ITÔ PROCESSES

$$X_t = X_0 + \underbrace{\int_0^t a_s ds}_{dt \text{ term}} + \underbrace{\int_0^t b_s dW_s}_{dW_t \text{ term}}$$

Decomposition unique:

- dW_t term is martingale
- dt term is drift

(Doob-Meyer decomposition)

UNDER RISK NEUTRAL MEASURE P^*

Discounted securities only have dW term:

$$d\widetilde{S}_t = \underbrace{\mu_t \widetilde{S}_t dt}_{=0} + \sigma_t \widetilde{S}_t dW_t$$

UNDISCOUNTED SECURITIES UNDER P^*:

$$d\widetilde{S}_t = \sigma_t \widetilde{S}_t dW_t$$

1) Numeraire $= B_t = \exp(\int_0^t r_u du)$

$$\begin{aligned} S_t &= \widetilde{S}_t B_t \Rightarrow \\ dS_t &= B_t d\widetilde{S}_t + \widetilde{S}_t dB_t \\ &= \underbrace{B_t \sigma_t \widetilde{S}_t dW_t} + \underbrace{\widetilde{S}_t B_t r_t} dt \\ &= \sigma_t S_t dW_t + r_t S_t dt \end{aligned}$$

2) Other numeraire:
$\Lambda_t \neq B_t$, Λ_t has $dW_t^{(2)}$ term

$$d\langle W, W^{(2)} \rangle_t = \rho_t dt$$

$d\widetilde{S}_t \Lambda_t =$ full use of Itô's formula

Not same P^*!!!

UNDISCOUNTED SECURITIES UNDER P^*:

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t$$

LOG SCALE: ITO'S FORMULA

$$\begin{aligned} d\log(S_t) &= \frac{1}{S_t} dS_t + \frac{1}{2}(-\frac{1}{S_t^2}) d[S,S]_t \\ &= \frac{1}{S_t}(r_t S_t dt + \sigma_t S_t dW_t) + \frac{1}{2}(-\frac{1}{S_t^2})\sigma_t^2 S_t dt \\ &= (r_t - \frac{1}{2}\sigma_t^2) dt + \sigma_t dW_t \end{aligned}$$

OPTIONS PRICES: PDE'S

PAYOFF: $f(S_T)$

DISCOUNTED PAYOFF: $e^{-rT} f(S_T) = \widetilde{f}(\widetilde{S}_T)$

$$\widetilde{S}_T = e^{-rT} S_T \quad \widetilde{f}(\widetilde{s}) = e^{-rT} f(e^{rT}\widetilde{s})$$

CALL: $f(s) = (s-K)^+ \ \widetilde{f}(\widetilde{s}) = (\widetilde{s} - e^{-rT}K)^+$

CANDIDATE PRICE: DISCOUNTED:

$$\widetilde{C}(\widetilde{S}_t, t) \text{ satisfies: } \widetilde{C}(\widetilde{S}, T) = \widetilde{f}(\widetilde{S})$$

AND (UNDER P^*):

$$d\widetilde{C}(\widetilde{S}_t, t)$$

$$\text{Hedge}\left\{ \quad = \widetilde{C}'_s(\widetilde{S}_t, t) d\widetilde{S}_t \quad \right\} \begin{array}{l}\text{MG term} \\ d\widetilde{S} = \sigma_t \widetilde{S}_t dW_t\end{array}$$

$$\begin{array}{l}\text{BS} \\ \text{PDE} \\ = 0\end{array} \left\{ \begin{array}{l} +\widetilde{C}'_t(\widetilde{S}_t, t)dt \\ +\frac{1}{2}\widetilde{C}''_{ss}(\widetilde{S}_t, t) \underbrace{d[\widetilde{S}, \widetilde{S}]_t}_{\sigma_t^2 \widetilde{S}_t^2 dt} \end{array} \right\} dt - \text{terms}$$

2 APPROACHES

THE BS PDE:

$$\text{Solve } (*) \begin{cases} \widetilde{C}'_t(\widetilde{s},t) + \dfrac{1}{2}\widetilde{C}''_t(\widetilde{s},t)\sigma^2\widetilde{s}^2 = 0 \\ \widetilde{C}(\widetilde{s},T) = \widetilde{f}(\widetilde{s}) \end{cases}$$

