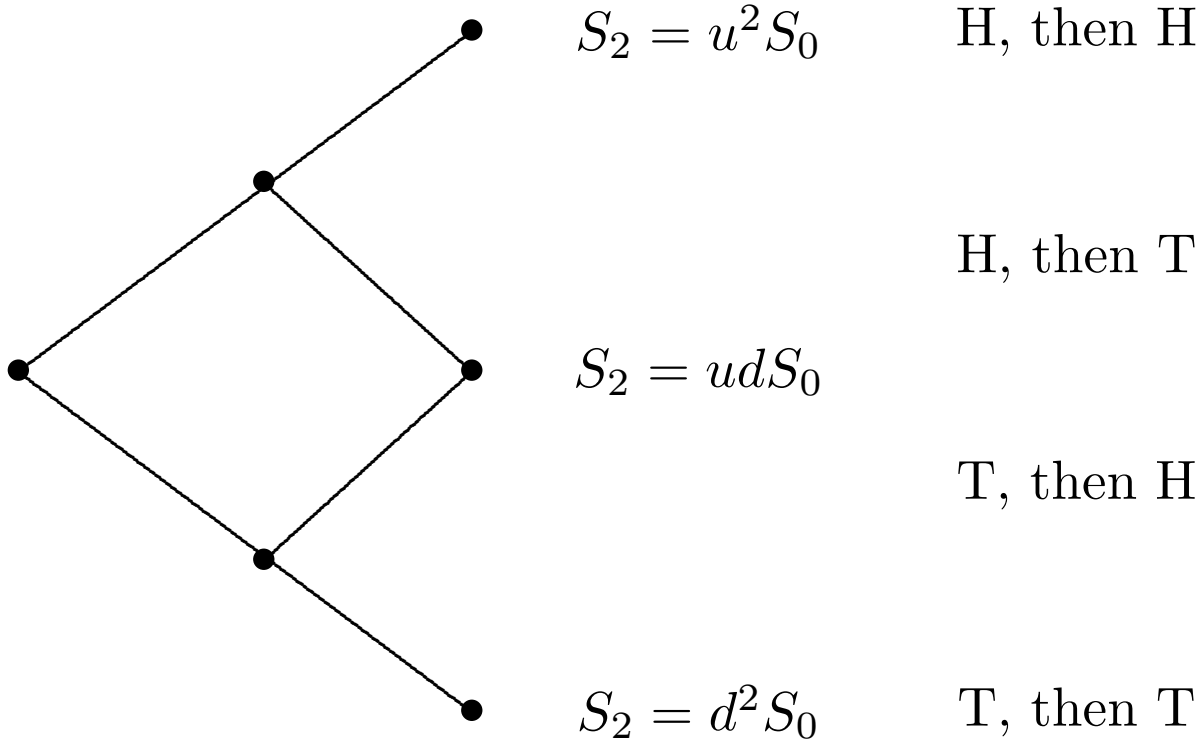
MULTIPERIOD MODELS: $t =$ TIME $= 0, 1, ..., T$

$K + 1$ ASSETS, PRICE S_t^j, $j = 0, \ldots, K$

$S^0 = B$ IS BOND: $S_0^0 = 1$, $S_t^0 = e^{rt}$

Randomness: $S_t^j = S_t^j(\omega)$, where ω is basic outcome

EXAMPLE: BINOMIAL TREE, $T = 2$, $p = 1$



ω = one of HH, HT, TH, and TT

4 basic outcomes, but 3 prices at $t = 2$: $S_2(HT) = S_2(TH)$

FOR BINOMIAL TREE, general T, $p = 1$:

2^T basic outcomes, but $T + 1$ prices at time $t = T$

PORTFOLIOS

$\Delta_0(t), \ldots, \Delta_K(t)$:

$$\Delta_j(t) = \#\text{ SHARES HELD IN } S^j$$
$$\text{FROM } t \text{ to } t+1$$

REALLOCATE AT EACH t

VALUE AT t:

$(B)_t \quad V_t = \Delta_0(t-1)S_t^0 + \cdots + \Delta_K(t-1)S_t^K$

$(A)_t \quad V_t = \Delta_0(t)S_t^0 + \cdots + \Delta_K(t)S_t^K$

BEFORE = AFTER : SELF FINANCING TRADING STRATEGY

PROFIT/LOSS : $\Delta S_t^j = S_{t+1}^j - S_t^j$

$\Delta V_t = V_{t+1} - V_t$

ONE STEP:

$$\Delta V_t = (B)_{t+1} - (A)_t = \sum_{j=0}^{K} \Delta_j(t)\Delta S_t^j$$

TOTAL P/L:

$$V_t - V_0 = \sum_{u=0}^{t-1}\sum_{j=0}^{K} \Delta_j(u)\Delta S_u^j = \sum_{j=0}^{K}\sum_{u=0}^{t-1} \Delta_j(u)\Delta S_u^j$$

NUMERAIRE INVARIANCE

$S^0, \ldots, S^K, V$: MEASURED IN \$\$, EUROs, ETC

$\widetilde{S}_t^j = \frac{S_t^j}{S_t^0}$ $\widetilde{V}_t = \frac{V_t}{S_t^0}$:

MEASURED IN NEW ACCOUNTING UNIT: # S^0

$S^0 > 0$: MONEY MARKET BOND: $S_t^0 = e^{rt}$
ZERO COUPON BOND
FOREIGN CURRENCY BOND
$\approx$ GOLD
...

NUMERAIRE INVARIANCE PRINCIPLE:

Δ SFS FOR V IN $S^0, \ldots, S^K$ IFF
Δ SFS FOR $\widetilde{V}$ IN $\widetilde{S}^1, \ldots, \widetilde{S}^K$ ($\widetilde{S}_t^0 = 1$ for all t)

PROOF:

$$(B_t): \quad V_t = \sum_{j=0}^{K} \Delta_j(t-1) S_t^j \quad (A)_t : V_t = \sum_{j=0}^{K} \Delta_j(t) S_t^j$$

$$(\widetilde{B}_t): \quad \widetilde{V}_t = \sum_{j=0}^{K} \Delta_j(t-1) \widetilde{S}_t^j \quad (\widetilde{A})_t : \widetilde{V}_t = \sum_{j} \Delta_j(t) \widetilde{S}_t^j$$

MAIN CONSEQUENCE OF NUMERAIRE INVARIANCE:

EXIT INTEREST

EXIT DIFFERENCE EQUATION $(B)_t = (A)_t$

TOTAL P/L, THIS TIME ON DISCOUNTED SCALE:

$$\widetilde{V}_t - \widetilde{V}_0 = \sum_{j=0}^{K} \sum_{u=0}^{t-1} \Delta_j(u) \Delta \widetilde{S}_u^j = \sum_{j=1}^{K} \sum_{u=0}^{t-1} \Delta_j(u) \Delta \widetilde{S}_u^j$$

since $\Delta \widetilde{S}_u^0 = \Delta \widetilde{B}_u = 0$

CONSEQUENCE:
SUPPOSE YOU HAVE REPRESENTATION

$$\widetilde{V}_t - \widetilde{V}_0 = \sum_{j=1}^{K} \sum_{u=0}^{t-1} \Delta_j(u) \Delta \widetilde{S}_u^j \qquad (*)$$

CHOOSE $\Delta_0(t)$ to satisfy $(\widetilde{A})_t : \widetilde{V}_t = \sum_j \Delta_j(t) \widetilde{S}_t^j$

A SELF FINANCING STRATEGY IS DEFINED BY (*)

A T PERIOD MARKET IS COMPLETE IF ALL POSSIBLE PAYOFFS V HAVE REPRESENTATION (*) FOR $t = T$

NUMERAIRE INVARIANCE PRESERVES

— ARBITRAGE/NO ARBITRAGE:

$$V_0 \leq 0 \Leftrightarrow \widetilde{V}_0 = \frac{V_0}{S_0^0} \leq 0$$

$$V_T \geq 0, > 0 \Leftrightarrow \widetilde{V}_T = \frac{V_T}{S_T^0} \geq 0, > 0$$

— COMPLETENESS (REPLICATION):

