AMERICAN OPTIONS

Case where the value only depends on the stock price at time of exercise (“non-path dependent” case)

Payoff: G_τ, at time τ decided by owner, $0 \leq \tau \leq T$
For example: $G_n = g(S_n)$ (non-path dependent)
Put: $g(s) = (K - s)^+$
In general: G_n is $(\mathcal{F}_n)$-adapted (= path dependent)
G_n: “Intrinsic value process”
Assume $G_n \geq 0$, otherwise no point in exercising. If necessary, take $G_n(s) \leftarrow G_n(s)^+$

VALUE OF OPTION at time n: V_n

• Boundary condition: If option has not been exercised before time T, then $V_T = G_T$

• Recursion: If option has not been exercised before time $n < T$, there is a choice:

(i) Exercise now. Value: G_n
(ii) Wait. Value: $e^{-r} E_\pi[V_{n+1} \mid \mathcal{F}_n]$
Decision at time n: pick the higher value
Overall value at n:

$$V_n = \max(G_n, e^{-r} E_\pi[V_{n+1} \mid \mathcal{F}_n])$$

Iterate until $n = 0$: this gives price V_0 at time 0

THE NON-PATH DEPENDENT CASE

$$G_n = g(S_n) \qquad \text{(such as } (K - S_n)^+\text{)}$$

$V_T = g(S_T)$ at final time: set $v_T(S_T) = g(S_T)$

If S_n is π-Markov:

Suppose by induction that $V_{n+1} = v_{n+1}(S_{n+1})$

$$\begin{aligned} e^{-r} E_\pi[V_{n+1} \mid \mathcal{F}_n] &= e^{-r} E_\pi[v_{n+1}(S_{n+1}) \mid \mathcal{F}_n] \\ &= e^{-r} E_\pi[v_{n+1}(S_{n+1}) \mid S_n] \end{aligned}$$

Hence:

$$\begin{aligned} V_n &= \max(G_n, e^{-r} E_\pi[V_{n+1} \mid \mathcal{F}_n]) \\ &= \max(g(S_n), e^{-r} E_\pi[v_{n+1}(S_{n+1}) \mid S_n]) \\ &= v_n(S_n) \text{ by definition} \end{aligned}$$

BINOMIAL MODEL: $S_{n+1} = u$ or $d \times S_n$:

$$E_\pi[v_{n+1}(S_{n+1}) \mid S_n = s]) = v_{n+1}(us)\pi(H) + v_{n+1}(ds)\pi(T)$$

