STOCHASTIC INTEGRALS

$X_t = \text{CONTINUOUS PROCESS} \begin{cases} S_t : \text{STOCK PRICE} \\ M_t : \text{MG} \\ W_t : \text{BROWNIAN MOTION} \end{cases}$

$\theta_t = \text{PORTFOLIO: \#} X_t \text{ HELD AT } t$

DISCRETE TIME: $0 = t_0 < t_1 < \ldots < t_n = t$

$$P/L_t = \sum_{i<n} \theta_{t_i} \frac{(X_{t_{i+1}} - X_{t_i})}{\Delta X_{t_i}}$$

GRID BECOMES “DENSE”: $\max_i \Delta t_i \to 0$

$$P/L_t \to \int_0^t \theta_u dX_u$$

INTEGRAL DEFINED AS LIMIT OF SUMS
PROPERTIES MOSTLY FROM SUMS:

\[
\sum_{i<n} (a\theta_{t_i} + b\eta_{t_i}) \Delta X_{t_i} = a \sum_{i<n} \theta_{t_i} \Delta X_{t_i} + b \sum_{i<n} \eta_{t_i} \Delta X_{t_i}
\]

\[
\int_0^t (a\theta_u + b\eta_u) dX_u = a \int_0^t \theta_u dX_u + b \int_0^t \eta_u dX_u
\]

⇒ LINEARITY OK

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TIME VARYING INTEGRAL:

\[
\int_0^t \theta_u dX_u = \text{limit of} \quad \sum_{t_i+1 \leq t} \theta_{t_i} (X_{t_{i+1}} - X_{t_i})
\]

Limit in probability
MARTINGALE PROPERTY:

If \( X_t = M_t = \text{MG} \):

\[
U_t^{(n)} = \sum_{t_i+1 \leq t} \theta_{t_i} \Delta X_{t_i} : \quad \Delta U_{t_{i+1}}^{(n)} = \theta_{t_i} \Delta X_{t_i}
\]

If on grid \( t_0, t_1, \ldots \):

\[
E(\Delta U_{t_{i+1}}^{(n)} \mid \mathcal{F}_{t_i}) = E(\theta_{t_i} \Delta X_{t_i} \mid \mathcal{F}_{t_i}) = 0
\]

\[\Rightarrow U_t^{(n)} \text{ is } \mathcal{F}_{t_i} - \text{MG} \]

Taking limits:

\[
E(U_t \mid \mathcal{F}_s) = U_s
\]
QUADRATIC VARIATION (Q. V.)

\[ U_t^{(n)} = \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta X_{t_i} \]

so: \[ \Delta U_{t_i}^{(n)} = U_{t_{i+1}}^{(n)} - U_{t_i}^{(n)} = \theta_{t_i} \Delta X_{t_i} \]

\[ (\Delta U_{t_i}^{(n)})^2 = \theta_{t_i}^2 (\Delta X_{t_i})^2 \]

Aggregate:

\[ [U^{(n)}, U^{(n)}]_t = \sum_{t_{i+1} \leq t} (\Delta U_{t_i}^{(n)})^2 \]

\[ = \sum_{t_{i+1} \leq t} \theta_{t_i}^2 (\Delta X_{t_i})^2 = \sum_{t_{i+1} \leq t} \theta_{t_i}^2 \Delta [X, X]_{t_i} \]

\[ [U, U]_t = \int_0^t \theta_u^2 d[X, X]_u \]

If \( X_t = W_t = \text{B.M.} \): \( d[X, X]_t = dt \)

IT FOLLOWS THAT \( [U, U]_t = \int_0^t \theta_u^2 du \)
Differential Notation

Integral:

\[ \Delta U_{ti}^{(n)} = \theta_{ti} \Delta X_{ti} \text{ vs. } U_t = \sum_{t_{i+1} \leq t} \theta_{ti} \Delta X_{ti} \]

becomes

\[ dU_t = \theta_t dX_t \text{ vs. } U_t = \int_0^t \theta_s dX_s \]