CAN REPLICATE ALL V IN S^j, $j = 0, ..., K$

IFF CAN REPLICATE ALL $\frac{V}{S_T^0}$ IN $\widetilde{S}_j$, $j = 1, ..., K$

EQUIVALENT STATEMENT (UNIQUE π) REQUIRES EXTENSION OF RISK NEUTRAL MEASURE

CASE OF THE BINOMIAL TREE

$$S_{t-1}(\omega_{t-1}) \begin{cases} \widetilde{S}_t(\omega_{t-1}, H) = \widetilde{u}\widetilde{S}_{t-1}(\omega_{t-1}) \\ \widetilde{S}_t(\omega_{t-1}, T) = \widetilde{d}\widetilde{S}_{t-1}(\omega_{t-1}) \end{cases}$$

where $\widetilde{u} = e^{-r}u$ and $\widetilde{d} = e^{-r}d$

ω_{t-1} = outcome of $t-1$ first coin tosses

FOR PAYOFF $V = V(\omega_T)$:
DEFINE RECURSIVELY $V_T(\omega_T) = V(\omega_T)$ AND

$$\widetilde{V}_{t-1}(\omega_{t-1}) = \pi(H)\widetilde{V}_t(\omega_{t-1}, H) + \pi(T)\widetilde{V}_t(\omega_{t-1}, T)$$

$$\text{where } \pi(T) = \frac{u - e^r}{u - d} \text{ and } \pi(H) = \frac{e^r - d}{u - d}$$

For each ω_{t-1}:

$$\Delta\widetilde{V}_{t-1} = \Delta(t-1)\Delta\widetilde{S}_{t-1} \qquad (**)$$

is a system of two equations, one unknown (scenario H and T): solve for $\Delta(t-1) = \Delta(t-1)(\omega_{t-1})$

Since (**) is true for each t, so is (*). Market is complete. Price of option: $\widetilde{V}_0$. Hedge ratio is solution to (**). (Cf Shreve, Thm 1.2.2.)

ONE PERIOD MODEL: RISK NEUTRAL MEASURE

$$\widetilde{S}_0^j = E_\pi \widetilde{S}_1^j \qquad j = 1, \ldots, K \qquad (*)$$

SAME (*) OK FOR OTHER NUMERAIRE, BUT: π DEPENDS ON S^1.

Q: HOW DOES (*) GENERALIZE TO T PERIODS?

A: MARTINGALES

NEED PROBABILITY THEORY FIRST

At this point:
• I assume you know Chapter 2.1-2.2 of Shreve
• If you don't, please read it
• We proceed to conditional expectations. Shreve does this for binomial model; here: a slightly more general version. Try to match the two approaches.

MODELING INFORMATION

Stochastic variable X:

- yesterday's closing price of S&P 500
- today's LIBOR
- #eyes on a die

Joint modeling of random variables:
Universe: the probability space $\Omega = \{\omega_0, ..., \omega_K\}$
all possible outcomes

Stochastic variables: functions

$$X : \ \Omega \to \mathbb{R} \quad \omega \to X(\omega)$$

If outcome ω, then value of X is $X(\omega)$, of Y is $Y(\omega)$, etc

System with time dependence:

$$S_t \ = \text{price of stock at time } t,\ t = 0, 1, ..., T$$

Two ways of thinking about it:

- S_t $(= \ S_t(\omega))$ is collection of random variables, or
- $S : \ \Omega \times \{0, 1, ..., T\} \to \mathbb{R} \quad (\omega, t) \to S_t(\omega)$

A market: $S_t^{(i)}$, $i = 0, ..., K$: stocks, bonds, exchange rates, other securities

Q: What information is available at time t?
A: if X is r.v., then X is observed at t if and only if there is a function f:

$$X = f(S_t^{(0)}, ..., S_t^{(K)})$$

Non time setup:
If $Y^{(1)}, ..., Y^{(p)}$ observable, then $X = f(Y^{(1)}, ..., Y^{(p)})$ observable

X, Y have the same information iff $X = f(Y)$, and f is 1-to-1

An awkward way of characterizing information

A better way: **partitions** of Ω

Definition: A set $\mathcal{P} = \{A_1, ..., A_p\}$ of subsets of Ω is a **partition** of Ω iff

