

RETURN ON RISK FREE ASSET
(in clock time) \( e^{rT} = e^{\rho n} \) (in time periods)
\[
in \text{other words}: \quad \rho = r \frac{T}{n} \quad (1)
\]
\( r \) IS FIXED – \( \rho \) DEPENDS ON \( n \)

RISK NEUTRAL MEASURE PER STEP:
\[
\pi_n(T) = \frac{u - e^\rho}{u - d} = \frac{u - e^{r\frac{T}{n}}}{u - d} \quad \text{and} \quad \pi_n(H) = \frac{e^\rho - d}{u - d} \quad (2)
\]
BEHAVIOR OF ADDITIVE INCREMENTS

MEAN:

\[ E(X) = \log(u)\pi(H) + \log(d)\pi(T) \]

TOTAL MEAN:

\[
E(\log(S_n) - \log(S_0)) = E(X_1) + \ldots + E(X_n) \\
= nE(X) \\
= n(\log(u)\pi(H) + \log(d)\pi(T))
\]

VARIANCE: \( X = \log d + (\log u - \log d)I_H \), and so

\[
Var(X) = (\log u - \log d)^2 Var(I_H) \\
= (\log u - \log d)^2 \pi(H)\pi(T)
\]

TOTAL VARIANCE:

\[
Var(\log(S_n)) = Var(X_1) + \ldots + Var(X_n) \\
= nVar(X_1) \\
= n(\log u - \log d)^2 \pi(H)\pi(T)
\]
WE WISH TO KEEP TOTAL MEAN, VARIANCE CONSTANT IN CLOCK TIME

\[ \nu T = E(\log S_n) \]
\[ = n(\log(u)\pi(H) + \log(d)\pi(T)) \quad (3) \]
\[ \sigma^2 T = \text{Var}(\log(S_n)) \]
\[ = n(\log u - \log d)^2\pi(H)\pi(T) \quad (4) \]

\( \sigma \) OR \( \sigma^2 \) IS VOLATILITY IN CLOCK TIME
NEED TO USE: \( \nu \approx r - \frac{1}{2}\sigma^2 \)

EQUATIONS (1)-(4) DEFINE A BINOMIAL TREE \((\rho, u, d, \pi(H), \pi(T))\) ON THE BASIS OF:

• VOLATILITY PER UNIT CLOCK TIME: \( \sigma^2 \)
• INTEREST PER UNIT CLOCK TIME: \( r \)
• # OF UNITS OF CLOCK TIME: \( T \)
• # OF TIME PERIODS IN COMPUTATION: \( n \)
AN APPROXIMATION FOR THE CASE $r = \rho = 0$
(THE DISCOUNTED PROCESS)

UP AND DOWN STEPS:

$$u = 1 + \sqrt{\frac{\sigma^2 T}{n}} \text{ AND } d = 1 - \sqrt{\frac{\sigma^2 T}{n}}$$

RISK NEUTRAL PROBABILITIES:

$$\pi_n(T) = \frac{u - e^{\rho}}{u - d} = \frac{1}{2} \text{ AND } \pi_n(H) = \frac{e^{\rho} - d}{u - d} = \frac{1}{2}$$

WE SHOW THAT EQUATIONS (3)-(4) ARE APPROXIMATELY SATISFIED

WILL USE THIS APPROXIMATE BINOMIAL TREE
APPROXIMATION TO CONDITION (4):

\[
\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots
\]

\[
x = \sqrt{\frac{\sigma^2 T}{n}} : \log(u) = \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots
\]

\[
x = -\sqrt{\frac{\sigma^2 T}{n}} : \log(d) = -\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots
\]

