MARTINGALES BASED ON IID:

ADDITIVE MG

\[ Y_1, \ldots, Y_t, \ldots: \text{ IID } \quad EY = 0 \]

\[ X_t = Y_1 + \ldots + Y_t \] is MG

MULTIPLICATIVE MG

\[ Y_1, \ldots, Y_t, \ldots: \text{ IID } \quad EY = 1 \]

\[ X_t = Y_1 \times \ldots \times Y_t : \quad X_{t+1} = X_t Y_{t+1} \]

\[
E(X_{t+1} \mid \mathcal{F}_t) = E(X_t Y_{t+1} \mid \mathcal{F}_t) \\
= X_t E(Y_{t+1} \mid \mathcal{F}_t) \\
= X_t \times 1 \quad = X_t
\]

MORE REALISTIC FOR STOCKS, FOR EXAMPLE

BINOMIAL MODEL: \( X_t = \tilde{S}_t, Y = e^{-r}u \) or \( e^{-r}d \) under \( \pi \)
MARKOV PROCESSES

\[ X_t, t = 0, \ldots, T \] is a process adapted to filtration \((\mathcal{F}_t)\).\n
\( (X_t) \) is a Markov process if for every function \( f \) there is a function \( g \) so that
\[
E(f(X_{t+1}) \mid \mathcal{F}_t) = g(X_t)
\]

In other words: all current info relevant to the future of \((X_t)\) is contained in the current value of \((X_t)\).

FOR EXAMPLE:

\((S_t)\) IS MARKOV IN THE BINOMIAL TREE

\[
S_{t-1}(\omega_{t-1}) 
\begin{cases}
S_t(\omega_{t-1}, H) = uS_{t-1}(\omega_{t-1}) & \\
S_t(\omega_{t-1}, T) = dS_{t-1}(\omega_{t-1})
\end{cases}
\]

For \( S_{t-1}(\omega_{t-1}) = s_{t-1} \), and if \( Z = u \) with probability \( p(H) \) and \( = d \) with probability \( p(T) \)

\[
E(f(S_t) \mid \mathcal{F}_{t-1})(\omega_{t-1}) = E(f(ZS_{t-1}) \mid \mathcal{F}_{t-1})(\omega_{t-1})
\]

\[
= E(f(Zs_{t-1}) \mid \mathcal{F}_{t-1})(\omega_{t-1})
\]

\[
= E(f(Zs_{t-1})) \text{ (since } Z \text{ is independent of } \mathcal{F}_{t-1})
\]

\[
= g(s_{t-1})
\]

where:

\[
g(s) = E(f(zs)) = f(us)p(H) + f(ds)p(T)
\]

This works for both actual and risk neutral probability.
General principle for taking out what is known

Lemma: Let \( X = (X_1, ..., X_K) \) and \( Y = (Y_1, ..., X_L) \) be random variables, and let \( f(x, y) \) be a function. If \( X = (X_1, ..., X_p) \) is \( \mathcal{G} \)-measurable, then on the set \( \omega \in \{ X = x \} \)

\[
E(f(X, Y) \mid \mathcal{G})(\omega) = E(f(x, Y) \mid \mathcal{G})(\omega).
\]

Proof of Lemma: Let \( \mathcal{P} \) be the partition corresponding to \( \mathcal{G} \). For each \( \omega \in \{ X = x \} \), there is a \( B \in \mathcal{P} \), so that \( \omega \in B \subseteq \{ X = x \} \). (This is since \( X \) is \( \mathcal{G} \)-measurable.) Then

\[
E(f(X, Y) \mid \mathcal{G})(\omega) = E(f(X, Y) \mid B) \\
= E(f(x, Y) \mid B) \text{ since } X(\omega) = x \text{ on } B \\
= E(f(x, Y) \mid \mathcal{G})(\omega)
\]

QED

When \( Y \) is independent of \( \mathcal{G} \)

\[
E(f(x, Y) \mid \mathcal{G})(\omega) = E(f(x, Y))
\]

consequently

\[
E(f(X, Y) \mid \mathcal{G})(\omega) = E(f(x, Y))
\]
THE STRUCTURE OF EUROPEAN OPTIONS IN THE BINOMIAL TREE

Payoff \( V = v_T(S_T) \).