- $i \neq j \Rightarrow A_i \bigcap A_j = \phi$ (disjoint sets)
- $\bigcup_i A_i = \Omega$

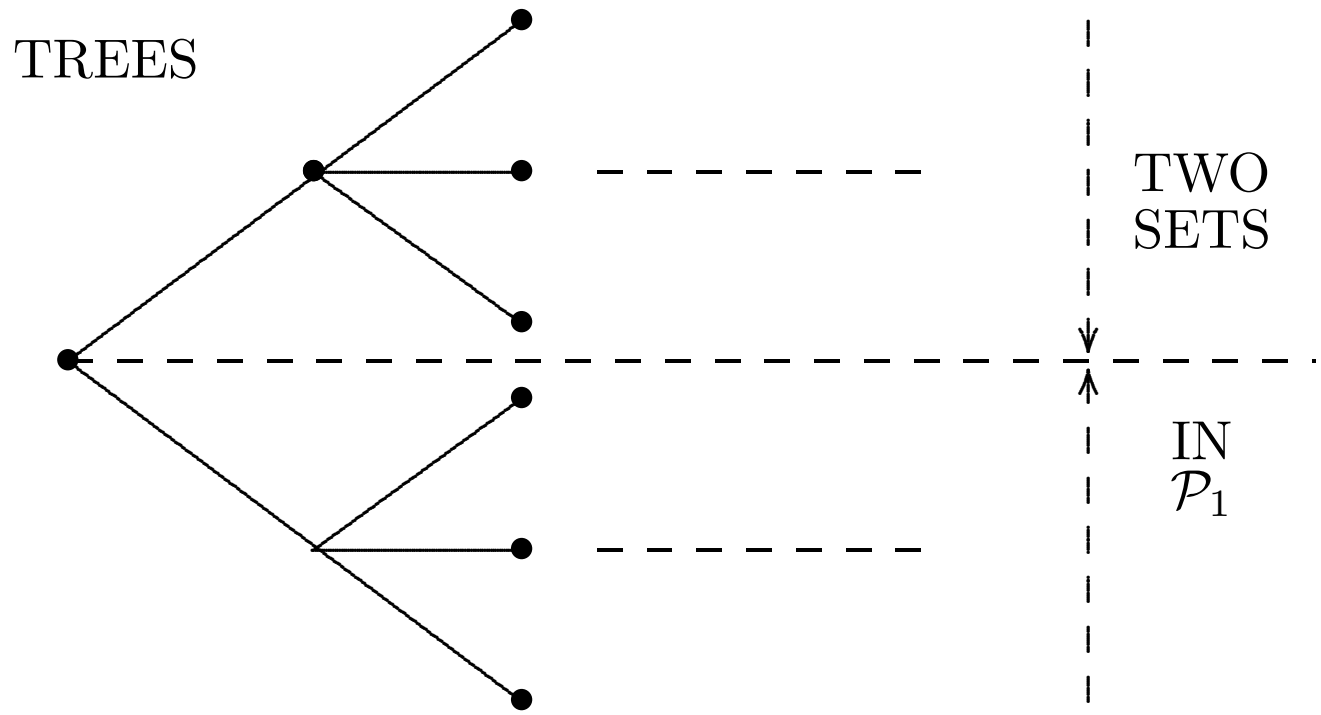
$\mathcal{P}$ is the information available to you
if and only if
$A \in \mathcal{P} \Leftrightarrow$ you can decide whether A happened or not

Example:
X: random variable
$Im(X) = \{x \in R : \exists \omega \in \Omega : x = X(\omega)\}$
For each $x \in Im(X)$: $A_x = \{\omega : X(\omega) = x\}$

