MULTIPERIOD MODELS: $t =$ TIME $= 0, 1, \ldots, T$

$K + 1$ ASSETS, PRICE $S^j_t$, $j = 0, \ldots, K$

$S^0 = B$ IS BOND: $S^0_0 = 1$, $S^0_t = e^{rt}$

Randomness: $S^j_t = S^j_t(\omega)$, where $\omega$ is basic outcome

EXAMPLE: BINOMIAL TREE, $T = 2$, $p = 1$

$\omega$ = one of $HH$, $HT$, $TH$, and $TT$

4 basic outcomes, but 3 prices at $t = 2$: $S^2_2(HT) = S^2_2(TH)$

FOR BINOMIAL TREE, general $T$, $p = 1$:

$2^T$ basic outcomes, but $T + 1$ prices at time $t = T$
PORTFOLIOS

$\Delta_0(t), \ldots, \Delta_K(t)$:

$\Delta_j(t) = \# \text{ SHARES HELD IN } S^j$

FROM $t$ to $t + 1$

REALLOCATE AT EACH $t$

VALUE AT $t$:

$(B)_t \quad V_t = \Delta_0(t-1)S_t^0 + \cdots + \Delta_K(t-1)S_t^K$

$(A)_t \quad V_t = \Delta_0(t)S_t^0 + \cdots + \Delta_K(t)S_t^K$

BEFORE = AFTER: SELF FINANCING TRADING STRATEGY

PROFIT/LOSS:

$\Delta S^j_t = S^j_{t+1} - S^j_t$

$\Delta V_t = V_{t+1} - V_t$

ONE STEP:

$\Delta V_t = (B)_{t+1} - (A)_t = \sum_{j=0}^{K} \Delta_j(t) \Delta S^j_t$

TOTAL $P/L$:

$V_t - V_0 = \sum_{u=0}^{t-1} \sum_{j=0}^{K} \Delta_j(u) \Delta S^j_u = \sum_{j=0}^{K} \sum_{u=0}^{t-1} \Delta_j(u) \Delta S^j_u$
NUMERAIRE INVARIANCE

$S^0, \ldots, S^K, V$: MEASURED IN $\$, EUROs, ETC

$\tilde{S}_t^j = \frac{S_t^j}{S_t^0}$ \quad $\tilde{V}_t = \frac{V_t}{S_t^0}$:

MEASURED IN NEW ACCOUNTING UNIT: $\# S^0$

$S^0 > 0$: MONEY MARKET BOND: $S_t^0 = e^{rt}$

ZERO COUPON BOND

FOREIGN CURRENCY BOND

$\approx$ GOLD

\ldots

NUMERAIRE INVARIANCE PRINCIPLE:

$\Delta \ SFS$ FOR $V$ IN $S^0, \ldots, S^K$ IFF

$\Delta \ SFS$ FOR $\tilde{V}$ IN $\tilde{S}^1, \ldots, \tilde{S}^K$ ($\tilde{S}_t^0 = 1$ for all $t$)

PROOF:

$(B_t) : \quad V_t = \sum_{j=0}^{K} \Delta_j(t-1) S_t^j$ \quad $(A_t) : \quad V_t = \sum_{j=0}^{K} \Delta_j(t) S_t^j$

$(\tilde{B}_t) : \quad \tilde{V}_t = \sum_{j=0}^{K} \Delta_j(t-1) \tilde{S}_t^j$ \quad $(\tilde{A}_t) : \quad \tilde{V}_t = \sum_j \Delta_j(t) \tilde{S}_t^j$
MAIN CONSEQUENCE OF NUMERAIRE INVARIANCE:

EXIT INTEREST

EXIT DIFFERENCE EQUATION \((B)_t = (A)_t\)

TOTAL \(P/L\), THIS TIME ON DISCOUNTED SCALE:

\[
\tilde{V}_t - \tilde{V}_0 = \sum_{j=0}^{K} \sum_{u=0}^{t-1} \Delta_j(u) \Delta \tilde{S}^j_u = \sum_{j=1}^{K} \sum_{u=0}^{t-1} \Delta_j(u) \Delta \tilde{S}^j_u
\]

since \(\Delta \tilde{S}^0_u = \Delta \tilde{B}_u = 0\)