Quadratic Variation:

\[ (\Delta U_{ti}^{(n)})^2 = \theta_{ti}^2 (\Delta X_{ti})^2 \text{ vs. } [U^{(n)}, U^{(n)}]_t = \sum \theta_{ti}^2 \Delta [X, X_{ti}] \]

\[ \Delta [U^{(n)}, U^{(n)}]_{ti} = \theta_{ti}^2 \Delta [X, X]_{ti} \]

becomes:

\[ (dU_t)^2 = \theta_t^2 (dX_t)^2 \text{ vs. } [U, U]_t = \int_0^t \theta_u^2 d[X, X]_u \]

\[ d[U, U]_t = \theta_t^2 d[X, X]_t \]

Brownian Motion:

\[ (dW_t)^2 = dt \quad \text{AND} \quad d[U, U]_t = \theta_t^2 dt \]
QUADRATIC COVARIATION:

\[ U, Z : \quad [U, Z]_t = \text{limit of} \sum_{t_{i+1} \leq t} \Delta U_{t_i} \Delta Z_{t_i} \]

CASE OF TWO INTEGRALS:

\[ U_t = \int_0^t \theta_s dX_s, \quad Z_t = \int_0^t \eta_s dY_s \]

THEN:

\[ [U, Z]_t = \int_0^t \theta_s \eta_s d[X, Y]_s \]

BECAUSE

\[ \Delta U_{t_i} \Delta Z_{t_i} = \theta_{t_i} \eta_{t_i} \Delta X_{t_i} \Delta Y_{t_i} \]

or

\[ \begin{aligned} dU_t dZ_t &= \theta_t \eta_t \begin{array}{c} \overbrace{dX_t dY_t} \end{array} \\ d[U, Z]_t &= \theta_t \eta_t d[X, Y]_t \end{aligned} \]

IF: \( X_t = Y_t = W_t \) THE \underline{SAME} B.M.:

\[ d[U, Z]_t = \theta_t \eta_t dt \]
**DETERMINISTIC INTEGRAND**

IF $\theta_t$ IS NONRANDOM:

$$\int_0^t \theta_s dW_s = \text{limit of } \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta W_{t_i}$$

$\sum_{t_{i+1} \leq t} \theta_{t_i} \Delta W_{t_i}$:

- **LINEAR COBIMINATION OF NORMAL RANDOM VARIABLES IS A NORMAL RANDOM VARIABLE**

- **MEAN**: $E \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta W_{t_i} = 0$

- **VARIANCE**: $\text{Var} \left( \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta W_{t_i} \right) = \sum_{t_{i+1} \leq t} \theta_{t_i}^2 \Delta t_i$

**IN THE LIMIT:**

$$\int_0^t \theta_s dW_s$$

- **NORMAL RANDOM VARIABLE**

- **MEAN IS ZERO**

- **VARIANCE**: $\text{Var} \left( \int_0^t \theta_s dW_s \right) = \int_0^t \theta_s^2 ds$
ITÔ’s FORMULA

$X_t$: CONTINUOUS PROCESS (SOME RESTRICTIONS):

$$\xi(X_t) = \xi(X_0) + \int_0^t \xi'(X_u) dX_u + \frac{1}{2} \int_0^t \xi''(X_u) d[X, X]_t$$

DIFFERENTIAL NOTATION:

$$d\xi(X_t) = \xi'(X_t) dX_t + \frac{1}{2} \xi''(X_t) d[X, X]_t$$

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EX: $X_t = W_t = B.M.$:

$$d\xi(W_t) = \xi'(W_t) dW_t + \frac{1}{2} \xi''(W_t) dt$$

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EX: $dX_t = V_t dt + \sigma_t dW_t$  ITÔ PROCESS

or: $X_t = X_0 + \int_0^t V_s ds + \int_0^t \sigma_s dW_s$

First question:

¿What is $d[X, X]_t$?
FIRST: CASE OF EXPLICIT INTEGRATION:

\( W_t = \text{B.M.} \) \( \text{WHAT IS } \int_0^t W_s dW_s? \)

\[
U_t = W_t^2 = \zeta(W_t), \zeta(x) = x^2 \]

\[
dU_t = \zeta'(W_t)dW_t + \frac{1}{2}\zeta''(W_t)dt \]

\[
= 2W_t dW_t + dt \]

so: \( W_t dW_t = \frac{1}{2}dU_t - \frac{1}{2}dt \)

\[
> \int_0^t W_s dW_s = \frac{1}{2}(U_t - U_0) - \frac{1}{2}\int_0^t ds \]

\[
= \frac{1}{2}W_t^2 - \frac{1}{2}t \]

DIFFERENT FROM ORDINARY INTEGRAL:

If \( X_t = g(t) \) \( g' \) exists, continuous, \( g(t) = 0 \)

\[
\int_0^t X_s dX_s = \int_0^t g(s)g'(s)ds \]

\[
= \frac{1}{2}g(t)^2 \]

\[
= \frac{1}{2}X_t^2 \]
ITÔ PROCESS:

\[ X_t = X_0 + \int_0^t \nu_s \, ds + \int_0^t \sigma_s \, dW_s \]

Grid:

\[ \Delta X_{t_i} = \Delta Z_{t_i} + \Delta U_{t_i} \]

so:

\[ (\Delta X_{t_i})^2 = (\Delta Z_{t_i})^2 + (\Delta U_{t_i})^2 + 2\Delta Z_{t_i} \Delta U_{t_i} \]

\[ \sum (\Delta X_{t_i})^2 = \sum (\Delta Z_{t_i})^2 + \sum (\Delta U_{t_i})^2 + \sum 2 \Delta Z_{t_i} \Delta U_{t_i} \]

\[ |\Delta Z_{t_i}| = \left| \int_{t_i}^{t_{i+1}} \nu_s \, ds \right| \leq \int_{t_i}^{t_{i+1}} |\nu_s| \, ds \]

\[ \leq \sup_s |\nu_s|(t_{i+1} - t_i) = \sup_s |\nu_s| \Delta t_i \]

\[ \sum (\Delta Z_{t_i})^2 \leq (\sup_s |\nu_s|)^2 \sum (\Delta t_i)^2 \]

\[ \leq (\sup_s |\nu_s|)^2 \sup \Delta t_i \sum \Delta t_i \to 0 \]

\[ \simeq t \]

\[ \subset: [Z, Z]_t = 0 \quad \text{ALSO: } [Z, U]_t = 0 \]

\[ \text{ONLY: } [U, U]_t = \int_0^t \sigma_s^2 \, ds \]
ITÔ PROCESS

\[ X_t = X_0 + \int_0^t \nu_s ds + \int_0^t \sigma_s dW_s \]

\[ d[Z, Z]_t = 0 \quad d[Z, U]_t = 0 \quad d[U, U]_t = \sigma_t^2 dt \]

USING DIFFERENTIALS:

ANY \( dt \)-TERM HAS ZERO Q.V.:

\[ (dZ_t)^2 = \nu_t^2 (dt)^2 = 0 \quad \text{ETC} \]

COMBINING TERMS:

\[ (dX_t)^2 = (dZ_t + dU_t)^2 \]

\[ = (dZ_t)^2 + 2dZ_t dU_t + (dU_t)^2 \]

\[ = (dU_t)^2 = \sigma_t^2 dt \]

RIGOROUS:

\[ (\Delta X_{t_i})^2 = (\Delta Z_{t_i}^2 + 2 \Delta Z_{t_i} \Delta U_{t_i} + (\Delta U_{t_i})^2 \]

SUM OVER \( t_i \), TAKE LIMITS, GET SAME RESULT
INTEGRALS WITH RESPECT TO AN ITÔ PROCESS

\[ X_t = X_0 + \int_0^t \nu_s \, ds + \int_0^t \sigma_s \, dW_s \]

