STOCHASTIC INTEGRALS

$$X_t = \begin{matrix}\text{CONTINUOUS}\\ \text{PROCESS}\end{matrix} \begin{cases} S_t: & \text{STOCK PRICE} \\ M_t: & \text{MG} \\ W_t: & \text{BROWNIAN MOTION} \end{cases}$$

$\theta_t =$ PORTFOLIO: $\#X_t$ HELD AT t

DISCRETE TIME: $0 = t_0 < t_1 < \ldots < t_n = t$:

$$P/L_t = \sum_{i<n} \theta_{t_i} \underbrace{(X_{t_{i+1}} - X_{t_i})}_{\Delta X_{t_i}}$$

GRID BECOMES "DENSE": $\max_i \Delta t_i \to 0$

$$P/L_t \to \int_0^t \theta_u dX_u$$

INTEGRAL <u>DEFINED</u> AS LIMIT OF SUMS

PROPERTIES MOSTLY FROM SUMS:

$$\underbrace{\sum_{i<n}(a\theta_{t_i}+b\eta_{t_i})\Delta X_{t_i}}_{\downarrow} = a\underbrace{\sum_{i<n}\theta_{t_i}\Delta X_{t_i}}_{\downarrow} + b\underbrace{\sum_{i<n}\eta t_i\Delta X_{t_i}}_{\downarrow}$$

$$\int_0^t (a\theta_u+b\eta_u)dX_u \quad a\int_0^t \theta_u dX_u \quad + \; b\int_0^t \eta_u dX_u$$

$\Rightarrow$ LINEARITY OK

TIME VARYING INTEGRAL:

$$\int_0^t \theta_u dX_u = \text{limit of}$$

$$\sum_{t_{i+1}\le t}\theta_{t_i}(X_{t_{i+1}}-X_{t_i})$$

Limit in probability

MARTINGALE PROPERTY:

If $X_t = M_t =$ MG:

$$U_t^{(n)} = \sum_{t_{i+1} \le t} \theta_{t_i} \Delta X_{t_i} : \quad \Delta U_{t_{i+1}}^{(n)} = \theta_{t_i} \Delta X_{t_i}$$

If on grid $t_0, t_1, \ldots$:

$$\begin{aligned} E(\Delta U_{t_{i+1}}^{(n)} \mid \mathcal{F}_{t_i}) &= E(\theta_{t_i} \Delta X_{t_i} \mid \mathcal{F}_{t_i}) \\ &= \theta_{t_i} E(\Delta X_{t_i} \mid \mathcal{F}_{t_i}) = 0 \end{aligned}$$

$\Rightarrow U_t^{(n)}$ is $\mathcal{F}_{t_i} -$ MG

Taking limits:

$$E(U_t \mid \mathcal{F}_s) = U_s$$

QUADRATIC VARIATION (Q. V.)

$$U_t^{(n)} = \sum_{t_{i+1} \le t} \theta_{t_i} \Delta X_{t_i}$$

so: $\Delta U_{t_i}^{(n)} = U_{t_{i+1}}^{(n)} - U_{t_i}^{(n)} = \theta_{t_i} \Delta X_{t_i}$

$$(\Delta U_{t_i}^{(n)})^2 = \theta_{t_i}^2 (\Delta X_{t_i})^2$$

Aggregate:

$$\begin{aligned}
\underbrace{[U^{(n)}, U^{(n)}]_t}_{\downarrow} &= \sum_{t_{i+1} \le t} (\Delta U_{t_i}^{(n)})^2 \\
&= \sum_{t_{i+1} \le t} \theta_{t_i}^2 (\Delta X_{t_i})^2 = \underbrace{\sum_{t_{i+1} \le t} \theta_{t_i}^2 \Delta [X, X]_{t_i}}_{\downarrow} \\
[U, U]_t &= \int_0^t \theta_u^2 d[X, X]_u
\end{aligned}$$

