APPROXIMATE NORMALITY

BINOMIAL MODEL: \[ S_{n+1} = \begin{cases} uS_n \\ dS_n \end{cases} \]

LOG SCALE (IID ADDITIVE INCREMENTS):
\[ \log S_n = \log S_0 + X_1 + \ldots + X_n \]
WITH \( X_i = \log(u) \) or \( = \log(d) \)

TWO TIME SCALES

CLOCK TIME: \( T \) – TIME PERIODS: \( n \)
\[ t_0 = 0 \quad t_1 = \frac{T}{n} \quad t_2 = \frac{2T}{n} \quad t_3 = \frac{3T}{n} \quad \ldots \quad t_k = \frac{kT}{n} \]

\( T \) IS FIXED – \( n \) IS A MATTER OF CHOICE

RETURN ON RISK FREE ASSET
(in clock time) \( e^{rT} = e^{\rho n} \) (in time periods)

\[ \text{in other words: } \rho = r \frac{T}{n} \] (1)

\( r \) IS FIXED – \( \rho \) DEPENDS ON \( n \)

RISK NEUTRAL MEASURE PER STEP:
\[ \pi_n(T) = \frac{u - e^\rho}{u - d} = \frac{u - e^{r \frac{T}{n}}}{u - d} \] and \( \pi_n(H) = \frac{e^\rho - d}{u - d} \) (2)
BEHAVIOR OF ADDITIVE INCREMENTS

MEAN:

\[ E(X) = \log(u)\pi(H) + \log(d)\pi(T) \]

TOTAL MEAN:

\[ E(\log(S_n)) = E(X_1) + \ldots + E(X_n) = nE(X) = n(\log(u)\pi(H) + \log(d)\pi(T)) \]

VARIANCE: \( X = \log d + (\log u - \log d)I_{\{H\}} \), and so

\[ \text{Var} (X) = (\log u - \log d)^2 \text{Var} (I_{\{H\}}) = (\log u - \log d)^2\pi(H)\pi(T) \]

TOTAL VARIANCE:

\[ \text{Var} (\log(S_n)) = \text{Var} (X_1) + \ldots + \text{Var} (X_n) = n \text{Var} (X_1) = n(\log u - \log d)^2\pi(H)\pi(T) \]
WE WISH TO KEEP TOTAL MEAN, VARIANCE CONSTANT IN CLOCK TIME

\[ \nu T = E(\log S_n) \]
\[ = n(\log(u)\pi(H) + \log(d)\pi(T)) \]  
\[ \sigma^2 T = \text{Var} (\log(S_n)) \]
\[ = n(\log u - \log d)^2\pi(H)\pi(T) \]  

\( \sigma \) OR \( \sigma^2 \) IS VOLATILITY IN CLOCK TIME

NEED TO USE: \( \nu \approx r - \frac{1}{2}\sigma^2 \)

EQUATIONS (1)-(4) DEFINE A BINOMIAL TREE \((\rho, u, d, \pi(H), \pi(T))\) ON THE BASIS OF:

- VOLATILITY PER UNIT CLOCK TIME: \(\sigma^2\)
- INTEREST PER UNIT CLOCK TIME: \(r\)
- \# OF UNITS OF CLOCK TIME: \(T\)
- \# OF TIME PERIODS IN COMPUTATION: \(n\)
AN APPROXIMATION FOR THE CASE $r = \rho = 0$
(THE DISCOUNTED PROCESS)

UP AND DOWN STEPS:

$$u = 1 + \sqrt{\frac{\sigma^2 T}{n}} \text{ AND } d = 1 - \sqrt{\frac{\sigma^2 T}{n}}$$

RISK NEUTRAL PROBABILITIES:

$$\pi_n(T) = \frac{u - e^\rho}{u - d} = \frac{1}{2} \text{ AND } \pi_n(H) = \frac{e^\rho - d}{u - d} = \frac{1}{2}$$