AND SO:

\[
\text{Var} \left( \log(S_n) \right) = n \left( \log u - \log d \right)^2 \pi(H)\pi(T)
\]

\[
= \frac{1}{4} \left( \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots 
\right)^2
\]

\[
= \frac{1}{4} \left( 2\sqrt{\frac{\sigma^2 T}{n}} + \frac{1}{n\sqrt{n}} \times \ldots \right)^2
\]

\[
= \sigma^2 T + \frac{1}{n} \times \ldots
\]
ABOUT EQUATION (3):

\[
\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots
\]

\[
\log(u) = \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots
\]

\[
\log(d) = -\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots
\]

AND SO:

\[
\nu T = E(\log(S_n) - \log(S_0))
\]

\[
= n(\log(u)\pi(H) - \log(d)\pi(T))
\]

\[
= \frac{1}{2} n \left( \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots 

+ (-\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots) \right)
\]

\[
= -\frac{1}{2} \sigma^2 T + \frac{1}{\sqrt{n}} \times \ldots
\]

AS PREDICTED
HOW MUCH DO OUR RESULTS DEPEND ON $n$?

TRYING THE MATTER OUT IN R

```r
M <- 1000  # number of simulation steps
sigma <- .2 # clock time volatility
T <- 1      # clock time duration
S0 <- 100   # initial value
piH <- 1/2  # risk neutral probability
n <- 10     # steps
u <- 1 + sqrt(T*sigma^2/n) # up step
d <- 1 - sqrt(T*sigma^2/n) # down step
H <- rbinom(M,n,piH)        # simulation
logS <- log(S0) + log(u)*H + log(d)*(n-H)
par(mfrow=c(2,1))  # check this command out!
hist(logS,freq=F)
# try again with a larger number of steps
n <- 1000
# define u, d, H, logS as above, with new n
hist(logS,freq=F)
```
THE DISTRIBUTION OF $\log S_T$ STABILIZES
THE CENTRAL LIMIT PHENOMENON

**THEOREM:** SUPPOSE THAT

- $X_i, i = 1, \ldots, n$ ARE IID $P_n$
  (DISTRIBUTION CAN DEPEND ON $n$)
- $n \text{Var}_n(X) \to \gamma^2$ AS $n \to \infty$

THEN

$$\sum_{i=1}^{n} X_i - nE_n(X) \xrightarrow{\mathcal{L}} N(0, \gamma^2)$$

IN WORDS:

$\sum_{i=1}^{n} X_i - nE_n(X)$ CONVERGES IN LAW TO $N(0, \gamma^2)$

THAT IS TO SAY:

THE DISTRIBUTION OF $\sum_{i=1}^{n} X_i - nE_n(X)$ IS APPROXIMATELY NORMAL $N(0, \gamma^2)$

DENSITY OF THE NORMAL DISTRIBUTION $N(\mu, \gamma^2)$

$$\frac{d}{dx} P(N(\mu, \gamma^2) \leq x) = \frac{1}{\gamma} \phi \left( \frac{x - \mu}{\gamma} \right)$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\}$$
IN OUR CASE

\[ \log(S_T) - \log(S_0) = \sum_{i=1}^{n} X_i \]

\[ \gamma^2 = \sigma^2 T \]

\[ E(\log(S_T) - \log(S_0)) = nE_n(X) \approx -\frac{1}{2} \sigma^2 T \]

SO THAT

\[ \log(S_T) - \left( \log(S_0) - \frac{1}{2} \sigma^2 T \right) \]

IS APPROXIMATELY NORMAL \( N(0, \sigma^2 T) \)

OR: \( \log(S_T) \) IS APPROXIMATELY NORMAL

\[ N(\log(S_0) - \frac{1}{2} \sigma^2 T, \sigma^2 T) \]