CONSEQUENCE:
SUPPOSE YOU HAVE REPRESENTATION

\[
\tilde{V}_t - \tilde{V}_0 = \sum_{j=1}^{K} \sum_{u=0}^{t-1} \Delta_j(u) \Delta \tilde{S}^j_u \quad (*)
\]

CHOOSE \(\Delta_0(t)\) to satisfy \((\tilde{A})_t\) : \(\tilde{V}_t = \sum_j \Delta_j(t) \tilde{S}^j_t\)

A SELF FINANCING STRATEGY IS DEFINED BY (*)

A \(T\) PERIOD MARKET IS COMPLETE IF ALL POSSIBLE PAYOFFS \(V\) HAVE REPRESENTATION (*) FOR \(t = T\)
NUMERAIRE INVARIANCE PRESERVES

— ARBITRAGE/NO ARBITRAGE:

\[ V_0 \leq 0 \Leftrightarrow \tilde{V}_0 = \frac{V_0}{S_0} \leq 0 \]

\[ V_T \geq 0, > 0 \Leftrightarrow \tilde{V}_T = \frac{V_T}{S_T} \geq 0, > 0 \]

— COMPLETENESS (REPLICATION):

CAN REPLICATE ALL \( V \) IN \( S^j, j = 0, \ldots, K \)

IFF CAN REPLICATE ALL \( \frac{V}{S_T} \) IN \( \tilde{S}_j, j = 1, \ldots, K \)

EQUIVALENT STATEMENT (UNIQUE \( \pi \)) REQUIRES EXTENSION OF RISK NEUTRAL MEASURE
CASE OF THE BINOMIAL TREE

\[ S_{t-1}(\omega_{t-1}) \leq \tilde{S}_t(\omega_{t-1}, H) = \tilde{u}\tilde{S}_{t-1}(\omega_{t-1}) \]
\[ \tilde{S}_t(\omega_{t-1}, T) = \tilde{d}\tilde{S}_{t-1}(\omega_{t-1}) \]

where \( \tilde{u} = e^{-r}u \) and \( \tilde{d} = e^{-r}d \)
\[ \omega_{t-1} = \text{outcome of } t-1 \text{ first coin tosses} \]

FOR PAYOFF \( V = V(\omega_T) \):
DEFINE RECURSIVELY \( V_T(\omega_T) = V(\omega_T) \) AND

\[ \tilde{V}_{t-1}(\omega_{t-1}) = \pi(H)\tilde{V}_t(\omega_{t-1}, H) + \pi(T)\tilde{V}_t(\omega_{t-1}, T) \]

where \( \pi(T) = \frac{u - e^r}{u - d} \) and \( \pi(H) = \frac{e^r - d}{u - d} \)

For each \( \omega_{t-1} \):

\[ \Delta\tilde{V}_{t-1} = \Delta(t-1)\Delta\tilde{S}_{t-1} \quad (***) \]

is a system of two equations, one unknown (scenario \( H \) and \( T \)): solve for \( \Delta(t-1) = \Delta(t-1)(\omega_{t-1}) \)

Since (***) is true for each \( t \), so is (**). Market is complete.
Price of option: \( \tilde{V}_0 \). Hedge ratio is solution to (**). (Cf Shreve, Thm 1.2.2.)
ONE PERIOD MODEL: RISK NEUTRAL MEASURE

\[ \tilde{S}_0^j = E_\pi \tilde{S}_1^j \quad j = 1, \ldots, K \quad (\ast) \]

SAME (\ast) OK FOR OTHER NUMERAIRE, BUT: \( \pi \) DEPENDS ON \( S^1 \).

Q: HOW DOES (\ast) GENERALIZE TO T PERIODS?