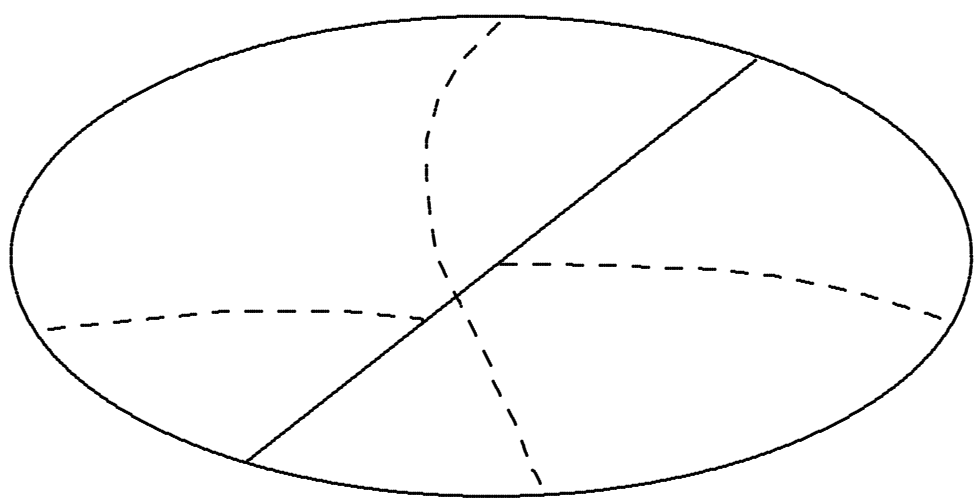
Properties:

- $x \neq y \Rightarrow A_x \bigcap A_y = \phi$ (X does not take two values at the same time)
- $\bigcup_x A_x = \Omega$ (X does take some value for all ω)

Hence: $\mathcal{P}^X = \{A_x : x \in Im(X)\}$ is a partition



FINER PARTITIONS



ORDERING OF PARTITIONS

Definition: $\mathcal{P}$ **has at least as much information as** $\mathcal{Q}$ ($\mathcal{P} \geq \mathcal{Q}$) if

$$\forall A \in \mathcal{Q}\ \exists B_1, ..., B_q \in \mathcal{P} : \ A = \bigcup_i B_i$$

$\mathcal{P}$ has more information than $\mathcal{Q}$: more sets are decidable in $\mathcal{P}$.

Theorem. If $Y = f(X)$ (f not necessarily 1-to-1):

$$\mathcal{P}^X \geq \mathcal{P}^Y$$

Corollary. If $Y = f(X)$ and f is 1-to-1:

$$\mathcal{P}^X = \mathcal{P}^Y$$

Because **Theorem:**
$\mathcal{P} \geq \mathcal{Q}$ and $\mathcal{P} \leq \mathcal{Q}$ => $\mathcal{P} = \mathcal{Q}$

EXAMPLE:
Throw 1 US\$, 1 CNY
X = total # of heads

Ω space of outcomes

	US\$	CNY
ω_1	T	T
ω_2	T	H
ω_3	H	T
ω_4	T	T

$$\begin{aligned}\mathcal{P}^X &= \{\{X=0\},\{X=1\},\{X=2\}\} \\ &= \{\{\omega_1\},\{\omega_2,\omega_3\},\{\omega_4\}\}\end{aligned}$$

The information in X does not include whether \$ was tail, CNY was head, or vice versa

The extreme cases:

- All is known: $\mathcal{P} = \{\{\omega_1\},...,\{\omega_K\}\}$
- Nothing is known: $\mathcal{P} = \{\Omega\}$

$\mathcal{P}$ is the info available to us
R.v. X is observable iff $\mathcal{P} \geq \mathcal{P}^X$
Terminology: **X is $\mathcal{P}$-measurable**, $X \textcircled{m} \mathcal{P}$

Time dependent system: $\mathcal{P}_t$ = all info in the market up to t

What is known at $t-1$ is not forgotten at t:
$\mathcal{P}_t \geq \mathcal{P}_{t-1}$

$\mathcal{P}_t,\ t = 0, 1, ..., T$ satisfying this: **filter, filtration**
$X_t \textcircled{m} \mathcal{P}_t$ for all t: (X_t) is (P_t)**-adapted**, $(X_t) \textcircled{m} (P_t)$

Insider trading:
$\mathcal{P}_t$ = market info at time t
$\mathcal{I}_t$ = your info at time t
$\mathcal{I}_t > P_t$ strictly, if and only if you have inside information

CONDITIONAL PROBABILITIES

$$P(A|B) \;=\; \frac{P(A \bigcap B)}{P(B)}$$

Useful abstraction:

$$P(A|\mathcal{P})(\omega) \;=\; P(A|B) \;\text{ when } \omega \in B \in \mathcal{P}$$