Induction: assume value of option at \( t+1 \) is \( v_{t+1}(S_{t+1}) \). Then

\[
\text{value of option at } t = E_{\pi}(e^{-r}v_{t+1}(S_{t+1}) \mid \mathcal{F}_t) \\
= E_{\pi}(e^{-r}v_{t+1}(ZS_t) \mid \mathcal{F}_t) \\
\text{where } Z = u \text{ or } d \\
= v_t(S_t)
\]

where

\[
v_t(s) = E_{\pi} v_{t+1}(Zs) = \pi(H)v_{t+1}(us) + \pi(T)v_{t+1}(ds)
\]
A NON-MARKOV PROCESS

\[ M_t = \max_{0 \leq u \leq t} S_u \]

Fig. 2.5.1. The maximum stock price to date.

(from Shreve p. 48)
CREATING A MARKOV PROCESS
BY ADDING STATE VARIABLES

\[ M_t = \max_{0 \leq u \leq t} S_u \]

CLAIM: \((S_t, M_t)\) is a Markov process

Proof: If \(S_t = s\) and \(M_t = m\): with \(Z = u\) or \(d\):

\[ S_{t+1} = sZ \text{ and } M_{t+1} = M_t \lor S_{t+1} = m \lor sZ \]

\((x \lor y = \max(x, y) \text{ and } x \land y = \min(x, y))\)

\[
E(f(S_{t+1}, M_{t+1}) \mid \mathcal{F}_t) = E(f(sZ, m \lor (sZ))) \\
= g(s, m)
\]

by appropriate definition of \(g\) \hspace{1cm} QED

It follows, by induction, that, for payoff \(v_T(S_T, M_T)\):

value of option at \(t = v_t(S_t, M_t)\)

General Theorem (Feynman-Kac): If \(X_0, X_1, \ldots, X_T\) is a Markov process under \(\pi\) (e.g. \(X_t = (S_t, M_t)\)). For payoff \(v_T(X_T)\),

value of option at \(t = v_t(X_t)\)

for some function \(v_t(x)\).
CHANGE OF MEASURE

$Q, R$: PROBABILITIES, $\mathcal{G}$: $\sigma$-field

$R$ IS ABSOLUTELY CONTINUOUS UNDER $Q$ (on $\mathcal{G}$):

$R \ll Q$: $\forall A \in \mathcal{G}: Q(A) = 0 \Rightarrow P(A) = 0$

$R$ IS EQUIVALENT TO $Q$ (on $\mathcal{G}$):

$R \sim Q$: $R \ll Q$ AND $Q \ll R$

ACTUAL AND RISK NEUTRAL:

NO ARBITRAGE $\Rightarrow P \sim \pi$

BECAUSE

(i) If $\pi(A) > 0$, $P(A) = 0$

$S_T = I_A = \begin{cases} 
1 & \text{if } A \\
0 & \text{if not}
\end{cases}$

so $P(S_T = 0) = 1$

But $\pi(S_T > 0) > 0 \Rightarrow \pi \left( \frac{1}{B_T} S_T > 0 \right) > 0$

$\Rightarrow S_0 = E_\pi \frac{1}{B_T} S_T > 0$ \hspace{1cm} ARBITRAGE

(ii) If $\pi(A) = 0$, $P(A) > 0$:

$P(S_T > 0) > 0$

$S_0 = E_\pi \frac{1}{B_T} S_T = 0$ \hspace{1cm} ARBITRAGE
BOND WITH DEFAULT

NUMERAIRE: \( B_t = e^{rt} \quad t = 0, 1 \)

**BOND:** \( V_t = 1 \) for \( t = 0 \)

\[
V_t = \begin{cases} 
  e^m & \text{no default} \\
  0 & \text{default}
\end{cases}
\]

for \( t = 1 \)

\[
\tilde{V}_t = \frac{V_t}{B_t} = 1 \quad \text{for} \quad t = 0
\]

\[
\tilde{V}_t = \begin{cases} 
  e^{m-r} & \text{no default} \\
  0 & \text{default}
\end{cases}
\]

for \( t = 1 \)

\[1 = \tilde{V}_0 = E_{\pi} \tilde{V}_1 = e^{m-r} \pi(\text{no default}) \quad (+0 \times \pi(\text{default}))\]

Condition: \( m > r \) or \( m \geq r? \) or ??

\[
P(\text{default}) = 0 \Rightarrow \pi(\text{default}) = 0 \quad (\pi \ll P)
\]

\[
\Rightarrow \pi(\text{no default}) = 1
\]

\[
\Rightarrow m = r
\]

\[
1 > P(\text{default}) > 0 \quad \Rightarrow 1 > \pi(\text{default}) > 0
\]

\[
\Rightarrow m > r
\]
RADON-NIKODYM DERIVATIVES

$Q, R$: PROBABILITIES, $Q \ll R$ (on $\mathcal{G}$)