If $X_t = W_t =$ B.M. : $d[X, X]_t = dt$

IT FOLLOWS THAT $[U, U]_t = \int_0^t \theta_u^2 du$

DIFFERENTIAL NOTATION

<u>INTEGRAL</u>:

$$\Delta U_{t_i}^{(n)} = \theta_{t_i} \Delta X_{t_i} \text{ vs. } U_t = \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta X_{t_i}$$

becomes

$$dU_t = \theta_t dX_t \text{ vs. } U_t = \int_0^t \theta_s dX_s$$

<u>QUADRATIC VARIATION:</u>

$$(\Delta U_{t_i}^{(n)})^2 = \theta_{t_i}^2 (\Delta X_{t_i})^2 \text{ vs. } [U^{(n)}, U^{(n)}]_t = \sum \theta_{t_i}^2 \Delta [X, X_{t_i}]$$

$$\text{"} \qquad \text{"}$$

$$\Delta [U^{(n)}, U^{(n)}]_{t_i} = \theta_{t_i}^2 \Delta [X, X]_{t_i}$$

becomes:

$$(dU_t)^2 = \theta_t^2 (dX_t)^2 \text{ vs. } [U, U]_t = \int_0^t \theta_u^2 d[X, X]_u$$

$$\text{"} \qquad \text{"}$$

$$d[U, U]_t = \theta_t^2 d[X, X]_t$$

BROWNIAN MOTION:

$$(dW_t)^2 = dt \quad \text{AND} \quad d[U, U]_t = \theta_t^2 dt$$

QUADRATIC COVARIATION:

$$U, Z: \quad [U, Z]_t = \text{limit of } \sum_{t_{i+1} \leq t} \Delta U_{t_i} \Delta Z_{t_i}$$

CASE OF TWO INTEGRALS:

$$U_t = \int_0^t \theta_s dX_s, \quad Z_t = \int_0^t \eta_s dY_s$$

THEN:

$$[U, Z]_t = \int_0^t \theta_s \eta_s d[X, Y]_s$$

BECAUSE

$$\Delta U_{t_i} \Delta Z_{t_i} = \theta_{t_i} \eta_{t_i} \Delta X_{t_i} \Delta Y_{t_i}$$

or

$$\underbrace{dU_t dZ_t}_{d[U,Z]_t} = \theta_t \eta_t \underbrace{dX_t dY_t}_{d[X,Y]_t}$$

IF: $X_t = Y_t = W_t$ THE <u>SAME</u> B.M.:

$$d[U, Z]_t = \theta_t \eta_t dt$$

DETERMINISTIC INTEGRAND

IF θ_t IS NONRANDOM:

$$\int_0^t \theta_s dW_s = \text{ limit of } \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta W_{t_i}$$

$\sum_{t_{i+1} \leq t} \theta_{t_i} \Delta W_{t_i}$:

- LINEAR COBIMNATION OF NORMAL RANDOM VARIABLES IS A NORMAL RANDOM VARIABLE
- MEAN: $E \sum_{t_{i+1} \leq t} \theta_{t_i} \Delta W_{t_i} = 0$
- VARIANCE: Var $(\sum_{t_{i+1} \leq t} \theta_{t_i} \Delta W_{t_i}) = \sum_{t_{i+1} \leq t} \theta_{t_i}^2 \Delta t_i$

IN THE LIMIT:

$\int_0^t \theta_s dW_s$:

- NORMAL RANDOM VARIABLE
- MEAN IS ZERO
- VARIANCE: Var $(\int_0^t \theta_s dW_s) = \int_0^t \theta_s^2 ds$

ITÔ's FORMULA

X_t: CONTINUOUS PROCESS (SOME RESTRICTIONS):

$$\xi(X_t) = \xi(X_0) + \int_0^t \xi'(X_u)dX_u + \frac{1}{2}\int_0^t \xi''(X_u)d[X,X]_t$$

DIFFERENTIAL NOTATION:

$$d\xi(X_t) = \xi'(X_t)dX_t + \frac{1}{2}\xi''(X_t)d[X,X]_t$$

EX: $X_t = W_t =$ B.M.:

$$d\xi(W_t) = \xi'(W_t)dW_t + \frac{1}{2}\xi''(W_t)dt$$

EX: $dX_t = V_t dt + \sigma_t dW_t$ ITÔ PROCESS

or: $X_t = X_0 + \int_0^t V_s ds + \int_0^t \sigma_s dW_s$

First question:

¿What is $d[X,X]_t$?