A: MARTINGALES

NEED PROBABILITY THEORY FIRST

At this point:
- I assume you know Chapter 2.1-2.2 of Shreve
- If you don’t, please read it
- We proceed to conditional expectations. Shreve does this for binomial model; here: a slightly more general version. Try to match the two approaches.
MODELING INFORMATION

Stochastic variable $X$:
- yesterday’s closing price of S&P 500
- today’s LIBOR
- #eyes on a die

Joint modeling of random variables:
Universe: the probability space $\Omega = \{\omega_0, \ldots, \omega_K\}$
all possible outcomes

Stochastic variables: functions

$$X : \Omega \to \mathbb{R} \quad \omega \to X(\omega)$$

If outcome $\omega$, then value of $X$ is $X(\omega)$, of $Y$ is $Y(\omega)$, etc

System with time dependence:

$$S_t = \text{price of stock at time } t, \ t = 0, 1, \ldots, T$$

Two ways of thinking about it:
- $S_t (= S_t(\omega))$ is collection of random variables, or
- $S : \Omega \times \{0, 1, \ldots, T\} \to \mathbb{R} \quad (\omega, t) \to S_t(\omega)$
A market: $S_t^{(i)}$, $i = 0, \ldots, K$: stocks, bonds, exchange rates, other securities

Q: What information is available at time $t$?
A: if $X$ is r.v., then $X$ is observed at $t$ if and only if there is a function $f$:

$$X = f(S_t^{(0)}, \ldots, S_t^{(K)})$$

Non time setup:
If $Y^{(1)}, \ldots, Y^{(p)}$ observable, then $X = f(Y^{(1)}, \ldots, Y^{(p)})$ observable

$X, Y$ have the same information iff $X = f(Y)$, and $f$ is 1-to-1

An awkward way of characterizing information
A better way: partitions of $\Omega$

Definition: A set $\mathcal{P} = \{A_1, \ldots, A_p\}$ of subsets of $\Omega$ is a partition of $\Omega$ iff

- $i \neq j \Rightarrow A_i \cap A_j = \emptyset$ (disjoint sets)
- $\bigcup_i A_i = \Omega$

$\mathcal{P}$ is the information available to you if and only if $A \in \mathcal{P} \iff$ you can decide whether $A$ happened or not

Example:

$X$: random variable

$\text{Im}(X) = \{x \in \mathbb{R} : \exists \omega \in \Omega : x = X(\omega)\}$

For each $x \in \text{Im}(X)$: $A_x = \{\omega : X(\omega) = x\}$

Properties:

- $x \neq y \Rightarrow A_x \cap A_y = \emptyset$ ($X$ does not take two values at the same time)
- $\bigcup_x A_x = \Omega$ ($X$ does take some value for all $\omega$)

Hence: $\mathcal{P}^X = \{A_x : x \in \text{Im}(X)\}$ is a partition
ORDERING OF PARTITIONS

Definition: \( \mathcal{P} \) has at least as much information as \( \mathcal{Q} \) (\( \mathcal{P} \geq \mathcal{Q} \)) if

\[ \forall A \in \mathcal{Q} \, \exists B_1, \ldots, B_q \in \mathcal{P} : A = \bigcup_i B_i \]

\( \mathcal{P} \) has more information than \( \mathcal{Q} \): more sets are decidable in \( \mathcal{P} \).

**Theorem.** If \( Y = f(X) \) (\( f \) not necessarily 1-to-1):

\[ \mathcal{P}^X \geq \mathcal{P}^Y \]

**Corollary.** If \( Y = f(X) \) and \( f \) is 1-to-1:

\[ \mathcal{P}^X = \mathcal{P}^Y \]

Because **Theorem:**

\( \mathcal{P} \geq \mathcal{Q} \) and \( \mathcal{P} \leq \mathcal{Q} \implies \mathcal{P} = \mathcal{Q} \)
**Example:**
Throw 1 US$, 1 CNY

$X =$ total # of heads

<table>
<thead>
<tr>
<th>US$</th>
<th>CNY</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>T</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>T</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>H</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>T</td>
</tr>
</tbody>
</table>

$\mathcal{P}^X = \{\{X = 0\}, \{X = 1\}, \{X = 2\}\}$

$= \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}\}$

The information in $X$ does not include whether $\$ was tail, CNY was head, or vice versa

The extreme cases:
- All is known: $\mathcal{P} = \{\{\omega_1\}, ..., \{\omega_K\}\}$
- Nothing is known: $\mathcal{P} = \{\Omega\}$
\( \mathcal{P} \) is the info available to us
R.v. \( X \) is observable iff \( \mathcal{P} \geq \mathcal{P}^X \\
Terminology: \( X \) is \( \mathcal{P} \)-measurable, \( X \in \mathcal{P} \)