WE SHOW THAT EQUATIONS (3)-(4) ARE APPROXIMATELY SATISFIED

WILL USE THIS APPROXIMATE BINOMIAL TREE
APPROXIMATION TO CONDITION (4):

\[
\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots
\]

\[
\log(u) = \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots
\]

\[
\log(d) = -\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots
\]

AND SO:

\[
\text{Var} (\log(S_n)) = n (\log u - \log d)^2 \pi(H)\pi(T)
\]

\[
= n \cdot \frac{1}{4} \left( \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots \right) \times \ldots
\]

\[
= \frac{1}{4} \left( 2\sqrt{\frac{\sigma^2 T}{n}} + \frac{1}{n\sqrt{n}} \times \ldots \right)^2
\]

\[
= \sigma^2 T + \frac{1}{n} \times \ldots
\]
ABOUT EQUATION (3):

\[
\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots
\]

\[
\log(u) = \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots
\]

\[
\log(d) = -\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots
\]

AND SO:

\[
E(\log(S_n)) = n(\log u - \log d)^2 \pi(H)\pi(T)
\]

\[
= \frac{1}{2n} \left( \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots 
+ (-\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n\sqrt{n}} \times \ldots) \right)
\]

\[
= -\frac{1}{2} \sigma^2 T + \frac{1}{n\sqrt{n}} \times \ldots
\]

AS PREDICTED
HOW MUCH DO OUR RESULTS DEPEND ON \( n \)?

TRYING THE MATTER OUT IN R

```r
M <- 1000   # number of simulation steps
sigma <- .2  # clock time volatility
T <- 1       # clock time duration
S0 <- 100    # initial value
piH <- 1/2   # risk neutral probability
n <- 10      # steps
u <- 1 + sqrt(T*sigma^2/n)  # up step
d <- 1 - sqrt(T*sigma^2/n)  # down step
H <- rbinom(M,n,piH)         # simulation
logS <- log(S0) + log(u)*H + log(d)*(n-H)
par(mfrow=c(2,1))             # check this command out!
hist(logS,freq=F)
# try again with a larger number of steps
n <- 1000
# define u, d, H, logS as above, with new n
hist(logS,freq=F)
```
THE DISTRIBUTION OF $\log S_T$ STABILIZES
THE CENTRAL LIMIT PHENOMENON

**THEOREM:** SUPPOSE THAT

- $X_i, i = 1, \ldots, n$ ARE IID $P_n$

  (DISTRIBUTION CAN DEPEND ON $n$)

- $n \text{ Var}_n (X) \to \gamma^2$ AS $n \to \infty$

THEN

$$\sum_{i=1}^{n} X_i - nE_n(X) \xrightarrow{\mathcal{L}} N(0, \gamma^2)$$

IN WORDS:

$\sum_{i=1}^{n} X_i - nE_n(X)$ CONVERGES IN LAW TO $N(0, \gamma^2)$

THAT IS TO SAY:

THE DISTRIBUTION OF $\sum_{i=1}^{n} X_i - nE_n(X)$ IS APPROXIMATELY NORMAL $N(0, \gamma^2)$

DENSITY OF THE NORMAL DISTRIBUTION $N(\mu, \gamma^2)$

$$\frac{1}{\gamma} \phi \left( \frac{x - \mu}{\gamma} \right)$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\}$$
IN OUR CASE

$$\log(S_T) - \log(S_0) = \sum_{i=1}^{n} X_i$$

$$\gamma^2 = \sigma^2 T$$

$$E(\log(S_T) - \log(S_0)) = nE_n(X) \approx -\frac{1}{2} \sigma^2 T$$

SO THAT

$$\log(S_T) - \left( \log(S_0) - \frac{1}{2} \sigma^2 T \right)$$

IS APPROXIMATELY NORMAL $N(0, \sigma^2 T)$

OR: $\log(S_T)$ IS APPROXIMATELY NORMAL

$$N(\log(S_0) - \frac{1}{2} \sigma^2 T, \sigma^2 T)$$
SUPERIMPOSING THE NORMAL CURVE ON THE HISTOGRAM