FIRST: CASE OF EXPLICIT INTEGRATION:

$W_t =$ B.M. ¿WHAT IS $\int_0^t W_s dW_s$?

$$U_t = W_t^2 = \zeta(W_t), \zeta(x) = x^2$$

$$\begin{aligned} dU_t &= \zeta'(W_t)dW_t + \frac{1}{2}\zeta''(W_t)dt \\ &= 2W_t dW_t + dt \end{aligned}$$

$$\text{so:} \quad W_t dW_t = \frac{1}{2}dU_t - \frac{1}{2}dt$$

$$\begin{aligned} >: \quad \int_0^t W_s dW_s &= \frac{1}{2}(U_t - U_0) - \frac{1}{2}\int_0^t ds \\ &= \frac{1}{2}W_t^2 - \frac{1}{2}t \end{aligned}$$

DIFFERENT FROM ORDINARY INTEGRAL:

If $X_t = g(t)$ g' exists, continuous, $g(t) = 0$

$$\begin{aligned} \int_0^t X_s dX_s &= \int_0^t g(s)g'(s)ds \\ &= \frac{1}{2}g(t)^2 \\ &= \frac{1}{2}X_t^2 \end{aligned}$$

ITÔ PROCESS:

$$X_t = X_0 + \underbrace{\int_0^t \nu_s ds}_{Z_t} + \underbrace{\int_0^t \sigma_s dW_s}_{U_t}$$

Grid:

$$\Delta X_{t_i} = \Delta Z_{t_i} + \Delta U_{t_i}$$

so:

$$(\Delta X_{t_i})^2 = (\Delta Z_{t_i})^2 + (\Delta U_{t_i})^2 + 2\Delta Z_{t_i}\Delta U_{t_i}$$

$$\sum(\Delta X_{t_i})^2 = \sum(\Delta Z_{t_i})^2 + \sum(\Delta U_{t_i})^2 + \sum 2\Delta Z_{t_i}\Delta U_{t_i}$$

$$|\Delta Z_{t_i}| = |\int_{t_i}^{t_{i+1}} \nu_s ds| \leq \int_{t_i}^{t_{i+1}} |\nu_s| ds$$

$$\leq \sup_s |\nu_s|(t_{i+1} - t_i) = \sup_s |\nu_s| \Delta t_i$$

$$\sum(\Delta Z_{t_i})^2 \leq (\sup_s |\nu_s|)^2 \sum_i (\Delta t_i)^2$$

$$\leq (\sup_s |\nu_s|)^2 \underbrace{\sup \Delta t_i}_{\to 0} \underbrace{\sum_i \Delta t_i}_{\simeq t} \to 0$$

$\subset$: $[Z, Z]_t = 0$ ALSO: $[Z, U]_t = 0$

ONLY: $[U, U]_t = \int_0^t \sigma_s^2 ds$

ITÔ PROCESS

$$X_t = X_0 + \underbrace{\int_0^t \nu_s d_s}_{Z_t} + \underbrace{\int_0^t \sigma_s dW_s}_{U_t}$$

$$\begin{array}{ccc} d[Z,Z]_t = 0 & d[Z,U]_t = 0 & d[U,U]_t = \sigma_t^2 dt \\ (dZ_t)^2 & dZ_t dU_t & \end{array}$$

USING DIFFERENTIALS:

ANY dt-TERM HAS ZERO Q.V.:

$$(dZ_t)^2 = \nu_t^2 (dt)^2 = 0 \text{ ETC}$$

COMBINING TERMS:

$$\begin{aligned} (dX_t)^2 &= (dZ_t + dU_t)^2 \\ &= (dZ_t)^2 + 2dZ_t dU_t + (dU_t)^2 \\ &= (dU_t)^2 = \sigma_t^2 dt \end{aligned}$$

RIGOROUS:

$$(\Delta X_{t_i})^2 = (\Delta Z_{t_i}^2 + 2\Delta Z_{t_i} \Delta U_{t_i} + (\Delta U_{t_i})^2$$