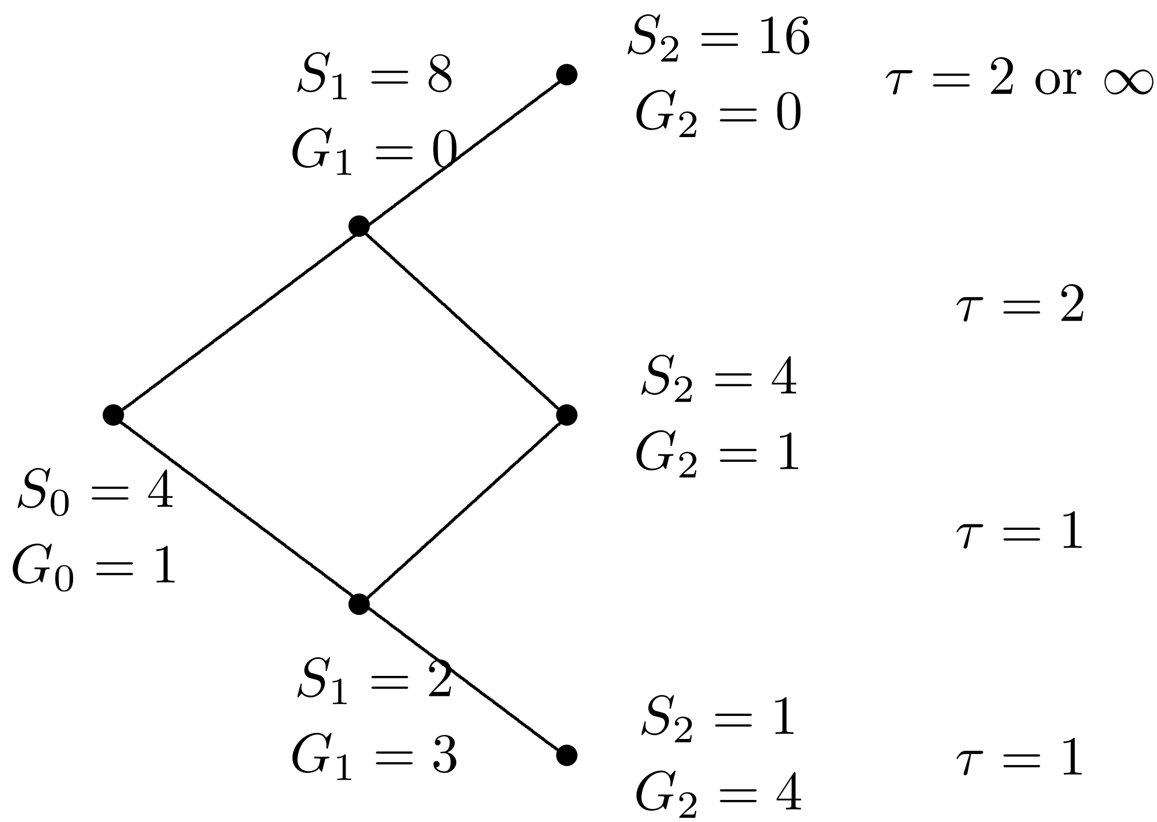
And so:

$$v_n(s) = \max(g(s), e^{-r}[v_{n+1}(us)\pi(H) + v_{n+1}(ds)\pi(T)]$$

EXAMPLE OF BINOMIAL TREE

structure: $S_0 = 4, u = 2, d = \frac{1}{2}, e^r = \frac{5}{4}$
(hence: $\pi(H) = \pi(T) = \frac{1}{2}$)

option: $G_n = g(S_n) = (5 - S_n)^+$, times: $n = 0, 1, 2$



$$v_n(s) = \max(g(s), e^{-r}[v_{n+1}(us)\pi(H) + v_{n+1}(ds)\pi(T)]$$

$$v_1(S_1 = 8) = \max(0, \frac{4}{5}[0 \times \frac{1}{2} + 1 \times \frac{1}{2}]) = \max(0, \frac{2}{5}) = 0.4$$

$$v_1(S_1 = 2) = \max(3, \frac{4}{5}[1 \times \frac{1}{2} + 4 \times \frac{1}{2}]) = \max(3, 2) = 3$$

$$v_0(S_0 = 4) = \max(1, \frac{4}{5}[0.4 \times \frac{1}{2} + 3 \times \frac{1}{2}]) = \max(1, 1.36) = 1.36$$

STOPPING TIMES τ

τ: random time

observable when it occurs:

$$\{\tau(\omega) = n\} \in \mathcal{F}_n \quad \text{or} \quad \{\tau(\omega) \leq n\} \in \mathcal{F}_n$$

$(\{\tau(\omega) \leq n\} = \{\tau(\omega) = 0\} \cup ... \cup \{\tau(\omega) = n\})$
EX:

$$\begin{aligned} \tau &= \inf\{t : \quad S_t \geq x\} \\ &= \text{first time stock price crosses barrier } x \end{aligned}$$

$\underline{\underline{\text{NOT}}}$ EX:

$$\inf\{t \leq T : S_t = \max_{0 \leq t \leq T} S_t\}$$

$=$ time when stock market reaches max NOT observable at the time

Make sure you understand connection to Shreve's Definition 4.3.1 (p. 97)

