MARTINGALES BASED ON IID:

ADDITIVE MG

$$Y_1, \ldots, Y_t, \ldots : \quad \text{IID} \quad EY = 0$$
$$X_t = Y_1 + \ldots + Y_t \quad \text{is MG}$$

MULTIPLICATIVE MG

$$Y_1, \ldots, Y_t, \ldots : \quad \text{IID} \quad EY = 1$$
$$X_t = Y_1 \times \ldots \times Y_t : \quad X_{t+1} = X_t Y_{t+1}$$

$$\begin{aligned} E(X_{t+1} \mid \mathcal{F}_t) &= E(X_t Y_{t+1} \mid \mathcal{F}_t) \\ &= X_t E(Y_{t+1} \mid \mathcal{F}_t) \\ &= X_t \times 1 \qquad = X_t \end{aligned}$$

MORE REALISTIC FOR STOCKS, FOR EXAMPLE

BINOMIAL MODEL: $X_t = \widetilde{S}_t$, $Y = e^{-r}u$ or $e^{-r}d$ under π

MARKOV PROCESSES

X_t, $t = 0, ..., T$ is process adapted to filtration $(\mathcal{F}_t)$

(X_t) is a Markov process if for every function f there is a function g so that

$$E(f(X_{t+1}) \mid \mathcal{F}_t) = g(X_t)$$

In other words: all current info relevant to the future of (X_t) is contained in the current value of (X_t).

FOR EXAMPLE: (S_t) IS MARKOV IN THE BINOMIAL TREE

$$S_{t-1}(\omega_{t-1}) \quad \begin{cases} S_t(\omega_{t-1}, H) = uS_{t-1}(\omega_{t-1}) \\ S_t(\omega_{t-1}, T) = dS_{t-1}(\omega_{t-1}) \end{cases}$$

For $S_{t-1}(\omega_{t-1}) = s_{t-1}$, and if $Z = u$ with probability $p(H)$ and $= d$ with probability $p(T)$

$$\begin{aligned} E(f(S_t) \mid \mathcal{F}_{t-1})(\omega_{t-1}) &= E(f(ZS_{t-1}) \mid \mathcal{F}_{t-1})(\omega_{t-1}) \\ &= E(f(Zs_{t-1}) \mid \mathcal{F}_{t-1})(\omega_{t-1}) \\ &= E(f(Zs_{t-1})) \text{ (since } Z \text{ is independent of } \mathcal{F}_{t-1}) \\ &= g(s_{t-1}) \end{aligned}$$

where:

$$g(s) = E(f(Zs)) = f(us)p(H) + f(ds)p(T)$$

This works for both actual and risk neutral probability

General principle for taking out what is known

Lemma: Let $X = (X_1, ..., X_K)$ and $Y = (Y_1, ..., X_L)$ be random variables, and let $f(x, y)$ be a function. If $X = (X_1, ..., X_p)$ is $\mathcal{G}$-measurable, then on the set $\omega \in \{X = x\}$

$$E(f(X,Y) \mid \mathcal{G})(\omega) = E(f(x,Y) \mid \mathcal{G})(\omega).$$

Proof of Lemma: Let $\mathcal{P}$ be the partition corresponding to $\mathcal{G}$. For each $\omega \in \{X = x\}$, there is a $B \in \mathcal{P}$, so that $\omega \in B \subseteq \{X = x\}$. (This is since X is $\mathcal{G}$-measurable.) Then

$$\begin{aligned} E(f(X,Y) \mid \mathcal{G})(\omega) &= E(f(X,Y) \mid B) \\ &= E(f(x,Y) \mid B) \text{ since } X(\omega) = x \text{ on } B \\ &= E(f(x,Y) \mid \mathcal{G})(\omega) \end{aligned}$$

QED

When Y is independent of $\mathcal{G}$

$$E(f(x,Y) \mid \mathcal{G})(\omega) = E(f(x,Y))$$

consequently

$$E(f(X,Y) \mid \mathcal{G})(\omega) = E(f(x,Y))$$

THE STRUCTURE OF EUROPEAN OPTIONS IN THE BINOMIAL TREE

Payoff $V = v_T(S_T)$.