SUM OVER t_i, TAKE LIMITS, GET SAME RESULT

INTEGRALS WITH RESPECT TO AN ITÔ PROCESS

$$X_t = X_0 + \int_0^t \nu_s ds + \int_0^t \sigma_s dW_s$$

CAN SHOW THAT:

$$\int_0^t \theta_s dX_s = \int_0^t \theta_s \nu_s ds + \int_0^t \theta_s \sigma_s dW_s$$

A NEW ITÔ PROCESS

BACK TO ITÔ'S FORMULA:

$$d\xi(X_t) = \xi'(X_t)dX_t + \frac{1}{2}\xi''(X_t)d[X,X]_t \qquad (*)$$

ITÔ PROCESS:

$$dX_t = \nu_t dt + \sigma_t dW_t$$

SO

$$d[X,X]_t = \sigma_t^2 dt$$

PLUG IN:

$$\begin{aligned} d\xi(X_t) &= \xi'(X_t)(\nu_t dt + \sigma_t dW_t) \\ &\quad + \frac{1}{2}\xi''(X_t)\sigma_t^2 dt \\ &= (\xi'(X_t)\nu_t + \frac{1}{2}\xi''(X_t)\sigma_t^2)dt \\ &\quad + \xi'(X_t)\sigma_t dW_t \end{aligned}$$

EASIER TO REMEMBER (*)...

"PROOF" OF ITÔ'S FORMULA:

$U_t = \xi(X_t)$:

$$\begin{aligned}\Delta U_{t_i} &= \xi(X_{t_{i+1}}) - \xi(X_{t_i}) \\ &= \xi(X_{t_i} + \Delta X_{t_i}) - \xi(X_{t_i}) \\ &= \zeta'(X_{t_i})\Delta X_{t_i} + \frac{1}{2}\zeta''(X_{t_i})\Delta X_{t_i}^2 \\ &\qquad + \frac{1}{3!}\zeta'''(X_{t_i})\Delta X_{t_i}^3 + \cdots\end{aligned}$$

sum up:

$$\begin{array}{ccccc} U_t - U_i = & \underbrace{\sum \zeta'(X_{t_i})\Delta X_{t_i}} & + & \underbrace{\frac{1}{2}\sum \zeta''(X_{t_i})\Delta X_{t_i}^2} \\ & \downarrow & & \downarrow \\ & \displaystyle\int_0^t \zeta'(X_s)dX_s & + & \displaystyle\frac{1}{2}\int_0^t \zeta''(X_s)d[X,X]_s \end{array}$$

OTHER "PROOF":

$$\begin{aligned}dU_t &= \zeta(X_t + dX_t) - \zeta(X_t) \\ &= \zeta'(X_t)dX_t + \frac{1}{2}\zeta''(X_t)\underbrace{(dX_t)^2}_{d[X,X]_t} + \cdots\end{aligned}$$

MULTIVARIATE FORMULA

$$
\begin{aligned}
U_t &= \zeta(X_t, Y_t) \\
dU_t &= \zeta'_x(X_t, Y_t)dX_t + \zeta'_y(X_t, Y_t)dY_t \\
&\quad + \frac{1}{2}\Big\{ \zeta''_{xx}(X_t, Y_t)d[X, X]_t \\
&\qquad + \zeta''_{yy}(X_t, Y_t)d[Y, Y]_t \\
&\qquad + 2\zeta''_{xy}(X_t, Y_t)d[X, Y]_t \Big\}
\end{aligned}
$$

etc.

EXAMPLE: GEOMETRIC BROWNIAN MOTION

$$S_t = S_0 \exp\{\int_0^t \sigma_s dW_s + \int_0^t \left(r_s - \frac{1}{2}\sigma_s^2\right) ds\}$$

SET

- $X_t = \int_0^t \sigma_s dW_s + \int_0^t \left(r_s - \frac{1}{2}\sigma_s^2\right) ds$
- $S_t = f(X_t)$ $(f(x) = S_0 \exp\{x\})$

