AMERICAN OPTIONS

The non-path dependent case

Payoff: $G_\tau$, at time $\tau$ decided by owner, $0 \leq \tau \leq T$
For example: $G_n = g(S_\tau)$ (non-path dependent)
Put: $g(s) = (K - s)^+$
In general: $G_n$ is ($\mathcal{F}_n$)-adapted (= path dependent)

$G_n$: “Intrinsic value process”

Assume $G_n \geq 0$, otherwise no point in exercising. If necessary, take $G_n(s) \leftarrow G_n(s)^+$

VALUE OF OPTION at time $n$: $V_n$

• Boundary condition: If option has not been exercised before time $T$, then $V_T = G_T$

• Recursion: If option has not been exercised before time $n < T$, there is a choice:
  
  (i) Exercise now. Value: $G_n$
  (ii) Wait. Value: $e^{-r}E_\pi[V_{n+1} \mid \mathcal{F}_n]$

Decision at time $n$: pick the higher value

Overall value at $n$:

$$V_n = \max(G_n, e^{-r}E_\pi[V_{n+1} \mid \mathcal{F}_n])$$

Iterate until $n = 0$: this gives price $V_0$ at time 0
THE NON-PATH DEPENDENT CASE

\[ G_n = g(S_n) \]

\( V_T = g(S_T) \) at final time: set \( v_T(S_T) = g(S_T) \)

If \( S_n \) is \( \pi \)-Markov:

Suppose by induction that \( V_{n+1} = v_{n+1}(S_{n+1}) \)

\[
e^{-r} E_\pi[V_{n+1} \mid \mathcal{F}_n] = e^{-r} E_\pi[v_{n+1}(S_{n+1}) \mid \mathcal{F}_n] = e^{-r} E_\pi[v_{n+1}(S_{n+1}) \mid S_n]
\]

Hence:

\[
V_n = \max(G_n, e^{-r} E_\pi[V_{n+1} \mid \mathcal{F}_n]) = \max(g(S_n), e^{-r} E_\pi[v_{n+1}(S_{n+1}) \mid S_n]) = v_n(S_n) \text{ by definition}
\]

BINOMIAL MODEL: \( S_{n+1} = u \) or \( d \times S_n \):

\[
E_\pi[v_{n+1}(S_{n+1}) \mid S_n = s] = v_{n+1}(us)\pi(H) + v_{n+1}(ds)\pi(T)
\]

And so:

\[
v_n(s) = \max(g(s), e^{-r}[v_{n+1}(us)\pi(H) + v_{n+1}(ds)\pi(T)])
\]
EXAMPLE OF BINOMIAL TREE

structure: $S_0 = 2, u = 4, d = \frac{1}{2}, e^r = \frac{5}{4}$
(hence: $\pi(H) = \pi(T) = \frac{1}{2}$)

option: $G_n = g(S_n) = (5 - S_n)^+$, times: $n = 0, 1, 2$

\[ S_1 = 8 \quad S_2 = 16 \quad \tau = 2 \text{ or } \infty \]
\[ G_1 = 0 \quad G_2 = 0 \]

\[ S_2 = 4 \]
\[ G_2 = 1 \quad \tau = 2 \]

\[ S_1 = 2 \quad S_2 = 1 \]
\[ G_1 = 3 \quad G_2 = 4 \quad \tau = 1 \]

\[ v_n(s) = \max(g(s), e^{-r}[v_{n+1}(us)\pi(H) + v_{n+1}(ds)\pi(T)]) \]

\[ v_1(S_1 = 8) = \max(0, \frac{4}{5}[0 \times \frac{1}{2} + 1 \times \frac{1}{2}]) = \max(0, \frac{2}{5}) = 0.4 \]

\[ v_1(S_1 = 2) = \max(3, \frac{4}{5}[1 \times \frac{1}{2} + 4 \times \frac{1}{2}]) = \max(3, 2) = 3 \]

\[ v_0(S_0 = 4) = \max(1, \frac{4}{5}[0.4 \times \frac{1}{2} + 3 \times \frac{1}{2}]) = \max(1, 1.36) = 1.36 \]
STopping times $\tau$

$\tau$: random time

observable when it occurs:

$$\{\tau(\omega) = n\} \in \mathcal{F}_n \quad \text{or} \quad \{\tau(\omega) \leq n\} \in \mathcal{F}_n$$

EX:

$$\tau = \inf\{t : S_t \geq x\}$$

= first time stock price crosses barrier $x$

NOT EX:

$$\inf\{t \leq \tau : S_t = \max_{0 \leq t \leq T} S_t\}$$

= time when stock market reaches max NOT observable at the time

Make sure you understand connection to Shreve’s Definition 4.3.1 (p. 97)
STopped Processes

\( \tilde{S}_t = \text{stock price} \)

\( \tilde{V}_t = \tilde{S}_{\tau \wedge t} \quad (\tau \wedge t = \text{MIN}(\tau, t)) \)

= hold \( \tilde{S} \) until \( \tau \)

\[ = \tilde{S}_0 + \sum_{u=0}^{t-1} \theta(u) \Delta \tilde{S}_u \]

\( \theta(u) = \begin{cases} 1 & \text{if } u < \tau \\ 0 & \text{if } u \geq \tau \end{cases} \)

If \( \tilde{S}_t \) is \( \pi - \text{MG} \), then \( \tilde{V}_t \) is \( \pi - \text{MG} \)