Induction: assume value of option at t+1 is $v_{t+1}(S_{t+1})$. Then

$$\begin{aligned}\text{value of option at } t &= E_\pi(e^{-r} v_{t+1}(S_{t+1}) \mid \mathcal{F}_t) \\ &= E_\pi(e^{-r} v_{t+1}(ZS_t) \mid \mathcal{F}_t) \\ &\qquad \text{where } Z = u \text{ or } = d \\ &= v_t(S_t)\end{aligned}$$

where

$$v_t(s) = E_\pi v_{t+1}(Zs) = \pi(H) v_{t+1}(us) + \pi(T) v_{t+1}(ds)$$

A NON-MARKOV PROCESS

$$M_t = \max_{0 \le u \le t} S_u$$

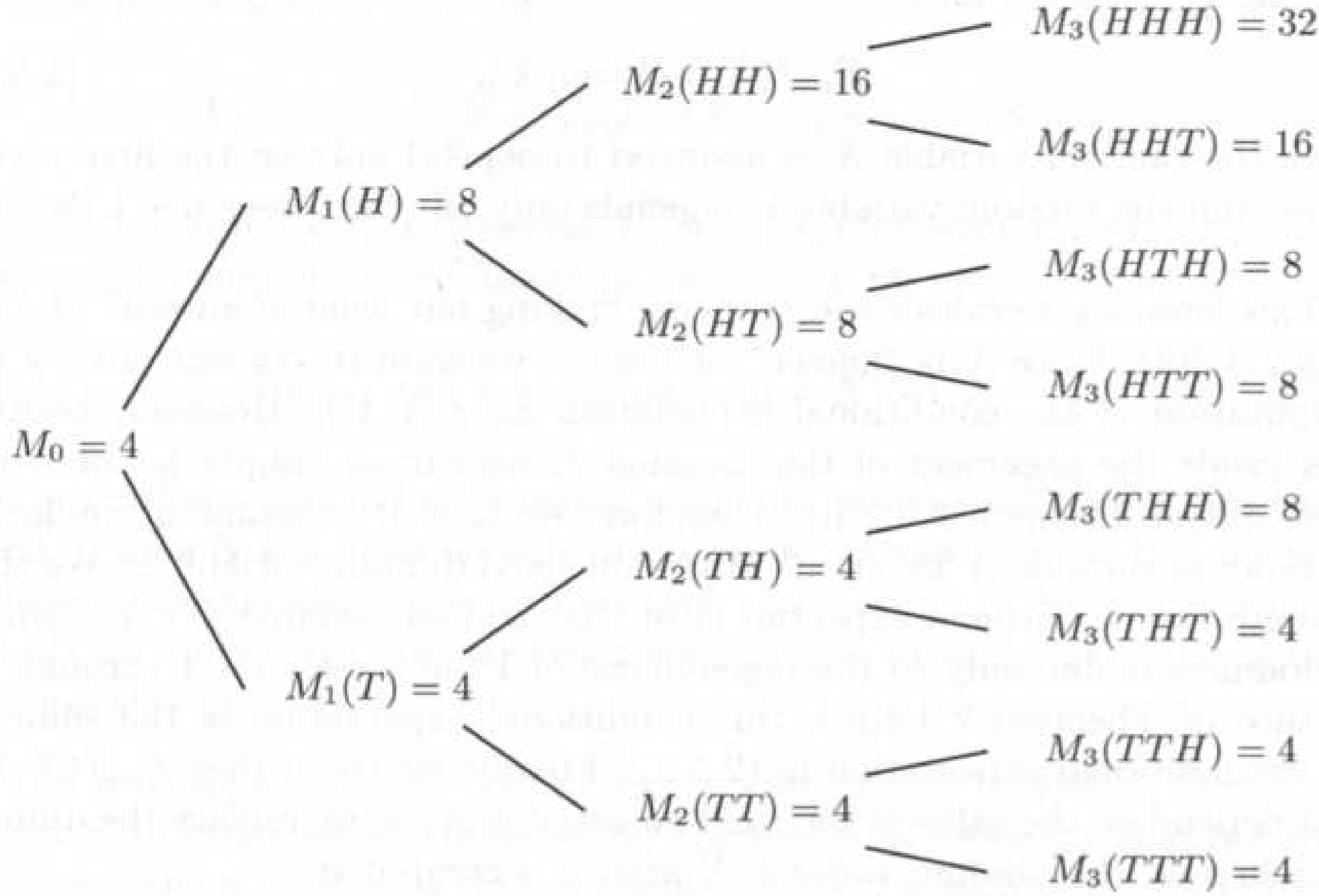


Fig. 2.5.1. The maximum stock price to date.