USE ITÔ'S FORMULA

$$dS_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X,X]_t$$

$f'(x) = f''(x) = f(x)$ AND $d[X,X]_t = \sigma_t^2 dt$, SO:

$$\begin{aligned} dS_t &= f(X_t)dX_t + \frac{1}{2}f(X_t)\sigma_t^2 dt \\ &= S_t dX_t + \frac{1}{2}S_t\sigma_t^2 dt \\ &= S_t\left(dX_t + \frac{1}{2}\sigma_t^2 dt\right) \\ &= S_t\left(\sigma_t dW_t + r_t dt\right) \\ &= S_t\sigma_t dW_t + S_t r_t dt \end{aligned}$$

DIFFERENTIAL REPRESENTATION OF S_t

VASICEK MODEL

$$dR_t = (\alpha - \beta R_t)dt + \sigma dW_t$$

- STEP 1: SET $U_t = R_t - \frac{\alpha}{\beta}$

EQUATION BECOMES:

$$dU_t = -\beta U_t dt + \sigma dW_t$$

- STEP 2: NOTE THAT (FROM ITO'S FORMULA)

$$\begin{aligned} d(\exp\{\beta t\}U_t) &= \exp\{\beta t\}dU_t + U_t d\exp\{\beta t\} \\ &= \exp\{\beta t\}dU_t + U_t\beta\exp\{\beta t\}dt \\ &= \exp\{\beta t\}\left(dU_t + U_t\beta dt\right) \\ &= \exp\{\beta t\}\sigma dW_t \end{aligned}$$

SO

$$\exp\{\beta t\}U_t = U_0 + \int_0^t \exp\{\beta s\}\sigma dW_s$$

OR

$$U_t = \exp\{-\beta t\}U_0 + \int_0^t \exp\{\beta(s-t)\}\sigma dW_s$$

IN OTHER WORDS: U_t IS NORMAL

- MEAN IS $\exp\{-\beta t\}U_0$
- VARIANCE IS

$$\begin{aligned}
&\int_0^t \left(\exp\{\beta(s-t)\}\sigma\right)^2 ds \\
&\quad = \int_0^t \exp\{2\beta(s-t)\}\sigma^2 ds \\
&\quad = \left[\frac{1}{2\beta}\exp\{2\beta(s-t)\}\sigma^2\right]_{s=0}^{s=t} \\
&\quad = \frac{1}{2\beta}\left(1-\exp\{-2\beta t\}\right)\sigma^2
\end{aligned}$$

DEDUCE FOR $R_t = U_t + \frac{\alpha}{\beta}$ THAT

- R_t IS NORMAL
- $E(R_t) = \exp\{-\beta t\}U_0 + \frac{\alpha}{\beta}$
- Var (R_t) =Var (U_t)

LEVY'S THEOREM

IF M_t IS A CONTINUOUS MARTINGALE, $M_0 = 0$, $[M, M]_t = t$ FOR ALL t, THEN M_t IS A CONTINUOUS BROWNIAN MOTION

PROOF: SET $f(x) = \exp\{hx\}$

ITO:

$$\begin{aligned} df(M_t) &= f'(M_t)dt + \frac{1}{2}f''(M_t)d[M, M]_t \\ &= f'(M_t)dt + \frac{1}{2}f''(M_t)dt. \end{aligned}$$

SINCE dM_t TERM IS MG, AND $f''(x) = h^2 f(x)$:

$$\begin{aligned} E(f(M_t)|\mathcal{F}_s) &= f(M_s) + \frac{1}{2}h^2 E\left(\int_s^t f(M_u)du|\mathcal{F}_s\right) \\ &= f(M_s) + \frac{1}{2}h^2 \int_s^t E(f(M_u)|\mathcal{F}_s)du \end{aligned}$$

Set $g(t) = E(\exp\{h(M_t - M_s)\}|\mathcal{F}_s)$:

$$g(t) = 1 + \frac{1}{2}h^2 \int_s^t g(u)du$$

SOLUTION:

$$g(t) = \exp\{\frac{1}{2}h^2(t-s)\}$$

IN OTHER WORDS:

$$E(\exp\{h(M_t - M_s)\}|\mathcal{F}_s) = \exp\{\frac{1}{2}h^2(t-s)\}$$

CHARACTERISTIC FUNCTION ARGUMENT GIVES:

- $M_t - M_s$ IS INDEPENDENT OF $\mathcal{F}_s$
- $M_t - M_s$ IS $N(0, t-s)$