n <- 10       # steps
u <- 1 + sqrt(T*sigma^2/n)  # up step
d <- 1 - sqrt(T*sigma^2/n)  # down step
H <- rbinom(M,n,piH)         # simulation
logS <- log(S0) + log(u)*H + log(d)*(n-H)
par(mfrow=c(2,1))            # check this command out!
hist(logS,freq=F)
# compare to normal distribution
xpoints<-c(-30:30)/10
mu<-log(S0) - (sigma^2*T)/2
gamma<-sqrt(sigma^2*T)
xpoints<-c(-30:30)/10
xpoints<-mu+sigma*xpoints
density<-dnorm(xpoints,mean=mu,sd=gamma)
lines(xpoints,density)
# try again with a larger number of steps
n <- 1000
# define u, d, H, logS as above, with new n
hist(logS,freq=F)
# mu, gamma, xpoints stay the same
lines(xpoints,density)
NORMAL CURVE SUPERIMPOSED ON HISTOGRAMS
THE CLASSICAL CENTRAL LIMIT THEOREM

(A digression. Just so you know.)

SETUP:

$Y_1, \ldots, Y_n$ ARE IID, $E(Y) = 0$ AND $\text{Var}(Y) = \gamma^2$

THEN:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i \xrightarrow{L} N(0, \gamma^2)$$

PROOF:

TAKE $X_i = \frac{1}{\sqrt{n}} Y_i$

IN EARLIER THEOREM

RESULT FOLLOWS
BEHAVIOR OF OPTIONS PRICES

STEP 1: CONTINUOUS FUNCTIONS

THEOREM: IF

• $Z_n \xrightarrow{\mathcal{L}} Z$ AS $n \to \infty$

• $x \to h(x)$ IS A CONTINUOUS FUNCTION

THEN $h(Z_n) \xrightarrow{\mathcal{L}} h(Z)$ AS $n \to \infty$

EXAMPLE

$$Z_n = \log(S_T^{(n)}) \xrightarrow{\mathcal{L}} Z = \mathcal{N} \left( \log(S_0) - \frac{1}{2}\sigma^2 T, \sigma^2 T \right)$$

CONTINUOUS FUNCTION #1:

$$S_T^{(n)} = \exp\{Z_n\} \xrightarrow{\mathcal{L}} S_T^{(\infty)} = \exp\{Z\}$$

CONTINUOUS FUNCTION #2:

$$V_T^{(n)} = (S_T^{(n)} - e^{-rT}K)^+ \xrightarrow{\mathcal{L}} (S_T^{(\infty)} - e^{-rT}K)^+$$

CHECK THIS IN R!
**BEHAVIOR OF OPTIONS PRICES**

**STEP 2: THE DOMINATED CONVERGENCE THEOREM**

**SETUP:**
- \((T_n, U_n) \xrightarrow{\mathcal{L}} (T, U)\) AS \(n \to \infty\)
- \(|T_n| \leq U_n\) a.s., FOR ALL \(n\)
- \(E(U_n) \to E(U)\) AS \(n \to \infty\)

**THEOREM:**

**UNDER THESE CONDITIONS:**
\[ E(T_n) \to E h(T) \] AS \(n \to \infty\)

- **CHECK THAT THEOREM IN SHREVE IS SPECIAL CASE**

- **GENERAL THEOREM:**
  - See Billingsley: *Probability and Measure*
  - Deduce using Skorokhod embedding
  - For final: need only to be able to use above Theorem
BEHAVIOR OF OPTIONS PRICES

STEP 3: COMBINE THEOREMS

TAKE: $T_n = (S_T^{(n)} - e^{-rT}K)^+ \text{ AND } U_n = S_T^{(n)}$

WE KNOW:

• $(T_n, U_n) \xrightarrow{\mathcal{L}} (T, U) \text{ AS } n \to \infty$
• $|T_n| \leq U_n \text{ a.s., FOR ALL } n: (S - e^{-rT}K)^+ \leq S$

WE NEED TO ESTABLISH

$$E(U_n) \to E(U) \text{ AS } n \to \infty \quad (5)$$

IF THIS IS THE CASE, WE CAN CONCLUDE THAT

$$n \text{ step options price } = E(S_T^{(n)} - e^{-rT}K)^+ \to E(S_T^{(\infty)} - e^{-rT}K)^+ \quad (6)$$

WHERE

$$S_T^{(\infty)} = \exp\{Z\}$$

AND

$$Z = N \left( \log(S_0) - \frac{1}{2}\sigma^2T, \sigma^2T \right)$$
COMPUTATION OF EXPECTED VALUES

\[
\log S_T = \log S_0 - \frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} N(0, 1)
\]

\[
E[f(S_T)] = E[f(\exp\{\log S_0 - \frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} N(0, 1)\})]
\]

\[
= E[f(S_0 \exp\{-\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} Z\})]
\]

\[
= \int_{-\infty}^{+\infty} f(S_0 \exp\{-\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} z\}) \phi(z) \, dz
\]

(7)

where \( \phi(z) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} z^2\} \)
IN PARTICULAR: \( f(s) = s \):

\[
E[S_T] = \int_{-\infty}^{+\infty} S_0 \exp\left\{ -\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} z \right\} \phi(z) \, dz
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S_0 \exp\left\{ -\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} z - \frac{1}{2} z^2 \right\} \, dz
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S_0 \exp\left\{ -\frac{1}{2} \left( z - \sqrt{\sigma^2 T} \right)^2 \right\} \, dz
\]

\[
= S_0 \int_{-\infty}^{+\infty} \phi(z - \sqrt{\sigma^2 T}) \, dz
\]

\[
= S_0 \int_{-\infty}^{+\infty} \phi(u) \, du \quad (u = z - \sqrt{\sigma^2 T})
\]

\[
= S_0
\]

IT FOLLOWS THAT EQUATION (5) IS SATISFIED
THE BLACK-SCHOLES-MERTON FORMULA

• THE OPTIONS PRICE FOR LARGE $n$ IS

$$E(\tilde{S}_T^{(\infty)} - e^{-rT}K)^+$$

• CAN COMPUTE IT EXPLICITELY USING EQUATION (7)

• THIS IS THE B-S-M FORMULA FOR THE PRICE OF A CALL OPTION

• YOU DON’T NEED TO USE A TREE IN THIS CASE
CONTINUOUS MARTINGALES

TWO CONTINUITIES: • TIME ITSELF:

\[ M_t, \quad 0 \leq t \leq T \quad (\text{or } 0 \leq t < \infty) \]

PROCESS PATH:

\[ t \rightarrow M_t = M_t(\omega) \quad \text{CONTINUOUS FUNCTION OF TIME} \]
(ADDITIVE) BROWNIAN MOTION $W_t$

1. $W_0 = 0$
2. $t \rightarrow W_t$ IS CONTINUOUS
3. HAS INDEPENDENT INCREMENTS
4. $W_{t+s} - W_s \sim N(0, t)$

PICTURE OF (3):

\[
\begin{array}{cccc}
\Delta W_{t_0} & \Delta W_{t_1} & \Delta W_{t_2} & \text{INDEPENDENT} \\
\end{array}
\]

ADDITIVE PROPERTY (4):

\[\Delta W_{t_0} \sim N(0, t_1), \; \Delta W_{t_1} \sim N(0, t_2 - t_1)\]