(from Shreve p. 48)

CREATING A MARKOV PROCESS BY ADDING STATE VARIABLES

$$M_t = \max_{0 \le u \le t} S_u$$

CLAIM: (S_t, M_t) is a Markov process

Proof: If $S_t = s$ and $M_t = m$: with $Z = u$ or d:

$S_{t+1} = sZ$ and $M_{t+1} = M_t \vee S_{t+1} = m \vee sZ$
$(x \vee y = \max(x, y)$ and $x \wedge y = \min(x, y))$

$$\begin{aligned} E(f(S_{t+1}, M_{t+1}) \mid \mathcal{F}_t) &= E(f(sZ, m \vee (sZ))) \\ &= g(s, m) \end{aligned}$$

by appropriate definition of g QED

It follows, by induction, that, for payoff $v_T(S_T, M_T)$:

$$\text{value of option at } t = v_t(S_t, M_t)$$

General Theorem (Feynman-Kac): If $X_0, X_1, ..., X_T$ is a Markov process under π (e.g. $X_t = (S_t, M_t)$). For payoff $v_T(X_T)$,

$$\text{value of option at } t = v_t(X_t)$$

for some function $v_t(x)$.

CHANGE OF MEASURE

Q, R: PROBABILITIES, $\mathcal{G}$: σ-field

R IS ABSOLUTELY CONTINUOUS UNDER Q (on $\mathcal{G}$):

$$R \ll Q: \forall A \in \mathcal{G}: \; Q(A) = 0 \Rightarrow P(A) = 0$$

R IS EQUIVALENT TO Q (on $\mathcal{G}$):

$$R \sim Q: \; R \ll Q \text{ AND } Q \ll R$$

ACTUAL AND RISK NEUTRAL:
NO ARBITRAGE $\Rightarrow P \sim \pi$

BECAUSE

(i) If $\pi(A) > 0$, $P(A) = 0$

$$S_T = I_A = \begin{cases} 1 \text{ if } A \\ 0 \text{ if not} \end{cases} \quad soP(S_T = 0) = 1$$

But $\pi(S_T > 0) > 0 \Rightarrow \pi\left(\frac{1}{B_T} S_T > 0\right) > 0$

$$\Rightarrow S_0 = E_\pi \frac{1}{B_T} S_T > 0 \quad \text{ARBITRAGE}$$

(ii) If $\pi(A) = 0$, $P(A) > 0$:

$$P(S_T > 0) > 0$$

$$S_0 = E_\pi \frac{1}{B_T} S_T = 0 \quad \text{ARBITRAGE}$$

BOND WIH DEFAULT

NUMERAIRE: $B_t = e^{rt} \quad t = 0, 1$

$$\text{BOND: } V_t = 1 \text{ for } t = 0$$
$$= \left\{ \begin{array}{cc} e^m & \text{no default} \\ 0 & \text{default} \end{array} \right\} \text{ for } t = 1$$
$$\widetilde{V}_t = \frac{V_t}{B_t} = 1 \text{ for } t = 0$$
$$= \left\{ \begin{array}{cc} e^{m-r} & \text{no default} \\ 0 & \text{default} \end{array} \right\} \text{ for } t = 1$$

$$1 = \widetilde{V}_0 = E_\pi \widetilde{V}_1 = e^{m-r}\pi(\text{ no default}) \quad (+0\times\pi(\text{ default}))$$

Condition: $m > r$ or $m \geq r$? or ??

$$\begin{aligned} P(\text{default}) = 0 &\Rightarrow \pi(\text{default}) = 0 \quad (\pi \ll P) \\ &\Rightarrow \pi(\text{no default}) = 1 \\ &\Rightarrow m = r \\ 1 > P(\text{default}) > 0 \quad &\Rightarrow 1 > \pi(\text{default}) > 0 \\ &\Rightarrow m > r \end{aligned}$$