Note: \( Z \sim N(\mu, \gamma^2) \iff Z - \mu \sim N(0, \gamma^2) \)

\[ Z - \mu \sim N(0, \gamma^2) \iff \frac{Z - \mu}{\gamma} \sim N(0, 1) \]
SUPERIMPOSING THE NORMAL CURVE ON THE HISTOGRAM

n <- 10     # steps
u <- 1 + sqrt(T*sigma^2/n) # up step
d <- 1 - sqrt(T*sigma^2/n) # down step
H<- rbinom(M,n,piH)         # simulation
logS <- log(S0) + log(u)*H + log(d)*(n-H)
par(mfrow=c(2,1)) # check this command out!
hist(logS,freq=F)
# compare to normal distribution
xpoints<-c(-30:30)/10
mu<-log(S0)-(sigma^2*T)/2
gamma<-sqrt(sigma^2*T)
xpoints<-mu+sigma*xpoints
density<-dnorm(xpoints,mean=mu,sd=gamma)
lines(xpoints,density)
# try again with a larger number of steps
n <- 1000
# define u, d, H, logS as above, with new n
hist(logS,freq=F)
# mu, gamma, xpoints stay the same
lines(xpoints,density)
NORMAL CURVE SUPERIMPOSED ON HISTOGRAMS
THE CLASSICAL CENTRAL LIMIT THEOREM

(A digression. Just so you know.)

SETUP:

\[ Y_1, \ldots, Y_n \text{ ARE IID, } E(Y) = 0 \text{ AND } \text{Var}(Y) = \gamma^2 \]

THEN:

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i \xrightarrow{\mathcal{L}} N(0, \gamma^2) \]

PROOF:

TAKE \( X_i = \frac{1}{\sqrt{n}} Y_i \)

IN EARLIER THEOREM

RESULT FOLLOWS
BEHAVIOR OF OPTIONS PRICES

STEP 1: CONTINUOUS FUNCTIONS

THEOREM: IF

- $Z_n \xrightarrow{\mathcal{L}} Z$ AS $n \to \infty$
- $x \to h(x)$ IS A CONTINUOUS FUNCTION

THEN $h(Z_n) \xrightarrow{\mathcal{L}} h(Z)$ AS $n \to \infty$

EXAMPLE

$Z_n = \log(S(n)^T) \xrightarrow{\mathcal{L}} Z = N\left(\log(S_0) - \frac{1}{2}\sigma^2 T, \sigma^2 T\right)$

CONTINUOUS FUNCTION #1: $h(x) = e^x$:

$S(n)^T = \exp\{Z_n\} \xrightarrow{\mathcal{L}} S(\infty)^T = \exp\{Z\}$

CONTINUOUS FUNCTION #2: $h(x) = (x - e^{-rT} K)^+$:

$V(n)^T = (S(n)^T - e^{-rT} K)^+ \xrightarrow{\mathcal{L}} (S(\infty)^T - e^{-rT} K)^+$

CHECK THIS IN R!
BEHAVIOR OF OPTIONS PRICES

STEP 2: THE DOMINATED CONVERGENCE THEOREM

SETUP:

• \((T_n, U_n) \xrightarrow{\mathcal{L}} (T, U)\) as \(n \to \infty\)
• \(|T_n| \leq U_n\) a.s., for all \(n\)
• \(E(U_n) \to E(U)\) as \(n \to \infty\)

THEOREM:

UNDER THESE CONDITIONS:

\[ E(T_n) \to E(T) \text{ as } n \to \infty \]

• CHECK THAT THEOREM IN SHREVE IS SPECIAL CASE

• GENERAL THEOREM:
  
  • See Billingsley: *Probability and Measure*
  
  • Deduce using Skorokhod embedding
  
  • For final: need only to be able to use above Theorem
BEHAVIOR OF OPTIONS PRICES

STEP 3: COMBINE THEOREMS

TAKE: \( T_n = (S_T^{(n)} - e^{-rT}K)^+ \) AND \( U_n = S_T^{(n)} \)

WE KNOW:
- \((T_n, U_n) \xrightarrow{\mathcal{L}} (T, U)\) AS \( n \to \infty \)
- \(|T_n| \leq U_n\) a.s., FOR ALL \( n \): \((S - e^{-rT}K)^+ \leq S\)