STOPPED PROCESSES

$$\begin{aligned}
\widetilde{S}_t &= \text{discounted stock price} \\
\widetilde{V}_t &= \widetilde{S}_{\tau\wedge t} \qquad (\tau \wedge t = \text{MIN}(\tau, t)) \\
&= \text{hold } \widetilde{S} \text{ until } \tau \\
&= \widetilde{S}_0 + \sum_{u=0}^{t-1} \theta(u)\widetilde{\Delta S}_u
\end{aligned}$$

$$\theta(u) = \begin{cases} 1 & \text{if } u < \tau \\ 0 & \text{if } u \geq \tau \end{cases}$$

If $\widetilde{S}_t$ is $\pi - \text{MG}$, then $\widetilde{V}_t$ is $\pi - \text{MG}$

What is true:

$$\text{for } s \leq t: \quad E_\pi(\widetilde{S}_{\tau\wedge t} \mid \mathcal{F}_s) = \widetilde{S}_{\tau\wedge s} \quad \Bigg| \text{ “optional stopping”}$$

What is <u>not</u> true, in general:

$$E_\pi(\widetilde{S}_\tau \mid \mathcal{F}_s) = \widetilde{S}_{\tau\wedge s} \quad \Bigg| \text{ “doubling strategies”}$$

SUB- AND SUPERMARTINGALES

A PROCESS X_t IS A

- SUB-MG IF $E(X_t \mid \mathcal{F}_s) \geq X_s$ WHEN $t \geq s$

- SUPER-MG IF $E(X_t \mid \mathcal{F}_s) \leq X_s$ WHEN $t \geq s$

X_t SUPER-MG IFF $-X_t$ SUB-MG

TIME DISCRETE CASE: X_n SUPER-MG SAME AS $E(X_{n+1} \mid \mathcal{F}_n) \leq X_n$, ALL t

A STOPPED SUPER-MG IS A SUPER-MG (SAME ARGUMENT AS FOR MG)

THE DOOB DECOMPOSITION

IF $\widetilde{V}_n$ SUPER-MG: $\widetilde{V}_n = \widetilde{M}_n + \widetilde{A}_n$

WHERE

- $\widetilde{M}_n$ IS MG
- $\widetilde{A}_n$ IS NONINCREASING ($\widetilde{A}_{n+1} \leq \widetilde{A}_n$)
- $\widetilde{A}_0 = 0$
- $\widetilde{A}_{n+1}$ IS $\mathcal{F}_n$-MEASURABLE for all n (PREDICTABLE)
- THE DECOMPOSITION IS UNIQUE

CONSEQUENCE IN COMPLETE MARKET

- V_n CAN BE FINANCED, WITH DIVIDEND
- M_n HAS EXACT SELF FINANCING STRATEGY
- DIVIDEND FROM TIME n TO $n+1$ (DISCOUNTED SCALE):

$$\begin{aligned}\Delta\widetilde{V}_n - \text{P/L FOR HEDGE} &= \Delta\widetilde{V}_n - \Delta\widetilde{M}_n \\ &= \Delta\widetilde{A}_n\end{aligned}$$

YOU EVEN KNOW THE DIVIDEND AT TIME n

PROOF OF THE DOOB DECOMPOSTION

$\widetilde{V}_n$: SUPER-MG

DEFINE: $\Delta\widetilde{A}_n = E_\pi[\Delta\widetilde{V}_n \mid \mathcal{F}_n]$

AND: $\widetilde{A}_n = \Delta\widetilde{A}_0 + \ldots + \Delta\widetilde{A}_{n-1}$

AND: $\widetilde{M}_n = \widetilde{V}_n - \widetilde{A}_n$

CHECK CONDITIONS

- $E_\pi[\Delta\widetilde{M}_n \mid \mathcal{F}_n] = E_\pi[\Delta\widetilde{V}_n \mid \mathcal{F}_n] - E_\pi[\Delta\widetilde{A}_n \mid \mathcal{F}_n] = 0$

(hence $\widetilde{M}_n$ is MG)

- $\Delta\widetilde{A}_n \leq 0$ since $\widetilde{V}_n$ is SUB-MG
- $\widetilde{A}_0 = 0$ by definition
- $\widetilde{A}_{n+1}$ is $\mathcal{F}_n$-measurable by definition
- UNIQUENESS: do it as exercise