THE MARTINGALE APPROACH:

Set

$$\widetilde{C}(\widetilde{s},t) = E^*[\widetilde{f}(\widetilde{S}_T) \mid \widetilde{S}_t = \widetilde{s}]$$

Markov: $\widetilde{C}(\widetilde{S}_t,t) = E^*[\widetilde{f}(\widetilde{S}_T) \mid \mathcal{F}_t] =$ price under P^*

This $\widetilde{C}$ either

1) Market is complete:
   $\widetilde{C}$ automatically satisfies (*)
2) Otherwise: check if $\widetilde{C}$ satisfies (*):
   yes: solution OK
   no: try something else

REVERSAL OF DISCOUNTING

Numeraire: $B_t = \exp\{rt\}$

$$\begin{aligned} C(S_t, t) &= B_t \widetilde{C}(\widetilde{S}_t, t) \\ &= B_t \widetilde{C}\left(\frac{S_t}{B_t}, t\right) \\ &= e^{rt} \widetilde{C}(e^{-rt} S_t, t) \\ \text{Hence: } C(s,t) &= e^{rt} \widetilde{C}(e^{-rt} s, t) \\ &= e^{rt} E^*[\widetilde{f}(\widetilde{S}_T) \mid \widetilde{S}_t = e^{-rt} s] \\ &= e^{r(T-t)} E^*[f(S_T) \mid S_t = s] \end{aligned}$$

since

$$\widetilde{f}(\widetilde{s}) = e^{-rT} f(e^{rT} \widetilde{s})$$

COMPUTATION OF EXPECTED VALUES

$$\log \widetilde{S}_T = \log \widetilde{S}_t + \underbrace{\nu(T-t)}_{\nu = -\frac{\sigma^2}{2}} + \sigma \underbrace{(W_T - W_t)}_{\substack{\sqrt{T-t}\,Z \\ Z \sim N(0,1)}}$$

and so : $\widetilde{S}_T = \widetilde{S}_t \exp(\nu(T-t) + \sigma\sqrt{T-t}\,Z)$

$$\begin{aligned}
\widetilde{C}(s,t) &= E[\widetilde{f}(\widetilde{S}_T) \mid \widetilde{S}_t = \widetilde{s}] \\
&= E[\widetilde{f}(\widetilde{s}\exp(\nu(T-t) + \sigma\sqrt{T-t}\,Z)] \\
&= \int_{-\infty}^{+\infty} \widetilde{f}(\widetilde{s}\exp(\nu(T-t) + \sigma\sqrt{T-t}\,z)\phi(z)dz \\
&\qquad \text{where } \phi(z) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)
\end{aligned}$$

MORE GENERAL

"EUROPEAN CONTINGENT CLAIMS" (ECC)

$$\eta = \text{PAYOFF} \qquad \text{AT } \underbrace{\text{TIME } T}_{\text{FIXED}}$$

$$\text{LOOKBACK: } \eta = \left(\max_{0 \le t \le T} S_t - K\right)^+$$

$$\text{ASIAN: } \eta = \left(\frac{1}{T}\int_0^T S_u du - K\right)^+$$

$$\text{BARRIER } \eta = \begin{cases} (S_T - K)^+ \\ \text{UNLESS} \\ \min\limits_{0 \le t \le T} S_t \le X \end{cases}$$

ONLY THE IMAGINATION IS THE LIMIT...

OPTION PRICES: GENERAL SCHEME

SELF FINANCING STRATEGIES:

$$\eta = C_T$$

$$dC_t = \theta_t^{(0)} dB_t + \sum_{i=1}^{K} \theta_t^{(i)} dS_t^{(i)}$$

$$C_t = \theta_t^{(0)} B_t + \sum_{i=1}^{K} \theta_t^{(i)} S_t^{(i)}$$

Same as (by numeraire invariance):

$$\left.\begin{array}{l} \widetilde{\eta} = \frac{1}{B_T}\eta \quad : \quad \widetilde{\eta} = \widetilde{C}_T \\ d\widetilde{C}_t = \sum_{i=1}^{K} \theta_t^{(i)} d\widetilde{S}_t^{(i)} \end{array}\right] \text{Key requirement}$$

$$\left.\widetilde{C}_t = \theta_t^{(0)} + \sum_{i=1}^{K} \theta_t^{(i)} \widetilde{S}_t^{(i)} \right] \text{Defines } \theta_t^{(0)}$$