**DELETE** $t_1$ : $W_{t_2} - W_{t_0} = \frac{\Delta W_{t_0} + \Delta W_{t_1}}{N(0, t_1) + N(0, t_2 - t_1)}$

BY INDEP: $N(0, t_2)$
\[(3) + (4) \iff W_t \text{ IS A MARTINGALE}\]

\[
E(W_{t+s} \mid \mathcal{F}_s) = E(W_{t+s} - W_s + W_s \mid \mathcal{F}_s)
\]
\[
= E(W_{t+s} - W_s \mid \mathcal{F}_s) + W_s
\]
\[
= E(W_{t+s} - W_s) + W_s \quad \text{(independence)}
\]
\[
= 0 \quad \text{since } W_{t+s} - W_s \sim N(0, t)
\]
\[
= W_s
\]
THE BLACK-SCHOLES MODEL: MULTIPLICATIVE BROWNIAN MOTION

\[ \tilde{S}_t = \tilde{S}_0 \times \exp(\sigma W_t - \frac{1}{2}\sigma^2 t) \]

EVOLUTION:

\[
\tilde{S}_t = \tilde{S}_0 \times \exp(\sigma W_u - \frac{1}{2}\sigma^2 u) \quad \text{\{ independent increment \}}
\]

\[
\times \exp(\sigma (W_t - W_u) - \frac{1}{2}\sigma^2 (t - u))
\]

\[
= \tilde{S}_u \times \exp(\sigma N(0, t - u) - \frac{1}{2}\sigma^2 (t - u))
\]

\[
\approx \tilde{S}_u \times \exp(\alpha Z - \frac{1}{2}\alpha^2) \quad \alpha^2 = \sigma^2 (t - u) \quad Z \sim N(0, 1)
\]

MARTINGALE:

\[
E(\tilde{S}_t \mid \mathcal{F}_u) = \tilde{S}_u E(\exp(\sigma Z - \frac{1}{2}\sigma^2) \mid \mathcal{F}_u)
\]

\[
= \tilde{S}_u E(\exp(\sigma Z - \frac{1}{2}\sigma^2)) \text{ BY INDEPENDENCE}
\]

\[
= \tilde{S}_u \times 1 \quad \text{(NORMAL)}
\]

\[
= \tilde{S}_u
\]
CLT FOR THE WHOLE PROCESS

\[ t_0 = 0 \quad t_1 = \frac{1}{n} \quad t_2 = \frac{2}{n} \quad t_3 = \frac{3}{n} \quad t_k = \frac{k}{n} \]

STOCK PRICE PROCESS

\[ \log(\tilde{S}_t^{(n)}) - \log(S_0) = \sum_{t_i \leq t} X_i \]

CONVERGENCE: AS \( n \to \infty \):

\[ \log(\tilde{S}_t^{(n)}) \xrightarrow{\mathcal{L}} \log(S_t) = \log(S_0) + \sigma W_t - \frac{1}{2} \sigma^2 t \]

GEOMETRIC BROWNIAN MOTION
APPLICATION TO OPTIONS

CONTINUOUS FUNCTIONALS

• \( x = \{x_t, 0 \leq t \leq T\} \) A REALIZATION OF THE PROCESS

• \( x \rightarrow h(x) \) TAKES REAL VALUES

• \( x \rightarrow h(x) \) IS CONTINUOUS:

\[
\sup_{0 \leq t \leq T} |x^{(n)}_t - x_t| \rightarrow 0 \implies h(x^{(n)}) \rightarrow h(x_t)
\]

FOR h CONTINUOUS:

\[
h(\log(\tilde{S}^{(n)}_t)) \xrightarrow{\mathcal{L}} h(\log(\tilde{S}_t))
\]

OR

\[
h(\tilde{S}^{(n)}_t) \xrightarrow{\mathcal{L}} h(\tilde{S}_t)
\]

EXAMPLE OF MEANINGFUL LIMIT:

\[
h(x) = \max_{0 \leq t \leq T} x_t
\]

LOOKBACK OPTIONS