QED

SUPERMARTINGALES AND AMERICAN OPTIONS

V_n: VALUE OF AMERICAN OPTION WITH INTRINSIC VALUE PROCESS G_n

$$V_n = \max(G_n, e^{-r} E_\pi[V_{n+1} \mid \mathcal{F}_n])$$

DISCOUNTED:

$$\begin{aligned}\widetilde{V}_n &= e^{-rn} \max(G_n, e^{-r} E_\pi[V_{n+1} \mid \mathcal{F}_n]) \\ &= \max(\widetilde{G}_n, E_\pi[\widetilde{V}_{n+1} \mid \mathcal{F}_n]) \\ &\geq E_\pi[\widetilde{V}_{n+1} \mid \mathcal{F}_n]\end{aligned}$$

$\widetilde{V}_n$ IS A SUPERMARTINGALE; $\widetilde{V}_n \geq \widetilde{G}_n$ for all n

WE SHALL SEE A CONVERSE:

$\widetilde{V}_n$ IS THE SMALLEST SUPERMARTINGALE

SATISFYING $\widetilde{V}_n \geq \widetilde{G}_n$ for all n

THE CONVERSE

$$\widetilde{V}_n = \max(\widetilde{G}_n, E_\pi[\widetilde{V}_{n+1} \mid \mathcal{F}_n]), \text{ AND } \widetilde{V}_T = \widetilde{G}_T$$

LET $\widetilde{Y}_n$ ANOTHER SUPERMARTINGALE,
$\widetilde{V}_n \geq \widetilde{Y}_n \geq \widetilde{G}_n$ FOR ALL n

WILL SHOW $\widetilde{Y}_n = \widetilde{V}_n$ FOR ALL n

INDUCTION: $\widetilde{Y}_T = \widetilde{V}_T$, and
Suppose $\widetilde{Y}_{n+1} = \widetilde{V}_{n+1}$. Then

$$\begin{aligned}
\widetilde{V}_n &= \max(\widetilde{G}_n, E_\pi[\widetilde{Y}_{n+1} \mid \mathcal{F}_n]) \\
&\leq \max(\widetilde{G}_n, \widetilde{Y}_n) \text{ since } \widetilde{Y}_n \text{ super-mg} \\
&= \widetilde{Y}_n \text{ since } \widetilde{Y}_n \geq \widetilde{G}_n
\end{aligned}$$

QED

COROLLARY: IF $\widetilde{Z}_n$ IS ANOTHER SUPERMARTINGALE, $\widetilde{Z}_n \geq \widetilde{G}_n$ FOR ALL n, THEN: $\widetilde{Z}_n \geq \widetilde{V}_n$ FOR ALL n

PROOF: SET $\widetilde{Y}_n = \min(\widetilde{V}_n, \widetilde{Z}_n)$ (SO $\widetilde{V}_n \geq \widetilde{Y}_n \geq \widetilde{G}_n$)

$$E_\pi[\widetilde{Y}_{n+1} \mid \mathcal{F}_n]) \leq \begin{cases} E_\pi[\widetilde{V}_{n+1} \mid \mathcal{F}_n]) \leq \widetilde{V}_n \\ E_\pi[\widetilde{Z}_{n+1} \mid \mathcal{F}_n]) \leq \widetilde{Z}_n \end{cases}$$

SO $E_\pi[\widetilde{Y}_{n+1} \mid \mathcal{F}_n]) \leq \min(\widetilde{V}_n, \widetilde{Z}_n) = \widetilde{Y}_n$: $\widetilde{Y}_n$ IS SUPER-MG

QED

STOPPING TIMES AND AMERICAN OPTIONS

$$\widetilde{V}_n = \max(\widetilde{G}_n, E_\pi[\widetilde{V}_{n+1} \mid \mathcal{F}_n])$$