CAN SHOW THAT:

\[ \int_0^t \theta_s \, dX_s = \int_0^t \theta_s \nu_s \, ds + \int_0^t \theta_s \sigma_s \, dW_s \]

A NEW ITÔ PROCESS
BACK TO ITÔ’S FORMULA:

\[ d\xi(X_t) = \xi'(X_t) dX_t + \frac{1}{2} \xi''(X_t) d[X, X]_t \quad (\ast) \]

ITÔ PROCESS:

\[ dX_t = \nu_t dt + \sigma_t dW_t \]

SO

\[ d[X, X]_t = \sigma_t^2 dt \]

PLUG IN:

\[
\begin{align*}
    d\xi(X_t) &= \xi'(X_t)(\nu_t dt + \sigma_t dW_t) \\
    &\quad + \frac{1}{2} \xi''(X_t) \sigma_t^2 dt \\
    &= \left(\xi'(X_t) \nu_t + \frac{1}{2} \xi''(X_t) \sigma_t^2\right) dt \\
    &\quad + \xi'(X_t) \sigma_t dW_t 
\end{align*}
\]

EASIER TO REMEMBER (\ast)…
"PROOF" OF ITÔ'S FORMULA:

\[ U_t = \xi(X_t) : \]

\[ \Delta U_{ti} = \xi(X_{ti+1}) - \xi(X_{ti}) = \xi(X_{ti} + \Delta X_{ti}) - \xi(X_{ti}) = \zeta'(X_{ti})\Delta X_{ti} + \frac{1}{2}\zeta''(X_{ti})\Delta X_{ti}^2 + \frac{1}{3!}\zeta'''(X_{ti})\Delta X_{ti}^3 + \cdots \]

sum up:

\[ U_t - U_i = \sum_{i} \zeta'(X_{ti})\Delta X_{ti} + \frac{1}{2} \sum_{i} \zeta''(X_{ti})\Delta X_{ti}^2 \]

\[ \int_0^t \zeta'(X_s)dX_s + \frac{1}{2} \int_0^t \zeta''(X_s)d[X,X]_s \]

OTHER "PROOF":

\[ dU_t = \zeta(X_t + dX_t) - \zeta(X_t) = \zeta'(X_t)dX_t + \frac{1}{2}\zeta''(X_t)(dX_t)^2 + \cdots \underbrace{d[X,X]_t}_{d[X,X]_t} \]
MULTIVARIATE FORMULA

\[ U_t = \zeta(X_t, Y_t) \]

\[ dU_t = \zeta'_x(X_t, Y_t)dX_t + \zeta'_y(X_t, Y_t)dY_t \]

\[ + \frac{1}{2} \left\{ \zeta''_{xx}(X_t, Y_t)d[X, X]_t \right. \]

\[ + \left. \zeta''_{yy}(X_t, Y_t)d[Y, Y]_t \right. \]

\[ + 2\zeta''_{xy}(X_t, Y_t)d[X, Y]_t \}

etc.
EXAMPLE: GEOMETRIC BROWNIAN MOTION

\[ S_t = S_0 \exp \left\{ \int_0^t \sigma_s dW_s + \int_0^t \left( r_s - \frac{1}{2} \sigma_s^2 \right) ds \right\} \]

SET

- \( X_t = \int_0^t \sigma_s dW_s + \int_0^t \left( r_s - \frac{1}{2} \sigma_s^2 \right) ds \)
- \( S_t = f(X_t) \) \( (f(x) = S_0 \exp\{x\}) \)

USE ITÔ’S FORMULA

\[ dS_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X, X]_t \]

\( f'(x) = f''(x) = f(x) \) AND \( d[X, X]_t = \sigma_t^2 dt \), SO:

\[ dS_t = f(X_t)dX_t + \frac{1}{2}f(X_t)\sigma_t^2 dt \]

\[ = S_t dX_t + \frac{1}{2}S_t \sigma_t^2 dt \]

\[ = S_t \left( dX_t + \frac{1}{2} \sigma_t^2 dt \right) \]

\[ = S_t (\sigma_t dW_t + r_t dt) \]

\[ = S_t \sigma_t dW_t + S_t r_t dt \]