PROOF: ITÔ'S FORMULA

ON THE DISCOUNTED SCALE

$$\left.\begin{aligned} \widetilde{\eta} &= \widetilde{C}_T \\ d\widetilde{C}_t &= \sum_{i=1}^{K} \theta_t^{(i)} d\widetilde{S}_t^{(i)} \end{aligned}\right]$$

UNDER P^*: SAME AS

$$\widetilde{\eta} = c + \sum_{i=1}^{K} \int_0^T \theta_t^{(i)} d\widetilde{S}_t^{(i)} \qquad (*)$$

BY TAKING

$$\widetilde{C}_t = E^*(\widetilde{\eta} \mid \mathcal{F}_t)$$

If $B_0 = 1$: $c = \widetilde{C}_0 =$ PRICE AT 0

(*): "MARTINGALE REPRESENTATION THEOREM"

WHEN DOES THE REPRESENTATION THEOREM HOLD?

THEOREM: *IF* $W^{(1)}, \ldots, W^{(K)}$ *INDEPENDENT B.M.'S*

$$\mathcal{F}_t = \mathcal{F}_t^{W^{(1)},\ldots,W^{(K)}}$$

$$\widetilde{\eta} \text{ @ } \mathcal{F}_T, \quad E^*|\widetilde{\eta}| < \infty :$$

$$\widetilde{\eta} = c + \sum_{i=1}^{P} \int_0^T f_t dW_t^{(i)}$$

Brownian motions are like binomial trees

FROM BROWNIAN MOTION TO STOCK PRICE

$$d\widetilde{S}_t = \sigma \widetilde{S}_t dW_t$$

or:

$$\log \widetilde{S}_t = \log \widetilde{S}_0 - \frac{1}{2}\sigma^2 t + \sigma \widetilde{W}_t$$

$$\widetilde{\eta} \text{ @ } \mathcal{F}_T^{\widetilde{S}} \Leftrightarrow \widetilde{\eta} \text{ @ } \mathcal{F}_T^W$$

$$\text{Get: } \widetilde{\eta} = c + \int_0^T f_t dW_t$$

$$= c + \int_0^T \underbrace{\frac{f_t}{\sigma \widetilde{S}_t}}_{\theta_t} \, d\widetilde{S}_t$$

More complex if σ_t or r_t random...

P AND P^*

$(B_t, S_t^{(t)}, \ldots, S_t^{(p)})$: securities

P: actual probability

P^*: risk neutral probability

Realtionship: mutual absolute continuity $P \sim P^*$

<u>For example</u>:

$P: \; dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$

$P^*: \; dS_t = \mu_t^* S_t dt + \sigma_t^* S_t dW_t^*$

Money market bond numeraire:

$$\mu_t^* = r_t$$

$Q: \; \sigma_t^* = ??$ $\qquad A: \; {\sigma_t^*}^2 = \sigma_t^2$. Why?

CONTINUOUS CASE: σ_t AND σ_t^*

$$P: \quad dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$
$$P^*: \quad dS_t = \mu_t^* S_t dt + \sigma_t^* S_t dW_t^*$$

$$P: \quad d\log S_t = \ldots dt + \sigma_t dW_t$$
$$>: \quad \underbrace{(d\log S_t)^2}_{d[\log S, \log S]_t} = \sigma_t^2 (dW_t)^2 = \sigma_t^2 dt$$

$$P^*: \quad \text{same argument}: \quad d[\log S, \log S]_t = {\sigma_t^*}^2 dt$$

Process same under P, P^*:

$${\sigma_t^*}^2 dt = d[\log S, \log S]_t = \sigma_t^2 dt$$
$$>: \quad {\sigma_t^*}^2 = \sigma_t^2$$

In particular: if σ is constant:

$$\sigma^* = \sigma$$

If numeraire = Money Market Bond:

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t^*$$
$$d\widetilde{S}_t = \sigma_t \widetilde{S}_t dW_t^*$$