Define

$$\tau = \min\{n : \widetilde{V}_n = \widetilde{G}_n\} \wedge T$$

$\widetilde{V}_{n\wedge\tau}$ IS A MARTINGALE:

On the set $A = \{\tau > n\}$ ($\in \mathcal{F}_n$):

$$\begin{aligned}
\widetilde{V}_{n\wedge\tau} I_A = \widetilde{V}_n I_A &= \max(\widetilde{G}_n, E_\pi[\widetilde{V}_{n+1} \mid \mathcal{F}_n]) I_A \\
&= E_\pi[\widetilde{V}_{n+1} \mid \mathcal{F}_n] I_A \\
&= E_\pi[\widetilde{V}_{n+1} I_A \mid \mathcal{F}_n] \\
&= E_\pi[\widetilde{V}_{(n+1)\wedge\tau} I_A \mid \mathcal{F}_n]
\end{aligned}$$

On the complement $A^c = \{\tau \leq n\}$:

$$\begin{aligned}
E_\pi[\widetilde{V}_{(n+1)\wedge\tau} I_{A^c} \mid \mathcal{F}_n] &= E_\pi[\widetilde{V}_{n\wedge\tau} I_{A^c} \mid \mathcal{F}_n] \\
&= \widetilde{V}_{n\wedge\tau} I_{A^c}
\end{aligned}$$

Add the two terms to show $\widetilde{V}_{n\wedge\tau}$ is MG

IN PARTICULAR ($\tau \leq T$):

$$\begin{aligned}
\widetilde{V}_0 &= E_\pi[\widetilde{V}_\tau] \\
&= E_\pi[\widetilde{G}_\tau]
\end{aligned}$$

STOPPING TIMES AND AMERICAN OPTIONS

$\mathcal{S}_n =$ set of stopping times τ which take values in $\{n, ..., T\}$

Define: $\widetilde{U}_n = \max_{\tau \in \mathcal{S}_n} E_\pi[\widetilde{G}_\tau \mid \mathcal{F}_n]$

THEOREM: $\widetilde{U}_n = \widetilde{V}_n$

PROOF: For optimal stopping time τ:

$\tau(\omega) = n$ on $\{\omega : \widetilde{G}_n(\omega) > \max_{\tau \in \mathcal{S}_{n+1}} E_\pi[\widetilde{G}_\tau \mid \mathcal{F}_n](\omega)\}$

$\tau(\omega) > n$ on $\{\omega : \widetilde{G}_n(\omega) < \max_{\tau \in \mathcal{S}_{n+1}} E_\pi[\widetilde{G}_\tau \mid \mathcal{F}_n](\omega)\}$

Otherwise redefine τ

Also (see next page):

$$\begin{aligned} \max_{\tau \in \mathcal{S}_{n+1}} E_\pi[\widetilde{G}_\tau \mid \mathcal{F}_n] &= E_\pi[\max_{\tau \in \mathcal{S}_{n+1}} E_\pi(\widetilde{G}_\tau \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n] \\ &= E_\pi[\widetilde{U}_{n+1} \mid \mathcal{F}_n] \end{aligned} \tag{$*$}$$

Therefore: $\widetilde{U}_n = \max(\widetilde{G}_n, E_\pi[\widetilde{U}_{n+1} \mid \mathcal{F}_n])$

Also: $\widetilde{U}_T = \widetilde{G}_T$ since $\mathcal{S}_T = \{T\}$

CONCLUSION:
$\widetilde{U}_n$ SATISFIES THE DEFINITION OF $\widetilde{V}_n$

QED

Justification for (*)

Take $\tau_1, \tau_2 \in \mathcal{S}_{n+1}$ so that

$$E_\pi[\widetilde{G}_{\tau_1} \mid \mathcal{F}_n] = \max_{\tau \in \mathcal{S}_{n+1}} E_\pi[\widetilde{G}_\tau \mid \mathcal{F}_n]$$

$$E_\pi(\widetilde{G}_{\tau_2} \mid \mathcal{F}_{n+1}) = \max_{\tau \in \mathcal{S}_{n+1}} E_\pi(\widetilde{G}_\tau \mid \mathcal{F}_{n+1})$$