RADON-NIKODYM DERIVATIVES

Q, R: PROBABILITIES, $Q \ll R$ (on $\mathcal{G}$)

$\mathcal{P}$ is partition associated with $\mathcal{G}$

DEFINITION OF R-N DERIVATIVE

$$\frac{dR}{dQ}(\omega) = \frac{R(B)}{Q(B)} \text{ when } \omega \in B \in \mathcal{P}$$

CAVEAT: $\frac{dR}{dQ}$ DEPENDS ON $\mathcal{G}$: $\frac{dR}{dQ} = \frac{dR}{dQ} |_{\mathcal{G}}$

PROPERTIES

- $Q(\frac{dR}{dQ} \geq 0) = 1$
- If $R \sim Q$: $Q(\frac{dR}{dQ} > 0) = 1$
- $E_Q\left(\frac{dR}{dQ}\right) = 1$
- For all $\mathcal{G}$-measurable Y: $E_R(Y) \; = \; E_Q\left(Y \frac{dR}{dQ}\right)$
- If $R \sim Q$: $\frac{dQ}{dR} = \left(\frac{dR}{dQ}\right)^{-1}$

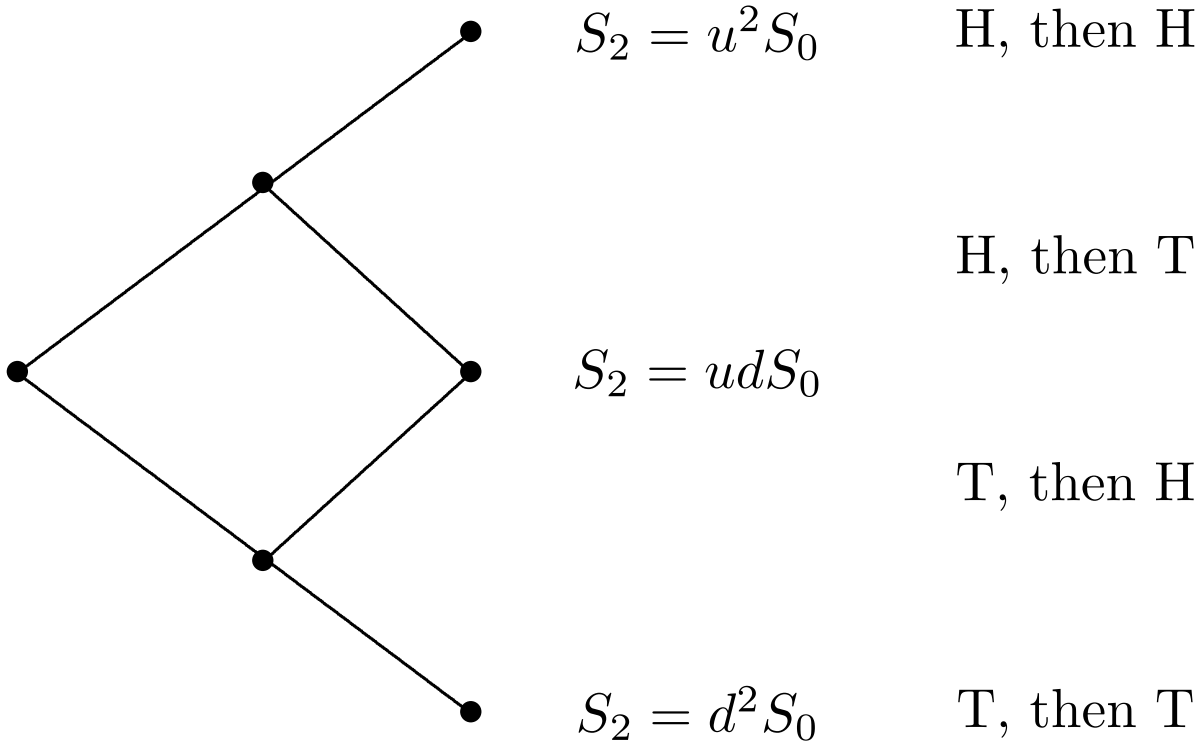
EXAMPLE OF PROOF

Since Y is $\mathcal{G}$- measurable:

for every $B \in \mathcal{P}$: $Y(\omega)$ is constant for $\omega \in B$: $Y(\omega) = Y(B)$. Hence:

$$\begin{aligned} E_R(Y) &= \sum_{B \in \mathcal{P}} Y(B) R(B) \\ &= \sum_{B \in \mathcal{P}} Y(B) \frac{dR}{dQ}(B) Q(B) \\ &= E_Q\left(Y \frac{dR}{dQ}\right) \end{aligned}$$