WE NEED TO ESTABLISH

\[
E(U_n) \to E(U) \text{ AS } n \to \infty \quad (5)
\]

IF THIS IS THE CASE, WE CAN CONCLUDE THAT

\[
n \text{ step options price } = E(S_T^{(n)} - e^{-rT}K)^+ \to E(S_T^{(\infty)} - e^{-rT}K)^+ \quad (6)
\]

WHERE

\[
S_T^{(\infty)} = \exp\{Z\}
\]

AND

\[
Z = N \left( \log(S_0) - \frac{1}{2}\sigma^2 T, \sigma^2 T \right)
\]
COMPUTATION OF EXPECTED VALUES

\[ \log S_T = \log S_0 - \frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} N(0, 1) \]

\[
E[f(S_T)] = E[f(\exp\{\log S_0 - \frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} N(0, 1)\})]
\]
\[
= E[f(S_0 \exp\{-\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} z\})]
\]
\[
= \int_{-\infty}^{+\infty} f(S_0 \exp\{-\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} z\}) \phi(z) dz
\]

where \( \phi(z) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} z^2\} \)
IN PARTICULAR: $f(s) = s$:

$$E[U] = E[S_T]$$

$$= \int_{-\infty}^{+\infty} S_0 \exp\left\{-\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} z\right\} \phi(z) dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S_0 \exp\left\{-\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} z - \frac{1}{2}z^2\right\} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S_0 \exp\left\{-\frac{1}{2} (z - \sqrt{\sigma^2 T})^2\right\} dz$$

$$= S_0 \int_{-\infty}^{+\infty} \phi(z - \sqrt{\sigma^2 T}) dz$$

$$= S_0 \int_{-\infty}^{+\infty} \phi(u) du \quad (u = z - \sqrt{\sigma^2 T})$$

$$= S_0$$

IT FOLLOWS THAT EQUATION (5) IS SATISFIED
THE BLACK-SCHOLES-MERTON FORMULA

- THE OPTIONS PRICE FOR LARGE $n$ IS
  \[ E(\tilde{S}_T^{(\infty)} - e^{-rT}K)^+ \]

- CAN COMPUTE IT EXPLICITELY USING EQUATION (7)

- THIS IS THE B-S-M FORMULA FOR THE PRICE OF A CALL OPTION

- YOU DON’T NEED TO USE A TREE IN THIS CASE
CONTINUOUS MARTINGALES

TWO CONTINUITIES:

• TIME ITSELF:
  \[ M_t, \quad 0 \leq t \leq T \quad \text{(or } 0 \leq t < \infty \text{)} \]

• PROCESS PATH:
  \[ t \rightarrow M_t = M_t(\omega) \quad \text{CONTINUOUS FUNCTION OF TIME} \]
(ADDITIVE) BROWNIAN MOTION $W_t$, $0 \leq t \leq T$

1. $W_0 = 0$
2. $t \to W_t(\omega)$ IS CONTINUOUS for each $\omega$
3. HAS INDEPENDENT INCREMENTS
4. $W_{t+s} - W_s \sim N(0, t)$

PICTURE OF (3):

\[
\begin{array}{cccc}
\Delta W_{t_0} & \Delta W_{t_1} & \Delta W_{t_2} & \text{INDEPENDENT} \\
= W_{t_1} - W_{t_0} & = W_{t_2} - W_{t_1} & \\
0 = t_0 & t_1 & t_2 & t_3 \quad \text{ANY GRID}
\end{array}
\]

ADDITIVE PROPERTY (4):