$\mathcal{P}$ is partition associated with $\mathcal{G}$

DEFINITION OF R-N DERIVATIVE

$$
\frac{dR}{dQ}(\omega) = \frac{R(B)}{Q(B)} \text{ when } \omega \in B \in \mathcal{P}
$$

CAVEAT: $\frac{dR}{dQ}$ DEPENDS ON $\mathcal{G}$: $\frac{dR}{dQ} = \frac{dR}{dQ} \mid \mathcal{G}$

PROPERTIES

- $Q\left(\frac{dR}{dQ} \geq 0\right) = 1$
- If $R \sim Q$: $Q\left(\frac{dR}{dQ} > 0\right) = 1$
- $E_Q \left(\frac{dR}{dQ}\right) = 1$
- For all $\mathcal{G}$-measurable $Y$: $E_R (Y) = E_Q \left(Y \frac{dR}{dQ}\right)$
- If $R \sim Q$: $\frac{dQ}{dR} = \left(\frac{dR}{dQ}\right)^{-1}$
EXAMPLE OF PROOF

Since $Y$ is $\mathcal{G}$-measurable:

for every $B \in \mathcal{P}$: $Y(\omega)$ is constant for $\omega \in B$: $Y(\omega) = Y(B)$. Hence:

$$E_R(Y) = \sum_{B \in \mathcal{P}} Y(B)R(B)$$

$$= \sum_{B \in \mathcal{P}} Y(B)\frac{dR}{dQ}(B)Q(B)$$

$$= E_Q \left( Y \frac{dR}{dQ} \right)$$
BINOMIAL TREES

\[ S_2 = u^2 S_0 \]  
H, then H

\[ S_2 = u d S_0 \]  
H, then T

\[ S_2 = d^2 S_0 \]  
T, then H

\[ S_2 = d S_0 \]  
T, then T

ACTUAL MEASURE: \( P(H), P(T) \)

RISK NEUTRAL MEASURE: \( \pi(H), \pi(T) \)

\( \sigma \)-FIELD \( \mathcal{F}_n \) OR PARTITION \( \mathcal{P}_n \): BASED ON \( n \) FIRST COIN TOSSES: \( (\omega_1, \ldots, \omega_n) \)
CALCULATION OF R-N DERIVATIVE IN BINOMIAL CASE

- For every sequence \( \omega = (\omega_1, ..., \omega_n) \): \( \{\omega\} = \{(\omega_1, ..., \omega_n)\} \) is a set in \( \mathcal{P}_n \)

- Value of derivative for this set:
  \[
  \frac{d\pi}{dP}(\omega_1, ..., \omega_n) = \frac{\pi(\omega_1, ..., \omega_n)}{P(\omega_1, ..., \omega_n)} = \frac{\pi(\omega_1) \times ... \times \pi(\omega_n)}{P(\omega_1) \times ... \times P(\omega_n)} = \left(\frac{\pi(H)}{P(H)}\right)^H \left(\frac{\pi(T)}{P(T)}\right)^T
  \]
  
where \( H = H(\omega) = \# \) of heads among \( \omega_1, ..., \omega_n \)

AMBIGUITY ABOUT “\( \omega \)”

Could consider \( \omega = (\omega_1, ..., \omega_n, ...) \) (the outcome of infinitely many coin tosses). Then

\[
\frac{d\pi}{dP} |_{\mathcal{F}_n} (\omega) = \frac{d\pi}{dP}(\omega_1, ..., \omega_n)
\]

Each set in \( \mathcal{P}_n \): one \( (\omega_1, ..., \omega_n) \), many \( (\omega_1, ..., \omega_n, ...) \)
RELATIONSHIP BETWEEN R-N DERIVATIVE AND STOCK PRICE IN BINOMIAL CASE

\[ S_n = S_0 \times u^H \times d^T \]
\[ = S_0 d^n \times \left( \frac{u}{d} \right)^H \]

since \(H + T = n\). Thus:

\[ \#H = \frac{\log S_n - \log S_0 - n \log d}{\log u - \log d} \]

ON THE OTHER HAND:

\[ \frac{d\pi}{dP}(\omega) = \left( \frac{\pi(H)}{P(H)} \right)^H \left( \frac{\pi(T)}{P(T)} \right)^T \]
\[ = f(S_n, n) \]

\[ \frac{d\pi}{dP} \] IS A FUNCTION OF \( S_n \)
STATE PRICE DENSITY

For payoffs at time $t$:

$$\xi_t(\omega) = e^{-rt} \frac{d\pi}{d\mathcal{F}_t}(\omega)$$

Price at time 0 for payoff $I_A(\omega)$:

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

at time $t$:

$$e^{-rt} E_{\pi}(I_A) = e^{-rt} E_P \left( I_A \frac{d\pi}{d\mathcal{F}_t} \right)$$

$$= E_P (I_A \xi_t)$$

State price corresponding to $\omega$: $A = \{\omega\}$:

$$E_P (I_{\{\omega\}} \xi_t) = \xi_t(\omega) P\{\omega\}$$

General valuation formula:

price for payoff $V$ at time $t = e^{-rt} E_{\pi}(V) = E_P (V \xi_t)$
THE RADON-NIKODYM DERIVATIVE IS A MARTINGALE

\[ Z_t = \frac{d\pi}{dP} | F_t \]

IN THE BINOMIAL CASE:

\[ Z_{t+1} = Z_t Y_{t+1} \]

WHERE \( Y_1, Y_2, \ldots \) ARE IID,

\[ Y_t = \begin{cases} \frac{\pi(H)}{P(H)} & \text{if outcome H at time } t \\ \frac{\pi(T)}{P(T)} & \text{if outcome T at time } t \end{cases} \]

AND

\[ E_P(Y) = \frac{\pi(H)}{P(H)} P(H) + \frac{\pi(T)}{P(T)} P(T) = \pi(H) + \pi(T) = 1 \]

\[ Z_0 = 1 \]

A TYPICAL MULTIPLICATIVE MARTINGALE
THE GENERAL CASE

\[ Z_t = \frac{d\pi}{dP} |_{\mathcal{F}_t} \]  
Filtration on partition form: \( \mathcal{P}_t \)

IF \( A \in \mathcal{P}_t \) AND \( A = \cup_q B_q \) WHERE \( B_q \in \mathcal{P}_{t+1} \):

\[
E(Z_{t+1} \mid A) = \sum_q Z_{t+1}(B_q)P(B_q \mid A)
\]

\[
= \sum_q \frac{\pi(B_q)}{P(B_q)} P(B_q \mid A)
\]

\[
= \sum_q \frac{\pi(B_q)}{P(B_q)} \frac{P(B_q \cap A)}{P(A)}
\]

\[
= \sum_q \frac{\pi(B_q)}{P(B_q)} \frac{P(B_q)}{P(A)} \quad \text{since } B_q \subseteq A
\]

\[
= \frac{1}{P(A)} \sum_q \pi(B_q)
\]

\[
= \frac{\pi(A)}{P(A)}
\]

\[
= Z_t(A)
\]

IF \( \omega \in A \):

\[
E(Z_{t+1} \mid \mathcal{P}_t)(\omega) = E(Z_{t+1} \mid A) = Z_t(A) = Z_t(\omega)
\]

THUS: \((Z_t) \) IS A MARTINGALE
THE R-N DERIVATIVE

REPRESENTATION BY FINAL VALUE

IF $T$ IS THE FINAL TIME, AND $\frac{d\pi}{dP}$ IS R-N DERIVATIVE ON $\mathcal{F}_T$:

$$Z_t = E(Z_T | \mathcal{F}_t) = E_P \left( \frac{d\pi}{dP} | \mathcal{F}_t \right) \text{ for } t \leq T$$
THE R-N DERIVATIVE:
CONDITIONAL EXPECTATIONS

**Theorem:** Suppose $Q \ll P$ on $\mathcal{F}$. Let $dQ/dP \otimes \mathcal{F}$ and $\mathcal{G} \subseteq \mathcal{F}$. Then

$$E_Q(X \mid \mathcal{G}) = \frac{E_P \left( X \frac{dQ}{dP} \mid \mathcal{G} \right)}{E_P \left( \frac{dQ}{dP} \mid \mathcal{G} \right)}.$$

IN THE TIME DEPENDENT SYSTEM: $\mathcal{G} = \mathcal{F}_t$

$$E_\pi(X \mid \mathcal{F}_t) = \frac{E_P \left( XZ_T \mid \mathcal{F}_t \right)}{E_P(Z_T \mid \mathcal{F}_t)}$$

$$= \frac{E_P \left( XZ_T \mid \mathcal{F}_t \right)}{Z_t}$$

PRICE AT TIME $t$ OF PAYOFF $V$ AT TIME $T$:

$$\tilde{V}_t = E_\pi(\tilde{V} \mid \mathcal{F}_t) = \frac{E_P \left( \tilde{V}Z_T \mid \mathcal{F}_t \right)}{Z_t}$$

$$V_t = \frac{E_P \left( e^{-rT}VZ_T \mid \mathcal{F}_t \right)}{e^{-rt}Z_t}$$

$$= \frac{E_P \left( V\xi_T \mid \mathcal{F}_t \right)}{\xi_t}$$
ANOTHER APPLICATION OF
THE CONDITIONAL EXPECTATION FORMULA:
MODIFYING PATH DEPENDENT OPTIONS
IN THE BINOMIAL MODEL