- $P(A|\mathcal{P})$ is the conditional probability of A given the info in $\mathcal{P}$ – no matter what actually occurred
- $P(A|\mathcal{P})$ is $\mathcal{P}$-measurable random variable

Same idea:

$$E(X|\mathcal{P})(\omega) \;=\; E(X|B) \;\text{ when } \omega \in B \in \mathcal{P}$$

Relationship:

$$P(A|\mathcal{P}) = E(I_A|\mathcal{P}) \quad \text{where}$$

$$I_A(\omega) = 1 \text{ if } \omega \in A, \quad 0 \text{ else}$$

Example of conditional expectation:

i	1	2	3	4	5
$X(\omega_i)$	1.2	3.5	-7.8	0	2.4
$p(\omega_i)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = 1.2\frac{1}{4} + 3.5\frac{1}{4} - 7.8\frac{1}{6} + 2.4\frac{1}{6} = 0.275$$

$\mathcal{P} = \{A, B, C\}$:

$$A = \{\omega_1, \omega_2\} \qquad B = \{\omega_3, \omega_5\} \qquad C = \{\omega_4\}$$

$$\begin{aligned} E(X \mid A) &= 1.2 \times P(\omega_1 \mid A) + 3.5 \times P(\omega_1 \mid A) \\ &= 1.2\frac{1}{2} + 3.5\frac{1}{2} = 2.35 \end{aligned}$$

$$\begin{aligned} E(X \mid B) &= -7.8 \times P(\omega_3 \mid B) + 2.4 \times P(\omega_5 \mid B) \\ &= -7.8 \times \frac{1}{2} + 2.4 \times \frac{1}{2} = -2.7 \end{aligned}$$

$$E(X \mid C) = 0$$

i	1	2	3	4	5
$E(X \mid \mathcal{P})(\omega)$	2.35	2.35	-2.7	0	-2.7

$$\begin{aligned} E\left(E(X \mid P)\right) &= 2.35 \times P(A) - 2.7 \times P(B) \\ &= 2.35 \times \frac{1}{2} - 2.7 \times \frac{1}{3} \\ &= 0.275 = E(X) \end{aligned}$$

PROPERTIES OF CONDITIONAL EXPECTATIONS

• Linearity: for constant c_1, c_2:

$$E(c_1X_1 + c_2X_2 \mid \mathcal{P}) = c_1E(X_1 \mid \mathcal{P}) + c_2E(X_2 \mid \mathcal{P})$$

• Conditional constants: if $Z \textcircled{m} \mathcal{P}$, then

$$E(ZX|\mathcal{P}) = ZE(X|\mathcal{P})$$

• Law of iterated expectations (iterated conditioning, tower property): if $\mathcal{Q} \leq \mathcal{P}$, then

$$E[E(X|\mathcal{P})|\mathcal{Q}] \;=\; E(X|\mathcal{Q})$$

• Independence: if X is independent of $\mathcal{P}$:

$$E(X|\mathcal{P}) = E(X)$$

• Jensen's inequality: if $\phi: \; x \to \phi(x)$ is convex:

$$E(\phi(X)|\mathcal{P}) \geq \phi(E(X|\mathcal{P}))$$

Note: ϕ is convex if $\phi(ax+(1-a)y) \leq a\phi(x)+(1-a)\phi(y)$ for $0 \leq a \leq 1$. For example: $\phi(x) = e^x$, $\phi(x) = (x-K)^+$. Or ϕ'' exists and is continuous, and $\phi''(x) \geq 0$.

PARTITIONS AND σ-FIELDS

$\mathcal{F}$: σ-field $\mathcal{P}$: partition

$\mathcal{F}$ and $\mathcal{P}$ contain some information if

$$\mathcal{F} = \{\cup A : A \in \mathcal{P}\}$$

Order relation: $\mathcal{Q} \leq \mathcal{P}$ if and only if $\mathcal{F}^{\mathcal{Q}} \subseteq \mathcal{F}^{\mathcal{P}}$

because: $\mathcal{Q} \leq \mathcal{P}$ => $Q \subseteq \mathcal{F}^{\mathcal{P}}$

σ-fields more useful in continuous settings.

