ITÔ PROCESSES

\[ X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dW_s \]

\text{dt term} \quad \text{dW}_t \text{ term}

Decomposition unique:

- \( dW_t \) term is martingale
- \( dt \) term is drift

(Doob-Meyer decomposition)

UNDER RISK NEUTRAL MEASURE \( P^* \)

Discounted securities only have \( dW \) term:

\[ d\tilde{S}_t = \mu_t \tilde{S}_t dt + \sigma_t \tilde{S}_t dW_t = 0 \]
**UNDISCOUNTED SECURITIES UNDER** $P^*$:

$$d\tilde{S}_t = \sigma_t \tilde{S}_t dW_t$$

1) **Numeraire** $= B_t = \exp(\int_0^t r_u du)$

$$S_t = \tilde{S}_t B_t \Rightarrow$$

$$dS_t = B_t d\tilde{S}_t + \tilde{S}_t dB_t$$

$$= B_t \sigma_t \tilde{S}_t dW_t + \tilde{S}_t B_t r_t dt$$

$$= \sigma_t S_t dW_t + r_t S_t dt$$

2) **Other numeraire:**

$\Lambda_t \neq B_t$, $\Lambda_t$ has $dW_t^{(2)}$ term

$$d\langle W, W^{(2)} \rangle_t = \rho_t dt$$

$$d\tilde{S}_t \Lambda_t = \text{full use of Itô’s formula}$$

Not same $P^*$!!!
UNDISCOUNTED SECURITIES UNDER $P^*$:

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t$$

LOG SCALE: ITO’S FORMULA

$$d \log(S_t) = \frac{1}{S_t} dS_t + \frac{1}{2} (-\frac{1}{S_t^2}) d[S, S]_t$$

$$= \frac{1}{S_t} (r_t S_t dt + \sigma_t S_t dW_t) + \frac{1}{2} (-\frac{1}{S_t^2}) \sigma_t^2 S_t dt$$

$$= (r_t - \frac{1}{2} \sigma_t^2) dt + \sigma_t dW_t$$
OPTIONS PRICES: PDE’S

PAYOFF: $f(S_T)$

DISCOUNTED PAYOFF: $e^{-rT}f(S_T) = \tilde{f}(\tilde{S}_T)$

$$\tilde{S}_T = e^{-rT}S_T \quad \tilde{f}(\tilde{s}) = e^{-rT}f(e^{rT}\tilde{s})$$

CALL: $f(s) = (s - K)^+ \quad \tilde{f}(\tilde{s}) = (\tilde{s} - e^{-rT}K)^+$

CANDIDATE PRICE: DISCOUNTED:

$$\tilde{C}(\tilde{S}_t, t) \text{ satisfies: } \tilde{C}(\tilde{S}, T) = \tilde{f}(\tilde{S})$$

AND (UNDER $P^*$):

$$d\tilde{C}(\tilde{S}_t, t)$$

Hedge \( \{ \)

\begin{align*}
= \tilde{C}'_s(\tilde{S}_t, t)d\tilde{S}_t & \quad \text{MG term} \\
= \tilde{C}'_t(\tilde{S}_t, t)dt & \\
+ \frac{1}{2} \tilde{C}''_{ss}(\tilde{S}_t, t) d[\tilde{S}, \tilde{S}]_t & \\
+ \tilde{C}''_{tt}(\tilde{S}_t, t) & \\
\end{align*}

$$d\tilde{S} = \sigma_t \tilde{S}_t dW_t$$

BS PDE \( \{ \)

$$= 0 \left\{ \right\}$$

$$dt - \text{terms}$$
2 APPROACHES

THE BS PDE:

Solve
\[
\begin{align*}
\frac{\partial \tilde{C}}{\partial t}(\tilde{s}, t) + \frac{1}{2} \frac{\partial^2 \tilde{C}}{\partial t^2}(\tilde{s}, t) \sigma^2 \tilde{s}^2 &= 0 \\
\tilde{C}(\tilde{s}, T) &= \tilde{f}(\tilde{s})
\end{align*}
\]

THE MARTINGALE APPROACH:

Set
\[
\tilde{C}(\tilde{s}, t) = E^*\left[ \tilde{f}(\tilde{S}_T) \mid \tilde{S}_t = \tilde{s} \right]
\]

Markov: \( \tilde{C}(\tilde{S}_t, t) = E^*\left[ \tilde{f}(\tilde{S}_T) \mid \mathcal{F}_t \right] = \text{price under } P^* \)

This \( \tilde{C} \) either

1) Market is complete: 
   \( \tilde{C} \) automatically satisfies (*)
2) Otherwise: check if \( \tilde{C} \) satisfies (*):
   yes: solution OK
   no: try something else
REVERSAL OF DISCOUNTING

Numeraire: \( B_t = \exp\{rt\} \)

\[
C(S_t, t) = B_t \tilde{C}(\tilde{S}_t, t) = B_t \tilde{C}\left(\frac{S_t}{B_t}, t\right) = e^{rt} \tilde{C}(e^{-rt}S_t, t)
\]

Hence: \( C(s, t) = e^{rt} \tilde{C}(e^{-rt} s, t) \)

\[
= e^{rt} E^* [\tilde{f}(\tilde{S}_T) \mid \tilde{S}_t = e^{-rt} s]
= e^{r(T-t)} E^* [f(S_T) \mid S_t = s]
\]

since

\[
\tilde{f}(\tilde{s}) = e^{-rT} f(e^{rT} \tilde{s})
\]
COMPUTATION OF EXPECTED VALUES

\[
\log \tilde{S}_T = \log \tilde{S}_t + \nu(T - t) + \sigma \left( W_T - W_t \right) \\
\nu = -\frac{\sigma^2}{2} \quad \sqrt{T - t} Z \\
Z \sim N(0, 1)
\]

and so: \( \tilde{S}_T = \tilde{S}_t \exp(\nu(T - t) + \sigma \sqrt{T - t} Z) \)

\[
\tilde{C}(s, t) = E[\tilde{f}(\tilde{S}_T) \mid \tilde{S}_t = \tilde{s}] \\
= E[\tilde{f}(\tilde{s} \exp(\nu(T - t) + \sigma \sqrt{T - t} Z)] \\
= \int_{-\infty}^{+\infty} \tilde{f}(\tilde{s} \exp(\nu(T - t) + \sigma \sqrt{T - t} z) \phi(z) dz
\]

where \( \phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2) \)
MORE GENERAL

“EUROPEAN CONTINGENT CLAIMS” (ECC)

\[ \eta = \text{PAYOFF AT TIME } T \]

LOOKBACK: \[ \eta = (\max_{0 \leq t \leq T} S_t - K)^+ \]

ASIAN: \[ \eta = \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \]

BARRIER \[ \eta = \begin{cases} (S_T - K)^+ & \text{UNLESS} \\ \min_{0 \leq t \leq T} S_t \leq X \end{cases} \]

ONLY THE IMAGINATION IS THE LIMIT...
OPTION PRICES:
GENERAL SCHEME

SELF FINANCING STRATEGIES:

\[ \eta = C_T \]

\[ dC_t = \theta_t^{(0)} dB_t + \sum_{i=1}^{K} \theta_t^{(i)} dS_t^{(i)} \]

\[ C_t = \theta_t^{(0)} B_t + \sum_{i=1}^{K} \theta_t^{(i)} S_t^{(i)} \]

Same as (by numeraire invariance):

\[ \tilde{\eta} = \frac{1}{B_T} \eta = \tilde{C}_T \]

\[ d\tilde{C}_t = \sum_{i=1}^{K} \theta_t^{(i)} d\tilde{S}_t^{(i)} \]

\[ \tilde{C}_t = \theta_t^{(0)} + \sum_{i=1}^{K} \theta_t^{(i)} \tilde{S}_t^{(i)} \]

PROOF: ITÔ’S FORMULA
ON THE DISCOUNTED SCALE

\[ \tilde{\eta} = \tilde{C}_T \]
\[ d\tilde{C}_t = \sum_{i=1}^{K} \theta_t^{(i)} d\tilde{S}_t^{(i)} \]