By definition of τ_2:

$$E_\pi(\widetilde{G}_{\tau_2} \mid \mathcal{F}_{n+1}) \geq E_\pi(\widetilde{G}_{\tau_1} \mid \mathcal{F}_{n+1}).$$

Hence

$$\begin{aligned} E_\pi[\widetilde{G}_{\tau_2} \mid \mathcal{F}_n] &= E_\pi[E_\pi(\widetilde{G}_{\tau_2} \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n] \\ &\geq E_\pi[E_\pi(\widetilde{G}_{\tau_1} \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n] \\ &= E_\pi[\widetilde{G}_{\tau_1} \mid \mathcal{F}_n] \\ &= \max_{\tau \in \mathcal{S}_{n+1}} E_\pi[\widetilde{G}_\tau \mid \mathcal{F}_n] \end{aligned}$$

by definition of τ_1. Hence we can replace τ_1 by τ_2. Thus

$$\begin{aligned} \max_{\tau \in \mathcal{S}_{n+1}} E_\pi[\widetilde{G}_\tau \mid \mathcal{F}_n] &= E_\pi[\widetilde{G}_{\tau_2} \mid \mathcal{F}_n] \\ &= E_\pi[E_\pi(\widetilde{G}_{\tau_2} \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n] \\ &= E_\pi[\max_{\tau \in \mathcal{S}_{n+1}} E_\pi(\widetilde{G}_\tau \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n]. \end{aligned}$$

QED

$\widetilde{V}_n$ CAN BECOME STRICT SUPERMARTINGALE IF OPTION IS NOT EXERCISED AT τ

$$\widetilde{V}_n = \max(\widetilde{G}_n, E_\pi[\widetilde{V}_{n+1} \mid \mathcal{F}_n])$$

$$\tau = \min\{n : \widetilde{V}_n = \widetilde{G}_n\} \wedge T$$

On the set $\{\omega : \tau(\omega) = N\} \cap \{\widetilde{G}_N > E_\pi[\widetilde{V}_{N+1} \mid \mathcal{F}_N]\}$:

$$E_\pi[\widetilde{V}_{N+1} \mid \mathcal{F}_N] < \widetilde{V}_N$$

(A tautology...)

CONVEX EUROPEAN AND AMERICAN OPTIONS

Let $s \to g(s)$ be convex, ≥ 0, with $g(0) = 0$,
say $g(s) = (s - X)^+$

S_n is a nonnegative security price

$$\begin{aligned} e^{-r} g(S_{n+1}) &= e^{-r} g(S_{n+1}) + (1 - e^{-r}) g(0) \\ &\geq g(e^{-r} S_{n+1} + (1 - e^{-r})0) = g(e^{-r} S_{n+1}) \end{aligned}$$

, and so

$$\begin{aligned} E_\pi[e^{-r} g(S_{n+1}) \mid \mathcal{F}_n] &\geq E_\pi[g(e^{-r} S_{n+1}) \mid \mathcal{F}_n] \\ &\geq g(S_n) \end{aligned}$$

by Jensen's inequality, since

$$E_\pi[e^{-r} S_{n+1} \mid \mathcal{F}_n] = e^{rn} E_\pi[\widetilde{S}_{n+1} \mid \mathcal{F}_n] = e^{rn} \widetilde{S}_n = S_n$$

Thus: $X_n = e^{-rn} g(S_n)$ is a submartingale

For any stopping time $\tau \geq n$ (see next page):

$$E_\pi[X_{\tau \vee (n+1)}] \geq E_\pi[X_\tau] \qquad (*)$$

By induction: the value of the American option is

$$\sup_{\tau \in \mathcal{S}_0} E_\pi[e^{-r\tau} g(S_\tau)] = E_\pi[e^{-rT} g(S_T)]$$

Same as for European option

Justification for (*) on previous page

Since $\tau \geq n$:

$$X_{\tau \vee (n+1)} - X_\tau = (X_{n+1} - X_n) I_{\{\tau = n\}}.$$

Hence

$$\begin{aligned} E[X_{\tau \vee (n+1)} - X_\tau \mid \mathcal{F}_n] &= E[(X_{n+1} - X_n) I_{\{\tau = n\}} \mid \mathcal{F}_n] \\ &= E[(X_{n+1} - X_n) \mid \mathcal{F}_n] I_{\{\tau = n\}} \\ &\quad \text{since } I_{\{\tau = n\}} \text{ is conditionally constant} \\ &\geq 0 \end{aligned}$$

since X_n is a sub-MG.