$V = \text{payoff of option (at } T\text{)}$

$V' = E_{\pi}(V \mid S_T)$

$= \frac{E\left(V \frac{d\pi}{dP} \mid S_T\right)}{E\left(\frac{d\pi}{dP} \mid S_T\right)}$  

$(G = \sigma(S_T))$

$\frac{d\pi}{dP} E\left(V \mid S_T\right)$

$= \frac{d\pi}{dP}$

(since $\frac{d\pi}{dP}$ is a function of $S_T$)

$= E(V \mid S_T)$

WE HAVE HERE USED $G = \sigma(S_T)$
ARE PATH DEPENDENT OPTIONS OPTIMAL?

\[ V = \text{payoff} \]
\[ V' = E_\pi(V \mid S_T) = E(V \mid S_T) \]

- **Price:**

  \[
  \text{Price of } V = e^{-rT} E_\pi V \\
  = e^{-rT} E_\pi V' \quad \text{tower property} \\
  = \text{price of } V'
  \]

- **Expected payoff of } V**

  \[
  e^{-rT} EV = e^{-rT} EV' \quad \text{tower property} \\
  = \text{expected payoff of } V'
  \]

- **The Rao-Blackwell inequality:**

  \[
  \text{Var } (V) = E \left( \text{Var } (V \mid S_T) \right) + \text{Var } (E(V \mid S_T)) \\
  = E \left( \text{Var } (V \mid S_T) \right) + \text{Var } (V') \\
  > \text{Var } (V')
  \]

  unless \( V = V' \).
UTILITY: TYPICAL ASSUMPTIONS

- more is better: $V_1 \geq V_2 \Rightarrow U(V_1) \geq U(V_2)$
- risk aversion: strict concavity:

  $$U(\alpha V_1 + (1 - \alpha) V_2) > \alpha U(V_1) + (1 - \alpha) U(V_2)$$

  non-strict concavity: replace “<” by “≤”

Jensen: $U$ strictly concave: unless $V$ is $\mathcal{G}$-measurable:

$$E(U(V) \mid \mathcal{G}) < U(E(V) \mid \mathcal{G}) \quad P\text{-a.s.}$$

PATH DEPENDENT OPTIONS AGAIN

$$V' = E_\pi(V \mid S_T) = E(V \mid S_T)$$

- Price of $V =$ price of $V'$

$$EU(V) = EE(U(V) \mid S_T) \;
< EU(E(V \mid S_T)) \quad \text{unless } V \in \mathfrak{M} \sigma(S_T)$$

$$= EU(V')$$
OPTIMAL INVESTMENT IN BIN. MODEL

MAXIMIZATION OF UTILITY SUBJECT TO INITIAL CAPITAL:

$$\max EU(V_T)$$

subject to constraints:

capital constraint: $$V_0 = v$$

replicability: $$\tilde{V}_T = V_0 + \sum_{t=0}^{T-1} \Delta_t \Delta \tilde{S}_t$$

BY COMPLETENESS: CONSTRAINTS EQUIVALENT TO (with $$\xi_T = e^{-rT}(d\pi/dP)$$)

$$E_\pi \tilde{V}_T = v$$

or

$$EV_T \xi_T = v$$

BY ARGUMENT ON PREVIOUS PAGE: $$V_T$$ IF FUNCTION OF $$S_T$$, OR, EQUIVALENTLY, $$\xi_T$$:

$$V_T = f(\xi_T)$$
REFORMULATION OF PROBLEM

$\xi_T$ can take values $x_1, \ldots, x_{T+1}$ Problem becomes:

$$\max_f \sum U(f(x_i))P(\xi = x_i)$$

subject to:

$$\sum f(x_i)x_iP(\xi = x_i) = v$$

Lagrangian:

$$L = \sum U(f(x_i))P(\xi = x_i) - \lambda \left( \sum f(x_i)x_iP(\xi = x_i) - v \right)$$

Want:

$$0 = \frac{\partial L}{\partial f(x_i)} = U'(f(x_i))P(\xi = x_i) - \lambda x_iP(\xi = x_i)$$

In other words:

$$U'(f(x_i)) = \lambda x_i$$

or:

$$U'(V_T) = \lambda \xi_T$$

Finally:

$$V_T = (U')^{-1}(\lambda \xi_T)$$