Time dependent system: \( \mathcal{P}_t = \) all info in the market up to \( t \)

What is known at \( t - 1 \) is not forgotten at \( t \):
\( \mathcal{P}_t \geq \mathcal{P}_{t-1} \)

\( \mathcal{P}_t, \ t = 0, 1, \ldots, T \) satisfying this: \textit{filter, filtration} \\
\( X_t \in \mathcal{P}_t \) for all \( t \): \( (X_t) \) is \( (P_t) \)-adapted, \( (X_t) \in \mathcal{P} \)

Insider trading:
\( \mathcal{P}_t = \) market info at time \( t \) \\
\( \mathcal{I}_t = \) your info at time \( t \) \\
\( \mathcal{I}_t > P_t \) strictly, if and only if you have inside information
CONDITIONAL PROBABILITIES

\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \]

Useful abstraction:

\[ P(A|\mathcal{P})(\omega) = P(A|B) \quad \text{when } \omega \in B \in \mathcal{P} \]

- \( P(A|\mathcal{P}) \) is the conditional probability of \( A \) given the info in \( \mathcal{P} \) – no matter what actually occurred
- \( P(A|\mathcal{P}) \) is \( \mathcal{P} \)-measurable random variable

Same idea:

\[ E(X|\mathcal{P})(\omega) = E(X|B) \quad \text{when } \omega \in B \in \mathcal{P} \]

Relationship:

\[ P(A|\mathcal{P}) = E(I_A|\mathcal{P}) \quad \text{where} \]

\[ I_A(\omega) = 1 \text{ if } \omega \in A, \quad 0 \text{ else} \]
Example of conditional expectation:

\[
\begin{array}{c|ccccc}
  i & 1 & 2 & 3 & 4 & 5 \\
\hline
  X(\omega_i) & 1.2 & 3.5 & -7.8 & 0 & 2.4 \\
  p(\omega_i) & \frac{1}{4} & \frac{1}{4} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\end{array}
\]

\[
E(X) = 1.2 \frac{1}{4} + 3.5 \frac{1}{4} - 7.8 \frac{1}{6} + 2.4 \frac{1}{6} = 0.275
\]

\[\mathcal{P} = \{A, B, C\}:\]

\[A = \{\omega_1, \omega_2\} \quad B = \{\omega_3, \omega_5\} \quad C = \{\omega_4\}\]

\[
E(X | A) = 1.2 \times P(\omega_1 | A) + 3.5 \times P(\omega_1 | A) \\
= 1.2 \frac{1}{2} + 3.5 \frac{1}{2} = 2.35
\]

\[
E(X | B) = -7.8 \times P(\omega_3 | B) + 2.4 \times P(\omega_5 | B) \\
= -7.8 \times \frac{1}{2} + 2.4 \times \frac{1}{2} = -2.7
\]

\[
E(X | C) = 0
\]

\[
\begin{array}{c|ccccc}
  i & 1 & 2 & 3 & 4 & 5 \\
\hline
  E(X | \mathcal{P})(\omega) & 2.35 & 2.35 & -2.7 & 0 & -2.7 \\
\end{array}
\]

\[
E(E(X | P)) = 2.35 \times P(A) - 2.7 \times P(B) \\
= 2.35 \times \frac{1}{2} - 2.7 \times \frac{1}{3} \\
= 0.275 = E(X)
\]
PROPERTIES OF CONDITIONAL EXPECTATIONS

- Linearity: for constant $c_1, c_2$:

$$E(c_1X_1 + c_2X_2 \mid \mathcal{P}) = c_1E(X_1 \mid \mathcal{P}) + c_2E(X_2 \mid \mathcal{P})$$