UNDER \( P^* \): SAME AS

\[ \tilde{\eta} = c + \sum_{i=1}^{K} \int_0^T \theta_t^{(i)} d\tilde{S}_t^{(i)} \]  

\[ (*) \]

BY TAKING

\[ \tilde{C}_t = E^*(\tilde{\eta} \mid \mathcal{F}_t) \]

If \( B_0 = 1 \): \( c = \tilde{C}_0 = \text{PRICE AT} \ 0 \)

\[ (*) : \text{"MARTINGALE REPRESENTATION THEOREM"} \]
WHEN DOES THE REPRESENTATION THEOREM HOLD?

**Theorem:** \( \text{IF } W^{(1)}, \ldots, W^{(K)} \text{ INDEPENDENT B.M.'S} \)

\[ \mathcal{F}_t = \mathcal{F}_t^{W^{(1)}, \ldots, W^{(K)}} \]

\[ \tilde{\eta} @ \mathcal{F}_T, \quad E^*|\tilde{\eta}| < \infty : \]

\[ \tilde{\eta} = c + \sum_{i=1}^{P} \int_{0}^{T} f_t dW_t^{(i)} \]

Brownian motions are like binomial trees
FROM BROWNIAN MOTION TO STOCK PRICE

\[ d\tilde{S}_t = \sigma \tilde{S}_t dW_t \]

or:

\[ \log \tilde{S}_t = \log \tilde{S}_0 - \frac{1}{2} \sigma^2 t + \sigma \tilde{W}_t \]

\[ \tilde{\eta} @ \mathcal{F}_T^{\tilde{S}} \iff \tilde{\eta} @ \mathcal{F}_T^W \]

Get:\n\[ \tilde{\eta} = c + \int_0^T f_t dW_t \]

\[ = c + \int_0^T \frac{f_t}{\sigma \tilde{S}_t} d\tilde{S}_t \]

More complex if \( \sigma_t \) or \( r_t \) random...
P AND P*

\((B_t, S_t^{(t)}, \ldots, S_t^{(p)})\): securities

\(P\): actual probability

\(P^*\): risk neutral probability

Relationship: mutual absolute continuity \(P \sim P^*\)

For example:

\(P\): \(dS_t = \mu_t S_t dt + \sigma_t S_t dW_t\)

\(P^*\): \(dS_t = \mu^*_t S_t dt + \sigma^*_t S_t dW^*_t\)

Money market bond numeraire:

\(\mu^*_t = r_t\)

\(Q\): \(\sigma^*_t = ??\) \(A\): \(\sigma^*_t^2 = \sigma^2_t\). Why?
CONTINUOUS CASE: $\sigma_t$ AND $\sigma^*_t$

\[ P : \quad dS_t = \mu_t S_t dt + \sigma_t S_t dW_t \]
\[ P^* : \quad dS_t = \mu^*_t S_t dt + \sigma^*_t S_t dW^*_t \]

\[ P : \quad d \log S_t = \ldots dt + \sigma_t dW_t \]
\[ >: \quad (d \log S_t)^2 = \sigma^2_t (dW_t)^2 = \sigma^2_t dt \]
\[ d[\log S, \log S]_t \]

\[ P^* : \quad \text{same argument} : \quad d[\log S, \log S]_t = \sigma^*_t dt \]

Process same under $P$, $P^*$:

\[ \sigma^*_t dt = d[\log S, \log S]_t = \sigma^2_t dt \]
\[ >: \quad \sigma^*_t = \sigma^2_t \]

In particular: if $\sigma$ is constant:

\[ \sigma^* = \sigma \]