DIFFERENTIAL REPRESENTATION OF \( S_t \)
VASICEK MODEL

\[ dR_t = (\alpha - \beta R_t)dt + \sigma dW_t \]

- **STEP 1:** SET \( U_t = R_t - \frac{\alpha}{\beta} \)

**EQUATION BECOMES:**

\[ dU_t = -\beta U_t dt + \sigma dW_t \]

- **STEP 2:** NOTE THAT (FROM ITO’S FORMULA)

\[
\begin{align*}
  d(\exp{\beta t}U_t) &= \exp{\beta t}dU_t + U_t d\exp{\beta t} \\
  &= \exp{\beta t}dU_t + U_t \beta \exp{\beta t} dt \\
  &= \exp{\beta t} (dU_t + U_t \beta dt) \\
  &= \exp{\beta t} \sigma dW_t
\end{align*}
\]

SO

\[ \exp{\beta t}U_t = U_0 + \int_0^t \exp{\beta s} \sigma dW_s \]

OR

\[ U_t = \exp{-\beta t}U_0 + \int_0^t \exp{\beta(s-t)} \sigma dW_s \]
IN OTHER WORDS: $U_t$ IS NORMAL

- MEAN IS $\exp\{-\beta t\} U_0$
- VARIANCE IS

$$
\int_0^t (\exp\{\beta(s - t)\}\sigma^2) \, ds
$$

$$
= \int_0^t \exp\{2\beta(s - t)\}\sigma^2 \, ds
$$

$$
= \left[ \frac{1}{2\beta} \exp\{2\beta(s - t)\}\sigma^2 \right]_{s=0}^{s=t}
$$

$$
= \frac{1}{2\beta} (1 - \exp\{-2\beta t\}) \sigma^2
$$

DEDUCE FOR $R_t = U_t + \frac{\alpha}{\beta}$ THAT

- $R_t$ IS NORMAL
- $E(R_t) = \exp\{-\beta t\} U_0 + \frac{\alpha}{\beta}$
- $\text{Var} (R_t) = \text{Var} (U_t)$
LEVY’S THEOREM

IF $M_t$ IS A CONTINUOUS MARTINGALE, $M_0 = 0$, $[M, M]_t = t$ FOR ALL $t$, THEN $M_t$ IS A CONTINUOUS BROWNIAN MOTION

PROOF: SET $f(x) = \exp\{hx\}$

ITO:

$$df(M_t) = f'(M_t)dt + \frac{1}{2}f''(M_t)d[M, M]_t$$

$$= f'(M_t)dt + \frac{1}{2}f''(M_t)dt.$$ 

SINCE $dM_t$ TERM IS MG, AND $f''(x) = h^2f(x)$:

$$E(f(M_t)|F_s) = f(M_s) + \frac{1}{2}h^2 E\left(\int_s^t f(M_u)du|F_s\right)$$

$$= f(M_s) + \frac{1}{2}h^2 \int_s^t E(f(M_u)|F_s)du$$

Set $g(t) = E(\exp\{h(M_t - M_s)\}|F_s)$:

$$g(t) = 1 + \frac{1}{2}h^2 \int_s^t g(u)du$$
SOLUTION:

\[ g(t) = \exp\left\{ \frac{1}{2} h^2 (t - s) \right\} \]

IN OTHER WORDS:

\[ E(\exp\{h(M_t - M_s)\}|\mathcal{F}_s) = \exp\left\{ \frac{1}{2} h^2 (t - s) \right\} \]

CHARACTERISTIC FUNCTION ARGUMENT GIVES:

- \( M_t - M_s \) IS INDEPENDENT OF \( \mathcal{F}_s \)
- \( M_t - M_s \) IS \( N(0, t - s) \)