- Conditional constants: if $Z \indep \mathcal{P}$, then

$$E(ZX \mid \mathcal{P}) = ZE(X \mid \mathcal{P})$$

- Law of iterated expectations (iterated conditioning, tower property): if $Q \leq \mathcal{P}$, then

$$E[E(X \mid \mathcal{P}) \mid Q] = E(X \mid Q)$$

- Independence: if $X$ is independent of $\mathcal{P}$:

$$E(X \mid \mathcal{P}) = E(X)$$

- Jensen’s inequality: if $\phi : x \to \phi(x)$ is convex:

$$E(\phi(X) \mid \mathcal{P}) \geq \phi(E(X \mid \mathcal{P}))$$

Note: $\phi$ is convex if $\phi(ax + (1-a)y) \leq a\phi(x) + (1-a)\phi(y)$ for $0 \leq a \leq 1$. For example: $\phi(x) = e^x$, $\phi(x) = (x-K)^+$. Or $\phi''$ exists and is continuous, and $\phi''(x) \geq 0$. 
PARTITIONS AND $\sigma$-FIELDS

$\mathcal{F}$: $\sigma$-field  \hspace{0.5cm} $\mathcal{P}$: partition

$\mathcal{F}$ and $\mathcal{P}$ contain some information if

$$\mathcal{F} = \{\bigcup A : A \in \mathcal{P}\}$$

Order relation: $Q \leq \mathcal{P}$ if and only if $\mathcal{F}^Q \subseteq \mathcal{F}^\mathcal{P}$

because: $Q \leq \mathcal{P} \implies Q \subseteq \mathcal{F}^\mathcal{P}$

$\sigma$-fields more useful in continuous settings.

Definition:

$\Omega, \phi \in \mathcal{F}$

$A \in \mathcal{F} \implies A^c \in \mathcal{F}$

$A_1, \ldots, A_n, \ldots \in \mathcal{F} \implies \bigcup_n A_n \in \mathcal{F}$

(countable union only)

Filtration: $(\mathcal{F}_t)$:

$\mathcal{F}_t$ is a $\sigma$-field for each $t$

$s \leq t \implies \mathcal{F}_s \subseteq \mathcal{F}_t$

For now:

$$E(X \mid \mathcal{F}) = E(X \mid \mathcal{P})$$
CONDITIONAL EXPECTATION GIVEN \( \sigma \)-FIELD

**IF** \( X \) **GENERATES** \( \mathcal{G} \):

\[
E(Y \mid \mathcal{G}) = E(Y \mid X) = f(X)
\]

**IF** \((S^0_t, \ldots, S^K_t)\) **GENERATES** \(\mathcal{F}_t\):

\[
S^0_u, \ldots, S^K_u, \quad u \leq t \ \text{GENERATES} \ \mathcal{F}_t:
\]

\[
E(Y \mid \mathcal{F}_t) = E(Y \mid S^j_u, u \leq t, 0 \leq j \leq K)
\]
CONDITIONAL EXPECTATION

TWO NOTIONS OF CONDITIONING

\[ f(x) = E(Y \mid X = x) \]

\[ E(Y \mid X) = f(X) \]

A NUMBER

RANDOM VARIABLE

SIMILAR: OPTIONS PRICE

\[ C(s, t) \quad C(S_t, t) \]

SIMILAR: RADON-NIKODYM DERIVATIVE X HAS DENSITY \( f_P \) UNDER \( P \), \( f_Q \) UNDER \( Q \)

\[ \frac{f_Q(x)}{f_P(x)} \]

\[ \frac{dQ}{dP} = \frac{f_Q(X)}{f_P(X)} \]

NON RANDOM

RANDOM
MARTINGALES (FAIR GAMES)

\( \tilde{S}_t \) IS MARTINGALE \( \mathcal{F}_t \), \( \pi \)

IF \( E_\pi(\tilde{S}_{t+1} \mid \mathcal{F}_t) = \tilde{S}_t \quad \forall t \) (discrete time definition)

\( \Leftrightarrow E_\pi(\tilde{S}_u \mid \mathcal{F}_t) = \tilde{S}_t \quad \forall u \geq t \) (also OK in continuous time)

MOTIVATION

\( \pi \) IS RISK NEUTRAL IN ALL TIME PERIODS (in \( B_t = e^{rt} \) numeraire) IF

for each \( j, t, \omega_{t-1} \): \( E_\pi(e^{-r}S_t^j \mid \mathcal{F}_{t-1})(\omega_{t-1}) = S_t^j(\omega_{t-1}) \)

SAME AS IF (and this is true in general numeraire)

for each \( j, t, \omega_{t-1} \): \( E_\pi(\tilde{S}_t^j \mid \mathcal{F}_{t-1})(\omega_{t-1}) = \tilde{S}_t^j(\omega_{t-1}) \)