Hence

$$\begin{aligned} E[X_{\tau \vee (n+1)} - X_\tau] &= E\{E[X_{\tau \vee (n+1)} - X_\tau \mid \mathcal{F}_n] I_{\{\tau = n\}}\} \\ &= E[\text{random variable } \geq 0] \\ &\geq 0 \end{aligned}$$

DOUBLING STRATEGIES

$$S_n = \sum_{i=1}^{n} Y_i \qquad Y_i = \begin{cases} +1\ \$ & \pi = \frac{1}{2} \\ -1\ \$ & \pi = \frac{1}{2} \end{cases}$$

$$\tau = \inf\{t : S_t = A\} \qquad A > 0$$

The facts:
1) $\pi(\tau < \infty) = 1$
2) $A = E_\pi(S_\tau) \neq S_0 = 0$

Realistic version: credit constraint

$$\tau = \inf t : \begin{cases} S_t = A \text{ retire} \\ S_t = -B \text{ go bankrupt} \end{cases}$$

WHY CALLED DOUBLING STRATEGIES?

Take gamble # i at time $t_i = 1 - \frac{1}{2^i}$

$ in play goes to infinity in fixed time interval

THE ODDS

$$0 = ES_{\tau\wedge n} \Rightarrow ES_{\tau} \qquad \text{as } n \to \infty$$

— use dominated convergence:

$$|S_{\tau\wedge n}| \leq \max(A, B)$$

— fails for $B = +\infty$ (no constraint)

$$\begin{aligned} 0 &= E_{\pi} S_{\tau} \\ &= A\pi(S_{\tau} = A) - B\pi(S_{\tau} = B) \end{aligned}$$

Also: $\pi(S_{\tau} = A) + \pi(S_{\tau} = -B) = 1$

like binomial tree:

$$\pi(S_{\tau} = A) = \frac{B}{A + B}$$

$(\rightarrow 1 \text{ for credit} \rightarrow +\infty)$

STOPPING TIME DESCRIPTION OF MARTINGALES

$\widetilde{S}_t$ is $(\mathcal{F}_t), \pi - \text{MG} \Updownarrow$

FOR ALL BOUNDED STOPPING TIMES τ:

$$E_\pi \widetilde{S}_\tau = \widetilde{S}_0$$

$\Downarrow$: OPTIONAL STOPPING

$\Uparrow$: for all $A \in \mathcal{F}_t$, set $\quad \tau = \begin{cases} t & \text{if } A^C \\ t+1 & \text{if } A \end{cases}$

$$0 = E\widetilde{S}_t = E\widetilde{S}_t I_A + E\widetilde{S}_t I_{A^C}$$

so

$$E\widetilde{S}_\tau I_{A^C} = E\widetilde{S}_t I_{A^C} = -E\widetilde{S}_t I_A$$

and so

$$\begin{aligned} \widetilde{S}_0 = 0 : 0 = E\widetilde{S}_\tau &= E\widetilde{S}_\tau I_A + E\widetilde{S}_\tau I_{A^C} \\ &= E\widetilde{S}_{t+1} I_A - E\widetilde{S}_t I_A \\ &= E(\widetilde{S}_{t+1} - \widetilde{S}_t) I_A = E\Delta\widetilde{S}_t I_A \end{aligned}$$

Since true for all $A \in \mathcal{F}_t$: $E[\Delta\widetilde{S}_t \mid \mathcal{F}_t] = 0$