$\Delta W_{t_0} \sim N(0, t_1)$, $\Delta W_{t_1} \sim N(0, t_2 - t_1)$

DELETE $t_1$:

\[
W_{t_2} - W_{t_0} = \Delta W_{t_0} + \Delta W_{t_1} \\
\overset{N(0, t_1) + N(0, t_2 - t_1)}{\text{BY INDEP: } N(0, t_2)}
\]
\[ (3) + (4) \implies W_t \text{ IS A MARTINGALE} \]

\[
E(W_{t+s} \mid \mathcal{F}_s) = E(W_{t+s} - W_s + W_s \mid \mathcal{F}_s) \\
= E(W_{t+s} - W_s \mid \mathcal{F}_s) + W_s \\
= \underbrace{E(W_{t+s} - W_s)}_{\text{independence}} + W_s \\
= 0 \quad \text{since } W_{t+s} - W_s \sim N(0, t) \\
= W_s
\]
THE BLACK-SCHOLES MODEL: MULTIPLICATIVE BROWNIAN MOTION

\[ \tilde{S}_t = \tilde{S}_0 \times \exp(\sigma W_t - \frac{1}{2} \sigma^2 t) \]

EVOLUTION:

\[ \tilde{S}_t = \tilde{S}_0 \times \exp(\sigma W_u - \frac{1}{2} \sigma^2 u) \quad \{ \tilde{S}_u \text{ independent multiplicative increment} \} \]

\[ \times \exp(\sigma (W_t - W_u) - \frac{1}{2} \sigma^2 (t - u)) \]

\[ = \tilde{S}_u \times \exp\left(\sigma N(0, t-u) - \frac{1}{2} \sigma^2 (t - u)\right) \]

\[ = \tilde{S}_u \times \exp(\alpha Z - \frac{1}{2} \alpha^2) \quad \alpha^2 = \sigma^2 (t - u) \quad Z \sim N(0, 1) \]

MARTINGALE:

\[ E(\tilde{S}_t \mid \mathcal{F}_u) = \tilde{S}_u E(\exp(\sigma Z - \frac{1}{2} \sigma^2) \mid \mathcal{F}_u) \]

\[ = \tilde{S}_u E(\exp(\sigma Z - \frac{1}{2} \sigma^2)) \text{ BY INDEPENDENCE} \]

\[ = \tilde{S}_u \times 1 \quad (\text{NORMAL}) \]

\[ = \tilde{S}_u \]
CLT FOR THE WHOLE PROCESS

\[ t_0 = 0 \quad t_1 = \frac{\sigma^2 T}{n} \quad t_2 = \frac{2 \sigma^2 T}{n} \quad t_3 = \frac{3 \sigma^2 T}{n} \quad t_k = \frac{k \sigma^2 T}{n} \]

STOCK PRICE PROCESS

\[ \log(\tilde{S}_t^{(n)}) - \log(S_0) = \sum_{t_i \leq t} X_i, \quad 0 \leq t \leq T \]

CONVERGENCE: AS \( n \to \infty \):

\[ \log(\tilde{S}_t^{(n)}) \xrightarrow{\mathcal{L}} \log(S_t) = \log(S_0) + \sigma W_t - \frac{1}{2} \sigma^2 t \]

GEOMETRIC BROWNIAN MOTION
APPLICATION TO OPTIONS

CONTINUOUS FUNCTIONALS

• \( x = \{x_t, 0 \leq t \leq T\} \) A REALIZATION OF THE PROCESS

• \( x \to h(x) \) TAKES REAL VALUES

• \( x \to h(x) \) IS CONTINUOUS:

\[
\sup_{0 \leq t \leq T} |x_t^{(n)} - x_t| \to 0 \implies h(x^{(n)}) \to h(x_t)
\]

FOR \( h \) CONTINUOUS:

\[
h(\log(\tilde{S}_t^{(n)})) \xrightarrow{\mathcal{L}} h(\log(\tilde{S}_t))
\]

OR

\[
h(\tilde{S}_t^{(n)}) \xrightarrow{\mathcal{L}} h(\tilde{S}_t)
\]

EXAMPLE OF MEANINGFUL LIMIT:

\[
h(x) = \sup_{0 \leq t \leq T} x_t
\]

LOOKBACK OPTIONS