COMPUTATION

TWO APPROACHES

• Exact computation
• Simulation

OUR COMPUTER PACKAGE

• Splus (proprietary software)
• R (freeware)
• generically: S

S will also be used for statistical analysis in other courses

Help with S:
• Get a book about it (such as Krause and Olson)
• the help command in S
EXACT COMPUTATION
BINOMIAL MODEL

\[ S_t = S_0 u^{H_t} d^{T_t} = S_0 \left( \frac{u}{d} \right)^{H_t} d^t \] and \( B_t = e^{r t} \)

WHERE

\( H_t = \# \) of heads up to time \( t \) and \( T_t = t - H_t \)

DISCOUNTED SCALE

\[ \tilde{S}_t = e^{-r t} S_t = S_0 \left( \frac{\tilde{u}}{\tilde{d}} \right)^{H_t} \tilde{d}^t \]

WHERE \( \tilde{u} = e^{-r} u \) AND \( \tilde{d} = e^{-r} d \)

LOG SCALE

\[ \log(\tilde{S}_t) = \log(S_0) + H_t \log \left( \frac{\tilde{u}}{\tilde{d}} \right) + t \log(\tilde{d}) \]
IN TERMS OF INDIVIDUAL OUTCOMES

\[ H_t = \# \text{ of heads up to time } t \]
\[ = I_1 + \ldots + I_t \]

WHERE \( I_1, \ldots, I_t, \ldots \) ARE IID

\[
I = \begin{cases} 
I = 1 \text{ with probability } \pi(H) = \frac{1-d}{u-d} \\
I = 0 \text{ with probability } \pi(T) = \frac{u-1}{u-d}
\end{cases}
\]

PROPERTIES OF I

\[ E(I) = \pi(H) \]
\[ \text{Var}(I) = E(I^2) - E(I)^2 = E(I) - E(I)^2 = I^2 = I \]
\[ \pi(H) - \pi(H)^2 = \pi(H)(1 - \pi(H)) \]

PROPERTIES OF \( H_t \)

\[ E(H_t) = E(I_1) + \ldots + E(I_T) = t \pi(H) \]
\[ \text{Var}(H_t) = \text{Var}(I_1) + \ldots \text{Var}(I_t) = t \pi(H)(1 - \pi(H)) \]
(by independence)
DISTRIBUTION OF $H_n$:

THE BINOMIAL FORMULA

$$P(H_n = k) = \binom{n}{k} p(H)^k (1 - p(H))^{n-k} \text{ for } k = 0, \ldots, n$$

WHERE:

$$\binom{n}{k} = \frac{n \times (n - 1) \times \ldots \times (n - k + 1)}{k \times (k - 1) \times \ldots \times 1}$$

$$= \frac{n!}{k!(n - k)!}$$

AND

$$n! \text{ ("n factorial") } = n \times (n - 1) \times \ldots \times 2 \times 1$$

(for convenience/by convention: $0! = 1$)
HENCE, IN A BINOMIAL TREE

price for payoff $f(S_n)$ is

$$V_0 = \exp\{-rn\} E_{\pi} f\left(\exp\{rn\} \tilde{S}_n\right)$$

$$= \exp\{-rn\} E_{\pi} f\left(\exp\{rn + \log(\tilde{S}_n)\}\right)$$

$$= \exp\{-rn\}$$

$$\times E_{\pi} f\left(\exp\{rn + \log(S_0) + H_n \log\left(\frac{u}{d}\right) + n \log(\tilde{d})\}\right)$$

$$= \exp\{-rn\} \sum_{k=0}^{n} [\pi(H_n = k)]$$

$$\times f\left(\exp\{rn + \log(S_0) + k \log\left(\frac{u}{d}\right) + n \log(\tilde{d})\}\right)$$

WE WILL NOW IMPLEMENT THIS IN S
GETTING STARTED WITH S

Open R, then...

R : Copyright 2002, The R Development Core Team
Version 1.5.0 (2002-04-29)

R is free software and comes with ABSOLUTELY NO WARRANTY. You are welcome to redistribute it under certain conditions. Type 'license()' or 'licence()' for distribution details.

R is a collaborative project with many contributors. Type 'contributors()' for more information.

Type 'demo()' for some demos, 'help()' for on-line help, or 'help.start()' for a HTML browser interface to help. Type 'q()' to quit R.