BINOMIAL TREES



ACTUAL MEASURE: $P(H)$, $P(T)$

RISK NEUTRAL MEASURE: $\pi(H)$, $\pi(T)$

σ-FIELD $\mathcal{F}_n$ OR PARTITION $\mathcal{P}_n$: BASED ON n FIRST COIN TOSSES: $(\omega_1, ...\omega_n)$

CALCULATION OF R-N DERIVATIVE IN BINOMIAL CASE

• For every sequence $\omega = (\omega_1, ..., \omega_n)$: $\{\omega\} = \{(\omega_1, ..., \omega_n)\}$ is a set in $\mathcal{P}_n$

• Value of derivative for this set:

$$\begin{aligned}\frac{d\pi}{dP}(\omega_1, ...\omega_n) &= \frac{\pi(\omega_1, ...\omega_n)}{P(\omega_1, ..., \omega_n)} \\ &= \frac{\pi(\omega_1) \times ... \times \pi(\omega_n)}{P(\omega_1) \times ... \times P(\omega_n)} \\ &= \left(\frac{\pi(H)}{P(H)}\right)^{\#H} \left(\frac{\pi(T)}{P(T)}\right)^{\#T}\end{aligned}$$

where $H = H(\omega) = \#$ of heads among $\omega_1, ..., \omega_n$

AMBIGUITY ABOUT "ω"

Could consider $\omega = (\omega_1, ..., \omega_n, ...)$ (the outcome of infinitely many coin tosses). Then

$$\frac{d\pi}{dP} |_{\mathcal{F}_n} (\omega) = \frac{d\pi}{dP}(\omega_1, ..., \omega_n)$$

Each set in $\mathcal{P}_n$: one $(\omega_1, ..., \omega_n)$, many $(\omega_1, ..., \omega_n, ...)$

RELATIONSHIP BETWEEN R-N DERIVATIVE AND STOCK PRICE IN BINOMIAL CASE

$$S_n = S_0 \times u^{\#H} \times d^{\#T}$$
$$= S_0 d^n \times \left(\frac{u}{d}\right)^{\#H}$$

since $\#H + \#T = n$. Thus:

$$\#H = \frac{\log S_n - \log S_0 - n \log d}{\log u - \log d}$$

ON THE OTHER HAND:

$$\frac{d\pi}{dP}(\omega) = \left(\frac{\pi(H)}{P(H)}\right)^{\#H} \left(\frac{\pi(T)}{P(T)}\right)^{\#T}$$
$$= f(S_n, n)$$

$\frac{d\pi}{dP}$ IS A FUNCTION OF S_n

STATE PRICE DENSITY

For payoffs at time t:

$$\xi_t(\omega) = e^{-rt} \tfrac{d\pi}{dP} \mid_{\mathcal{F}_t} (\omega)$$

Price at time 0 for payoff

$$I_A(\omega) = \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ otherwise} \end{cases}$$

at time t:

$$\begin{aligned} e^{-rt} E_\pi(I_A) &= e^{-rt} E_P \left(I_A \frac{d\pi}{dP} \right) \\ &= E_P \left(I_A \xi_t \right) \end{aligned}$$

State price corresponding to ω: $A = \{\omega\}$:

$$E_P \left(I_{\{\omega\}} \xi_t \right) = \xi_t(\omega) P\{\omega\}$$

General valuation formula:

price for payoff V at time $t \;=\; e^{-rt} E_\pi (V) \;=\; E_P (V \xi_t)$

THE RADON-NIKODYM DERIVATIVE IS A MARTINGALE

DEFINE $Z_t = \frac{d\pi}{dP} \mid_{\mathcal{F}_t}$

IN THE BINOMIAL CASE:

$$Z_{t+1} = Z_t Y_{t+1}$$

WHERE $Y_1, Y_2, ...$ ARE IID,

$$Y_t = \begin{cases} \frac{\pi(H)}{P(H)} \text{ if outcome H at time } t \\ \frac{\pi(T)}{P(T)} \text{ if outcome T at time } t \end{cases}$$