If numeraire = Money Market Bond:

\[ dS_t = r_t S_t dt + \sigma_t S_t dW^*_t \]
\[ d\tilde{S}_t = \sigma_t \tilde{S}_t dW^*_t \]
CONTINUOUS CASE: THE MARKET PRICE OF RISK

\[ dS_t = \mu_t S_t dt + \sigma_t S_t dW_t \]

\[ \]

\[ dS_t = \mu^*_t S_t dt + \sigma_t S_t dW_t^* \]

\[ \mu_t S_t dt + \sigma_t S_t dW_t = \mu^*_t S_t dt + \sigma_t S_t dW_t^* \]

\[ >: \mu_t dt + \sigma_t dW_t = \mu^*_t dt + \sigma_t dW_t^* \]

\[ >: \frac{\mu_t - \mu^*_t}{\sigma_t} dt + dW_t = dW_t^* \]

\( \lambda_t \)

Change from \( P \) to \( P^* \):

Market price of risk

\[ P^* - BM = dW_t^* + \lambda_t dt \]

\[ P - BM \]
OTHER NUMERAIRE

\[
\begin{align*}
    d \log S_t &= \ldots dt + \sigma_t dW_t \\
    d \log \Lambda_t &= \ldots dt + \gamma_t dV_t
\end{align*}
\quad \quad P, P^*
\]

\[
\tilde{S}_t = \frac{S_t}{\Lambda_t} \Rightarrow \log \tilde{S}_t = \log S_t - \log \Lambda_t
\]

\[
\Rightarrow
\]

\[
(d \log \tilde{S}_t)^2 = (d \log S_t)^2 + (d \log \Lambda_t)^2 - 2(d \log S_t)(d \log \Lambda_t)
\]

\[
= \sigma_t^2 (dW_t)^2 + \gamma_t^2 (dV_t)^2 - 2\sigma_t \gamma_t \rho_t dt
\]

\[
= (\sigma_t^2 + \gamma_t^2 - 2\sigma_t \gamma_t \rho_t) dt
\]

\[
\rho_t = \frac{1}{dt} d[W, V]_t = \text{correlation} \ (dW, dV)
\]

(can be random)

\[
\tilde{\sigma}_t^2 = \sigma_t^2 + \gamma_t^2 - 2\sigma_t \gamma_t \rho_t \quad P, P^*
\]

\[
\neq \sigma_t^2 \text{ ex } \gamma_t = 0 : \text{ MONEY MARKET BOND}
\]
OTHER NUMERAIRES

\[ d \log S_t = \ldots dt + \sigma_t dW_t \]

\[ d \log \Lambda_t = \ldots dt + \gamma_t dV_t \]

so \( d \log \tilde{S}_t = \ldots dt + \sigma_t + \sigma_t dW_t - \gamma_t dV_t \)

\[ = \ldots dt + \tilde{\sigma}_t d\tilde{W}_t \]

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or \( d\tilde{S}_t = \ldots \tilde{S}_t dt + \tilde{\sigma}_t \tilde{S}_t d\tilde{W}_t \)

\[ = 0 \text{ if } P^* \text{ for numeraire } \Lambda \]

\[ \tilde{\sigma}_t d\tilde{W}_t = \sigma_t dW_t - \gamma_t dV_t \]

or

\[ d\tilde{W}_t = \frac{\sigma_t}{\tilde{\sigma}_t} dW_t - \frac{\gamma_t}{\tilde{\sigma}_t} dV_t \]

Yet another Brownian motion
$P^*$ DEPENDS ON NUMERAIRE

$\Lambda_t, B_t$: NUMERAIRE, SUPPOSE $P^*$ SAME

$$M_t = \frac{\Lambda_t}{B_t} = P^* - \text{MG}$$

$$\frac{1}{M_t} = \frac{B_t}{\Lambda_t} = P^* - \text{MG}$$

$$d \frac{1}{M_t} = -\frac{1}{M_t^2} dM_t + \frac{1}{M_t^3} d[M, M]_t$$

$\underbrace{d\text{MG}}_{d\text{MG}} - \underbrace{d\text{MG}}_{d\text{MG}} + \underbrace{d\text{MG}}_{d\text{MG}}$??

NEED $d[M, M]_t = d\text{MG}$

$dt$ term

UNIQUENESS OF DOOB-MEYER $\Rightarrow$

$$[M, M]_t = 0$$

$\Rightarrow M_t$ constant if continuous

$\Rightarrow \Lambda, B$ are the same