>
TO FIND OUT ABOUT THE BINOMIAL DISTRIBUTION

> help(rbinom)

Binomial package:base R Documentation

The Binomial Distribution

Description:

Density, distribution function, quantile function and random generation for the binomial distribution with parameters ‘size’ and ‘prob’.

Usage:

dbinom(x, size, prob, log = FALSE)
pbinom(q, size, prob, lower.tail = TRUE, log.p = FALSE)
qbinom(p, size, prob, lower.tail = TRUE, log.p = FALSE)
rbinom(n, size, prob)

Arguments:

x, q: vector of quantiles.

p: vector of probabilities.

and so on... scroll down.. give “q” to quit help
> n<-100
# 100 time periods
# comments are, incidentally, preceded by a "#"

> r<-log(1.05)/n
# interest rate for the whole period is therefore given by
> exp(r*n)
[1] 1.05

> u<-1.01
> d<-0.99
# This defines the up and down movements
# the most extreme outcomes are
> u^100
[1] 2.704814
# and, of course,
> d^100
[1] 0.3660323

# since we do a lot of discounting: u-tilde and d-tilde
> ut <- exp(-r)*u
> dt <- exp(-r)*d

# the risk neutral probabilities are
> piH <- (1-dt)/(ut -dt)
> piT <- (ut-1)/(ut -dt)
# and so
> piH
[1] 0.524401

# we take out initial stock price to be
> S0 <- 100
# this sets up our basic system
LET'S HAVE A LOOK AT WHAT OUR DISTRIBUTIONS LOOK LIKE

# let's generate random numbers representing the number of heads

# we generate a sample of 1000 realizations of H
> M<-1000
> H<- rbinom(M,n,piH)

> hist(H,freq=F)

# this produces the graph below
# to find out about the hist function, give the command
> help(hist)

# to save a copy of your graph, you can for example do this

> postscript("histogram.ps")
> hist(H,freq=F)
> graphics.off()
WHAT ABOUT THE DISTRIBUTION OF $S_n$?

# S-tilde and S
> St<- S0 * exp(H*log(ut/dt) + n*log(dt))
> S <- exp(r*n)*St
> hist(S,freq=F)
PRICING THE CALL OPTION
EXACT CALCULATION

# strike price
K <- 105

# the probability distribution of H:
# the outcome space:
omega <- c(0:100)
> omega
[1]  0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20
[40] 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57
[60] 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75
[80] 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93
[99] 94 95 96 97 98 99 100

# probabilities of the outcomes:
> p <- dbinom(omega,n,piH)
# consists of 101 elements
# this is indeed a probability
> sum(p)
[1] 1

# the outcomes for the stock price (101 elements)
> st <- S0 * exp(omega*log(ut/dt) + n*log(dt))
> s <- exp(r*n)*st

# the payoff of the call option
> v <- pmax(s-K,0)
# max(s-K,0) gives a different answer

# the price (the discounted risk neutral expectation)
> exp(-r*n) * sum(v*p)
[1] 3.989224
PRICING THE CALL OPTION
MONTE CARLO SIMULATION

IF $S^{(1)},...,S^{(M)}$ ARE IID COPIES WITH THE DISTRIBUTION OF $S$, THEN

$$\hat{E}h(S) = \frac{1}{M} \sum_{i=1}^{M} h(S^{(i)})$$

APPROXIMATES THE EXPECTATION

$$\hat{E}h(S) \approx E(S)$$

# our pi is a risk neutral measure: $E(S)$ is estimated to be
> exp(-r*n)*mean(S)
[1] 99.72981

# the options payoff
> V<-pmax(S-K,0)
# the estimated options price
> exp(-r*n)*mean(V)
[1] 3.991318

# by comparison to the true price
> 3.991318/3.989224
[1] 1.000525

THE APPROXIMATION GETS BETTER AS $M \to \infty$

HOW BIG DOES $M$ NEED TO BE?
HOW DOES ONE ASSESS THE ERROR?
A LATER LECTURE

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PATH DEPENDENT OPTIONS

# generating one realization of a path
# the process of heads and tails
> II <- rbinom(n,1,piH)
# H is now a process
> HH <- cumsum(II)
> HH