Definition:

$\Omega, \phi \in \mathcal{F}$

$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

$A_1, \ldots, A_n, \ldots \in \mathcal{F} \Rightarrow \bigcup_n A_n \in \mathcal{F}$

(countable union only)

Filtration: $(\mathcal{F}_t)$:

$\mathcal{F}_t$ is a σ-field for each t

$s \leq t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t$

For now:

$$E(X \mid \mathcal{F}) = E(X \mid \mathcal{P})$$

CONDITIONAL EXPECTATION GIVEN σ-FIELD

IF X GENERATES $\mathcal{G}$:

$$E(Y \mid \mathcal{G}) = E(Y \mid X) = f(X)$$

IF $(S_t^0, \ldots, S_t^K)$ GENERATES $(\mathcal{F}_t)$:

$S_u^0, \ldots, S_u^K, \quad u \leq t$ GENERATES $\mathcal{F}_t$:

$$E(Y \mid \mathcal{F}_t) = E(Y \mid S_u^j, u \leq t, 0 \leq j \leq K)$$

CONDITIONAL EXPECTATION

TWO NOTIONS OF CONDITIONING

$$\underbrace{f(x) = E(Y \mid X = x)}_{\text{A NUMBER}} \qquad \underbrace{E(Y \mid X) = f(X)}_{\text{RANDOM VARIABLE}}$$

SIMILAR: OPTIONS PRICE

$$C(s,t) \qquad C(S_t,t)$$

SIMILAR: RADON-NIKODYM DERIVATIVE X HAS DENSITY f_P UNDER P, f_Q UNDER Q

$$\underbrace{\frac{f_Q(x)}{f_P(x)}}_{\text{NON RANDOM}} \qquad \underbrace{\frac{dQ}{dP} = \frac{f_Q(X)}{f_P(X)}}_{\text{RANDOM}}$$

MARTINGALES (FAIR GAMES)

$\widetilde{S}_t$ IS MARTINGALE $\not/$. $(\mathcal{F}_t)$, π

IF $E_\pi(\widetilde{S}_{t+1} \mid \mathcal{F}_t) = \widetilde{S}_t \quad \forall t$ (discrete time definition)

$\Leftrightarrow$ $E_\pi(\widetilde{S}_u \mid \mathcal{F}_t) = \widetilde{S}_t \quad \forall u \geq t$ (also OK in continuous time)

MOTIVATION

π IS RISK NEUTRAL IN ALL TIME PERIODS (in $B_t = e^{rt}$ numeraire) IF

for each j, t, ω_{t-1}: $E_\pi(e^{-r} S_t^j \mid \mathcal{F}_{t-1})(\omega_{t-1}) = S_t^j(\omega_{t-1})$

SAME AS IF (and this is true in general numeraire)

for each j, t, ω_{t-1}: $E_\pi(\widetilde{S}_t^j \mid \mathcal{F}_{t-1})(\omega_{t-1}) = \widetilde{S}_t^j(\omega_{t-1})$

SAME AS IF

$\widetilde{S}^1, \ldots, \widetilde{S}^K$ ARE $(\mathcal{F}_t)$, $\pi -$ MARTINGALES

ADAPT THE LATTER
AS DEFINITION OF RISK NEUTRAL MEASURE

CAUTION: π depends on choice of numeraire

MARTINGALES (MG)

(1) $E(X_{t+1} \mid \mathcal{F}_t) = X_t \quad \forall^t$ | discrete time

$\Leftrightarrow$

(2) $E(X_u \mid \mathcal{F}_t) = X_t \qquad \forall^{u>t}$

BECAUSE: TOWER PROPERTY: (1) $\Rightarrow$ (2)

$$E(X_{t+k} \mid \mathcal{F}_t) = E\Big(\underbrace{E(X_{t+k} \mid \mathcal{F}_{t+k-1})}_{X_{t+k-1}} \mid \mathcal{F}_t\Big)$$
$$= E(X_{t+k-1} \mid \mathcal{F}_t) \quad \text{INDUCTION}$$

REPRESENTATION BY FINAL VALUE:

* Any $X_t = E(X \mid \mathcal{F}_t)$ is MG:

$$E[X_u \mid \mathcal{F}_t] = E\Big[E(X \mid \mathcal{F}_u) \mid \mathcal{F}_t\Big]$$
$$= E[X \mid \mathcal{F}_t] = X_t$$

* $X_t = E(X_T \mid \mathcal{F}_t)$

ALL DISCOUNTED PORTFOLIOS ARE MGs

$$\widetilde{P/L} : \widetilde{V}_t = \widetilde{V}_0 + \sum_{u=0}^{t-1} \Delta(u)\Delta\widetilde{S}_u \qquad (***)$$

$$\Delta\widetilde{V}_t = \widetilde{V}_{t+1} - \widetilde{V}_t = \Delta(t)\Delta\widetilde{S}_t$$

$$E_\pi(\Delta\widetilde{V}_t \mid \mathcal{F}_t) = \underbrace{\Delta(t)}_{\substack{\text{KNOWN} \\ \text{AT } t}} \underbrace{E_\pi(\widetilde{\Delta S}_t \mid \mathcal{F}_t)}_{\text{MG}} = 0$$

THE IMPORTANT VALUATION FORMULA:

$$\widetilde{V}_0 = E_\pi \widetilde{V}_T$$

COMPLETENESS: ALL RANDOM VARIABLES $\widetilde{V}$ CAN BE REPRESENTED

$$\widetilde{V} = \widetilde{V}_0 + \sum_{u=0}^{T-1} \Delta(u)\Delta\widetilde{S}_u$$

(cf p. 4; this assures that a self financing strategy exists)

BY REPRESENTATION OF MARTINGALES BY FINAL VALUE: COMPLETENESS =

ALL π-MG's ARE DISCOUNTED PORTFOLIOS (***)

("MARTINGALE REPRESENTATION THEOREM")

THIS IS THE SAME AS

π is UNIQUE

IN THIS CASE, FOR ALL RANDOM PAYOFFS:

$$\widetilde{V}_0 = E_\pi \widetilde{V}$$

RISK NEUTRAL MEASURES π:

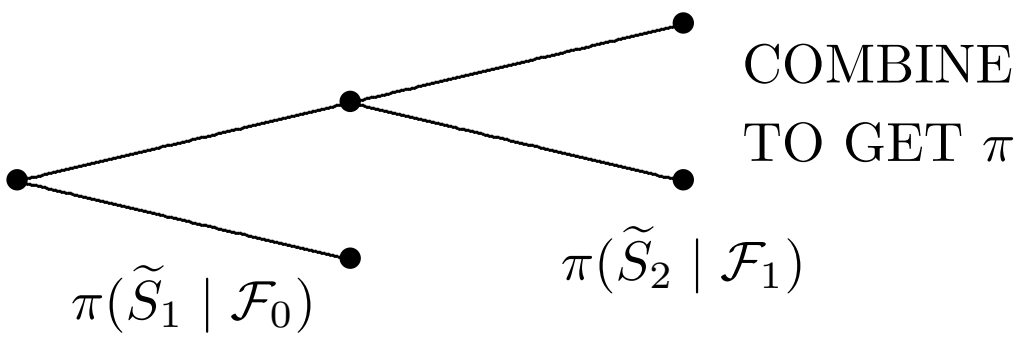
DEF: ALL $\widetilde{S}_t^j$ ARE π-MG, $j = 1, \ldots, K$

FOR EACH PERIOD $t \to t+1$:

$$E_\pi(\widetilde{S}_{t+1}^j \mid \mathcal{F}_t) = \widetilde{S}_t^j \qquad j = 1, \ldots, K$$

* $\pi(\widetilde{S}_{t+1} \mid \mathcal{F}_t)$ exists $\Leftrightarrow$ NO ARBITRAGE $t \Rightarrow t+1$

* OVERALL π:



MULTIPERIOD π EXISTS $\Leftrightarrow$ NO ARBITRAGE

REPLACE "exists" BY "is unique" IN CONSTRUCTION:

MULTIPERIOD π UNIQUE $\Leftrightarrow$ COMPLETENESS

BINOMIAL TREE CASE: π IS AS ON p. 6