CONTINUOUS CASE: THE MARKET PRICE OF RISK

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$

„

$$dS_t = \mu_t^* S_t dt + \sigma_t S_t dW_t^*$$

$$\mu_t S_t dt + \sigma_t S_t dW_t = \mu_t^* S_t dt + \sigma_t S_t dW_t^*$$

$$>: \ \mu_t dt + \sigma_t dW_t = \mu_t^* dt + \sigma_t dW_t^*$$

$$>: \ \underbrace{\frac{\mu_t - \mu_t^*}{\sigma_t}}_{\lambda_t} dt + dW_t = dW_t^*$$

Change from P to P^*:

Market price of risk

$$\underbrace{dW_t^*}_{P^* - BM} = \underbrace{dW_t}_{P - BM} + \lambda_t dt$$

OTHER NUMERAIRE

$$\left.\begin{aligned} d\log S_t &= \ldots dt + \sigma_t dW_t \\ d\log \Lambda_t &= \ldots dt + \gamma_t dV_t \end{aligned}\right] P, P^*$$

$$\widetilde{S}_t = \frac{S_t}{\Lambda_t} \Rightarrow \log \widetilde{S}_t = \log S_t - \log \Lambda_t$$

$$\Rightarrow$$

$$\begin{aligned} (d\log \widetilde{S}_t)^2 &= (d\log S_t)^2 + (d\log \Lambda_t)^2 - 2(d\log S_t)(d\log \Lambda_t) \\ &= \sigma_t^2 (dW_t)^2 + \gamma_t^2 (dV_t)^2 - 2\sigma_t\gamma_t \underbrace{(dW_t)(dV_t)}_{\rho_t dt} \\ &= (\sigma_t^2 + \gamma_t^2 - 2\sigma_t\gamma_t\rho_t)dt \end{aligned}$$

$$\rho_t = \frac{1}{dt}\, d[W,V]_t = \text{correlation } (dW, dV)$$

(can be random)

$$\widetilde{\sigma}_t^2 = \sigma_t^2 + \gamma_t^2 - 2\sigma_t\gamma_t\rho_t \quad P, P^*$$

$$\neq \sigma_t^2 \text{ ex } \gamma_t = 0: \text{ MONEY MARKET BOND}$$

OTHER NUMERAIRE

$$d\log S_t = \ldots dt + \sigma_t dW_t$$

$$d\log \Lambda_t = \ldots dt + \gamma_t dV_t$$

so $\underbrace{d\log \widetilde{S}_t}_{= \ldots dt + \widetilde{\sigma}_t d\widetilde{W}_t \text{ previous page}} = \ldots dt + \sigma_t + \sigma_t dW_t - \gamma_t dV_t$

or $d\widetilde{S}_t = \underbrace{\ldots \widetilde{S}_t dt}_{= 0 \text{ if } P^* \text{ for numeraire } \Lambda} + \widetilde{\sigma}_t \widetilde{S}_t d\widetilde{W}_t$

$$\widetilde{\sigma}_t d\widetilde{W}_t = \sigma_t dW_t - \gamma_t dV_t$$

or

$$d\widetilde{W}_t = \frac{\sigma_t}{\widetilde{\sigma}_t} dW_t - \frac{\gamma_t}{\widetilde{\sigma}_t} dV_t$$

Yet another Brownian motion

P^* DEPENDS ON NUMERAIRE

Λ_t, B_t: NUMERAIRES, SUPPOSE P^* SAME

$$M_t = \frac{\Lambda_t}{B_t} = P^* - \text{MG}$$

$$\frac{1}{M_t} = \frac{B_t}{\Lambda_t} = P^* - \text{MG}$$

$$\underbrace{d\,\frac{1}{M_t}}_{d\text{MG}} = \underbrace{-\frac{1}{M_t^2}\,dM_t}_{d\text{MG}} + \underbrace{\frac{1}{M_t^3}\,d[M,M]_t}_{d\text{MG??}}$$

NEED $\underbrace{d[M,M]_t}_{dt \text{ term}} = d\text{MG}$

UNIQUENESS OF DOOB-MEYER $\Rightarrow$

$$[M,M]_t = 0$$

$\Rightarrow M_t$ constant if continuous

$\Rightarrow \Lambda, B$ are the same