[1] 1 1 1 2 2 2 3 3 3 4 5 6 6 7 8 8 9 9 10 11 12 12
[26] 13 14 14 14 15 16 17 18 18 19 20 21 22 22 23 24 24 24 25 26 26
[51] 28 29 30 31 32 33 34 34 34 35 35 35 35 36 36 36 37 37 37 37
[76] 40 40 41 41 42 42 43 44 45 46 46 47 47 48 49 49 50 50 50 51 51 52

# to start thye cumulative process in 0:
> HH <- c(0,HH)
# S-tilde and S
> SSt <- S0 * exp(HH*log(ut/dt) + c(0:100)*log(dt))
> SS <- exp(r*c(0:100))*SSt
# the maximum
> MM <- max(SS)

# have a look at the process
> plot(SS,type="l",xlab="time periods",ylab="stock price")
# cumulate final values of the stock price in SSS,  
# final values of the maximum in MMM

# first initialize
SSS <- c(1:M)*0
MMM <- c(1:M)*0

# the loop
for(i in 1:M){
  II <- rbinom(n,1,piH)
  HH <- cumsum(II)
  HH <- c(0,HH)
  SSt<- S0 * exp(HH*log(ut/dt) + c(0:100)*log(dt))
  SS <- exp(r*c(0:100))*SSt
  SSS[i]<-SS[n]
  MMM[i]<- max(SS)
}

# the lookback option with strike K=1.1:  
K <- 1.1  
# payoff
V <- pmax(MMM/SSS - K ,0)
# price
exp(-r*n)*mean(V)
[1] 0.0097791

# or you can use the apply function in S

# or you can build a tree
RADON-NIKODYM DERIVATIVES:
IMPORTANCE SAMPLING

Often convenient to simulate under a different measure

\[ E_\pi(S_n - K)^+ = E_Q(S_n - K)^+ \frac{d\pi}{dQ} \]

For example, in the previous sampling scheme, suppose we sample the heads and tail under a fair coin, so \( Q(H) = Q(T) = 1/2 \):

\[ \frac{d\pi}{dQ} = \left( \frac{\pi(H)}{Q(H)} \right)^H \left( \frac{\pi(T)}{Q(T)} \right)^T \]

# the fair coin probabilities
QH <- 1/2
QT <- 1/2
# sample under these probabilities
H <- rbinom(M,n,QH)

# S-tilde and S
> St<- S0 * exp(H*log(ut/dt) + n*log(dt))
> S <- exp(r*n)*St

# and... the Radon-Nikodym derivative
> RN <- ((piH/QH)^H)*((piT/QT)^n-H))

# the Radon-Nikodym derivative is a random variable
hist(RN)
# it has mean 1
> mean(RN)
[1] 1.011647

# and the R-N derivative is a function of S
> plot(S,RN)
USING THE RELATION

\[ E_\pi(S_n - K)^+ = E_Q(S_n - K)^+ \frac{d\pi}{dQ} \]

# strike price
> K <- 105
# the options payoff
> V<-pmax(S-K,0)
# the estimated options price
> exp(-r*n)*mean(V*RN)
[1] 4.075278
# by comparison to the true price
> 4.075278/3.989224
[1] 1.021573

This is more off, but will get better as \( M \to \infty \)
PATH DEPENDENT OPTIONS AND R-N DERIVATIVES

# first initialize
SSS <- c(1:M)*0
MMM <- c(1:M)*0
HHH <- c(1:M)*0

# the loop
for(i in 1:M){
  II <- rbinom(n,1,QH)
  HH <- cumsum(II)
  HH <- c(0,HH)
  SSt<- S0 * exp(HH*log(ut/dt) + c(0:100)*log(dt))
  SS <- exp(r*c(0:100))*SSt
  SSS[i]<-SS[n]
  MMM[i]<- max(SS)
  HHH[i] <- HH[n]
}

# the Radon-Nikodym derivative
> RN <- ((piH/QH)^HHH)*((piT/QT)^n-HHH)
# a diagnostic, should be close to 1:
> mean(RN)
[1] 0.9582251

# the lookback option with strike K=1.1:
> K <- 1.1
# payoff
> V <- pmax(MMM/SSS - K ,0)
# price
> exp(-r*n)*mean(V*RN)
[1] 0.008485798
# Ratio to simulation without the RN-derivative
> 0.008485798/0.0097791
[1] 0.8677484 # not great, but you get the idea