AND

$$E_P(Y) = \frac{\pi(H)}{P(H)} P(H) + \frac{\pi(T)}{P(T)} P(T) = \pi(H) + \pi(T) = 1$$

$$Z_0 = 1$$

A TYPICAL MULTIPLICATIVE MARTINGALE

THE GENERAL CASE

$$Z_t = \frac{d\pi}{dP} |_{\mathcal{F}_t} \text{ Filtration on partition form: } \mathcal{P}_t$$

IF $A \in \mathcal{P}_t$ AND $A = \cup_q B_q$ WHERE $B_q \in \mathcal{P}_{t+1}$:

$$\begin{aligned}
E(Z_{t+1} \mid A) &= \sum_q Z_{t+1}(B_q) P(B_q \mid A) \\
&= \sum_q \frac{\pi(B_q)}{P(B_q)} P(B_q \mid A) \\
&= \sum_q \frac{\pi(B_q)}{P(B_q)} \frac{P(B_q \cap A)}{P(A)} \\
&= \sum_q \frac{\pi(B_q)}{P(B_q)} \frac{P(B_q)}{P(A)} \text{ since } B_q \subseteq A \\
&= \frac{1}{P(A)} \sum_q \pi(B_q) \\
&= \frac{\pi(A)}{P(A)} \\
&= Z_t(A)
\end{aligned}$$

IF $\omega \in A$:

$$E(Z_{t+1} \mid \mathcal{P}_t)(\omega) = E(Z_{t+1} \mid A) = Z_t(A) = Z_t(\omega)$$

THUS: (Z_t) IS A MARTINGALE

THE R-N DERIVATIVE

REPRESENTATION BY FINAL VALUE

IF T IS THE FINAL TIME, AND $\frac{d\pi}{dP}$ IS R-N DERIVATIVE ON $\mathcal{F}_T$:

$$Z_t = E\left(Z_T \mid \mathcal{F}_t\right) = E_P\left(\frac{d\pi}{dP} \mid \mathcal{F}_t\right) \text{ for } t \leq T$$

THE R-N DERIVATIVE: CONDITIONAL EXPECTATIONS

Theorem: *Suppose* $Q \ll P$ *on* $\mathcal{F}$. *Let* dQ/dP ⓜ $\mathcal{F}$ *and* $\mathcal{G} \subseteq \mathcal{F}$. *Then*

$$E_Q(X \mid \mathcal{G}) = \frac{E_P\left(X\frac{dQ}{dP} \,\middle|\, \mathcal{G}\right)}{E_P\left(\frac{dQ}{dP} \,\middle|\, \mathcal{G}\right)}.$$

IN THE TIME DEPENDENT SYSTEM: $\mathcal{G} = \mathcal{F}_t$

$$E_\pi(X \mid \mathcal{F}_t) = \frac{E_P\left(XZ_T \,\middle|\, \mathcal{F}_t\right)}{E_P\left(Z_T | \mathcal{F}_t\right)} = \frac{E_P\left(XZ_T \,\middle|\, \mathcal{F}_t\right)}{Z_t}$$

PRICE AT TIME t OF PAYOFF V AT TIME T:

$$\widetilde{V}_t = E_\pi(\widetilde{V} \mid \mathcal{F}_t) = \frac{E_P\left(\widetilde{V}Z_T \,\middle|\, \mathcal{F}_t\right)}{Z_t}$$

$$V_t = \frac{E_P\left(e^{-rT}VZ_T \,\middle|\, \mathcal{F}_t\right)}{e^{-rt}Z_t} = \frac{E_P\left(V\xi_T \,\middle|\, \mathcal{F}_t\right)}{\xi_t}$$

ANOTHER APPLICATION OF THE CONDITIONAL EXPECTATION FORMULA:

MODIFYING PATH DEPENDENT OPTIONS IN THE BINOMIAL MODEL

$$V = \text{payoff of option (at } T)$$

$$
\begin{aligned}
V' &= E_\pi(V \mid S_T) \\
&= \frac{E\left(V \frac{d\pi}{dP} \,\middle|\, S_T\right)}{E\left(\frac{d\pi}{dP} \,\middle|\, S_T\right)} \qquad (\mathcal{G} = \sigma(S_T)) \\
&= \frac{\frac{d\pi}{dP} E\left(V \,\middle|\, S_T\right)}{\frac{d\pi}{dP}} \\
&\quad \left(\text{since } \frac{d\pi}{dP} \text{ is a function of } S_T\right) \\
&= E(V \mid S_T)
\end{aligned}
$$

WE HAVE HERE USED $\mathcal{G} = \sigma(S_T)$

ARE PATH DEPENDENT OPTIONS OPTIMAL?

$$V = \text{payoff}$$
$$V' = E_\pi(V \mid S_T) = E(V \mid S_T)$$

- Price:

$$\begin{aligned}\text{Price of } V &= e^{-rT} E_\pi V \\ &= e^{-rT} E_\pi V' \qquad \text{tower property} \\ &= \text{price of } V'\end{aligned}$$

- Expected payoff of V

$$\begin{aligned}e^{-rT} EV &= e^{-rT} EV' \qquad \text{tower property} \\ &= \text{expected payoff of } V'\end{aligned}$$

- The Rao-Blackwell inequality:

$$\begin{aligned}\text{Var}\,(V) &= E\left(\text{Var}\,(V \mid S_T)\right) + \text{Var}\,(E(V \mid S_T)) \\ &= E\left(\text{Var}\,(V \mid S_T)\right) + \text{Var}\,(V') \\ &> \text{Var}\,(V')\end{aligned}$$

unless $V = V'$.

UTILITY: TYPICAL ASSUMPTIONS

- more is better: $V_1 \geq V_2 \Rightarrow U(V_1) \geq U(V_2)$
- risk aversion: strict concavity:

$$U(\alpha V_1 + (1-\alpha)V_2) > \alpha U(V_1) + (1-\alpha)U(V_2)$$

non-strict concavity: replace “$<$” by “$\leq$”

Jensen: U strictly concave: unless V is $\mathcal{G}$-measurable:

$$E(U(V) \mid \mathcal{G}) < U(E(V \mid \mathcal{G})) \qquad P\text{-a.s.}$$

PATH DEPENDENT OPTIONS AGAIN

$$V' = E_\pi(V \mid S_T) = E(V \mid S_T)$$

- Price of V = price of V'

$$\begin{aligned} EU(V) &= EE(U(V) \mid S_T) \\ &< EU(E(V \mid S_T)) \qquad \text{unless } V \textcircled{m} \sigma(S_T) \\ &= EU(V') \end{aligned}$$

OPTIMAL INVESTMENT IN BIN. MODEL

MAXIMIZATION OF UTILITY SUBJECT TO INITIAL CAPITAL:

$$\max EU(V_T)$$

subject to constraints:

$$\text{capital constraint: } V_0 = v$$

$$\text{replicability: } \widetilde{V}_T = V_0 + \sum_{t=0}^{T-1} \Delta_t \Delta \widetilde{S}_t$$

BY COMPLETENESS: CONSTRAINTS EQUIVALENT TO (with $\xi_T = e^{-rT}(d\pi/dP)$)

$$E_\pi \widetilde{V}_T = v$$

or

$$EV_T \xi_T = v$$

BY ARGUMENT ON PREVIOUS PAGE:
V_T IF FUNCTION OF S_T, OR, EQUIVALENTLY, ξ_T:

$$V_T = f(\xi_T)$$

REFORMULATION OF PROBLEM

ξ_T can take values x_1, ..., x_{T+1} Problem becomes:

$$\max_f \sum U(f(x_i))P(\xi = x_i)$$

subject to:

$$\sum f(x_i)x_i P(\xi = x_i) = v$$

Lagrangian:

$$L = \sum U(f(x_i))P(\xi = x_i) - \lambda \left(\sum f(x_i)x_i P(\xi = x_i) - v \right)$$

Want:

$$0 = \frac{\partial L}{\partial f(x_i)} = U'(f(x_i))P(\xi = x_i) - \lambda x_i P(\xi = x_i)$$

In other words:

$$U'(f(x_i)) = \lambda x_i$$

or:

$$U'(V_T) = \lambda \xi_T$$

Finally:

$$V_T = (U')^{(-1)}(\lambda \xi_T)$$