SAME AS IF

\( \tilde{S}^1, \ldots, \tilde{S}^K \) ARE (\( \mathcal{F}_t \)), \( \pi - \) MARTINGALES

ADAPT THE LATTER
AS DEFINITION OF RISK NEUTRAL MEASURE

CAUTION: \( \pi \) depends on choice of numeraire
MARTINGALES (MG)

(1) \( E(X_{t+1} \mid \mathcal{F}_t) = X_t \quad \forall t \) \text{ discrete time}

\( \Leftrightarrow \)

(2) \( E(X_u \mid \mathcal{F}_t) = X_t \quad \forall u > t \)

BECAUSE: TOWER PROPERTY: (1) \( \Rightarrow \) (2)

\[
E(X_{t+k} \mid \mathcal{F}_t) = E\left( E(X_{t+k} \mid \mathcal{F}_{t+k-1}) \mid \mathcal{F}_t \right)
\]

\[
= E(X_{t+k-1} \mid \mathcal{F}_t)
\]

INDUCTION

REPRESENTATION BY FINAL VALUE:

* Any \( X_t = E(X \mid \mathcal{F}_t) \) is MG:

\[
E[X_u \mid \mathcal{F}_t] = E\left[ E(X \mid \mathcal{F}_u) \mid \mathcal{F}_t \right]
\]

\[
= E[X \mid \mathcal{F}_t] = X_t
\]

* \( X_t = E(X_T \mid \mathcal{F}_t) \)
ALL DISCOUNTED PORTFOLIOS ARE MGs

\[ \frac{\widetilde{P}}{L}: \widetilde{V}_t = \widetilde{V}_0 + \sum_{u=0}^{t-1} \Delta(u) \Delta \widetilde{S}_u \]  
\[ \Delta \widetilde{V}_t = \widetilde{V}_{t+1} - \widetilde{V}_t = \Delta(t) \Delta \widetilde{S}_t \]

\[ E_\pi(\Delta \widetilde{V}_t \mid \mathcal{F}_t) = \Delta(t) E_\pi(\Delta \widetilde{S}_t \mid \mathcal{F}_t) = 0 \]

KNOWN MG AT \( t \)

THE IMPORTANT VALUATION FORMULA:

\[ \widetilde{V}_0 = E_\pi \widetilde{V}_T \]
**Completeness**: All random variables $\tilde{V}$ can be represented

$$\tilde{V} = \tilde{V}_0 + \sum_{u=0}^{T-1} \Delta(u) \Delta \tilde{S}_u$$

(cf p. 4; this assures that a self financing strategy exists)

**By Representation of Martingales by Final Value**: Completeness =

All $\pi$-MG’s ARE DISCOUNTED PORTFOLIOS (***) ("Martingale Representation Theorem")

This IS THE SAME AS

$\pi$ is UNIQUE

In this case, for all random payoffs:

$$\tilde{V}_0 = E_\pi \tilde{V}$$
RISK NEUTRAL MEASURES \( \pi \):

DEF: ALL \( \tilde{S}^j_t \) ARE \( \pi \)-MG, \( j = 1, \ldots, K \)

FOR EACH PERIOD \( t \rightarrow t + 1 \):

\[
E_\pi(\tilde{S}^j_{t+1} \mid \mathcal{F}_t) = \tilde{S}^j_t \quad j = 1, \ldots, K
\]

* \( \pi(\tilde{S}_{t+1} \mid \mathcal{F}_t) \) exists \( \Leftrightarrow \) NO ARBITRAGE \( t \Rightarrow t + 1 \)

* OVERALL \( \pi \):

\[
\pi(\tilde{S}_1 \mid \mathcal{F}_0) \quad \pi(\tilde{S}_2 \mid \mathcal{F}_1)
\]

MULTIPERIOD \( \pi \) EXISTS \( \Leftrightarrow \) NO ARBITRAGE

REPLACE “exists” BY “is unique” IN CONSTRUCTION:

MULTIPERIOD \( \pi \) UNIQUE \( \Leftrightarrow \) COMPLETENESS

BINOMIAL TREE CASE: \( \pi \) IS AS ON p. 6