What is true:

\[ E(\tilde{S}_{\tau \wedge t} \mid \mathcal{F}_s) = \tilde{S}_{\tau \wedge s} \quad \text{“optional stopping”} \]

What is not true, in general:

\[ E(\tilde{S}_\tau \mid \mathcal{F}_s) = \tilde{S}_{\tau \wedge s} \quad \text{“doubling strategies”} \]
SUB- AND SUPERMARTINGALES

A PROCESS $X_t$ IS A

- **SUB-MG** IF $E(X_t \mid \mathcal{F}_s) \geq X_s$ WHEN $t \geq s$

- **SUPER-MG** IF $E(X_t \mid \mathcal{F}_s) \leq X_s$ WHEN $t \geq s$

$X_t$ SUPER-MG IFF $-X_t$ SUB-MG

TIME DISCRETE CASE: $X_n$ SUPER-MG SAME AS $E(X_{n+1} \mid \mathcal{F}_n) \leq X_n$, ALL $t$

A STOPPED SUPER-MG IS A SUPER-MG (SAME ARGUMENT AS FOR MG)
THE DOOB DECOMPOSITION

IF $\tilde{V}_n$ SUPER-MG: $\tilde{V}_n = \tilde{M}_n + \tilde{A}_n$

WHERE

• $\tilde{M}_n$ IS MG
• $\tilde{A}_n$ IS NONINCREASING ($\tilde{A}_{n+1} \leq \tilde{A}_n$)
• $\tilde{A}_0 = 0$
• $\tilde{A}_{n+1}$ IS $\mathcal{F}_n$-MEASURABLE for all $n$
• THE DECOMPOSITION IS UNIQUE

CONSEQUENCE IN COMPLETE MARKET

• $V_n$ CAN BE FINANCED, WITH DIVIDEND
• $M_n$ HAS EXACT SELF FINANCING STRATEGY
• DIVIDEND FROM TIME $n$ TO $n+1$ (DISCOUNTED SCALE):

$$\Delta \tilde{V}_n - P/L \text{ FOR HEDGE} = \Delta \tilde{V}_n - \Delta \tilde{M}_n$$
$$= \Delta \tilde{A}_n$$

YOU EVEN KNOW THE DIVIDEND AT TIME $n$
PROOF OF THE DOOB DECOMPOSITION

\( \tilde{V}_n \): SUPER-MG

DEFINE: \( \Delta \tilde{A}_n = E_\pi[\Delta \tilde{V}_n \mid \mathcal{F}_n] \)

AND: \( \tilde{A}_n = \Delta \tilde{A}_0 + \cdots + \Delta \tilde{A}_{n-1} \)

AND: \( \tilde{M}_n = \tilde{V}_n - \tilde{A}_n \)

CHECK CONDITIONS

- \( E_\pi[\Delta \tilde{M}_n \mid \mathcal{F}_n] = E_\pi[\Delta \tilde{V}_n \mid \mathcal{F}_n] - E_\pi[\Delta \tilde{A}_n \mid \mathcal{F}_n] = 0 \)

(hence \( \tilde{M}_n \) is MG)

- \( \Delta \tilde{A}_n \leq 0 \) since \( \tilde{V}_n \) is SUB-MG

- \( \Delta \tilde{A}_0 = 0 \) by definition

- \( \tilde{A}_n \) is \( \mathcal{F}_n \)-measurable by definition

- **UNIQUENESS**: do it as exercise

QED
SUPERMARTINGALES AND AMERICAN OPTIONS

$V_n$: VALUE OF AMERICAN OPTION WITH INTRINSIC VALUE PROCESS $G_n$

$$V_n = \max(G_n, e^{-r} E_\pi [V_{n+1} \mid F_n])$$

DISCOUNTED:

$$\tilde{V}_n = e^{-rn} \max(G_n, e^{-r} E_\pi [V_{n+1} \mid F_n])$$

$$= \max(\tilde{G}_n, E_\pi [\tilde{V}_{n+1} \mid F_n])$$

$$\leq E_\pi [\tilde{V}_{n+1} \mid F_n]$$

$\tilde{V}_n$ IS A SUPERMARTINGALE; $\tilde{V}_n \geq \tilde{G}_n$ for all $n$

WE SHALL SEE A CONVERSE:

$V_n$ IS THE SMALLEST SUPERMARTINGALE SATISFYING $\tilde{V}_n \geq \tilde{G}_n$ for all $n$
THE CONVERSE

\[ \widetilde{V}_n = \max(\widetilde{G}_n, E_\pi[\widetilde{V}_{n+1} | \mathcal{F}_n]), \text{ AND } \widetilde{V}_T = \widetilde{G}_T \]

LET \( \widetilde{Y}_n \) ANOTHER SUPERMARTINGALE,

\[ \widetilde{V}_n \geq \widetilde{Y}_n \geq \widetilde{G}_n \text{ FOR ALL } n \]

WILL SHOW \( \widetilde{Y}_n = \widetilde{V}_n \) FOR ALL \( n \)

INDUCTION: \( \widetilde{Y}_T = \widetilde{V}_T, \) and

Suppose \( \widetilde{Y}_{n+1} = \widetilde{V}_{n+1} \). Then

\[ \widetilde{V}_n = \max(\widetilde{G}_n, E_\pi[\widetilde{Y}_{n+1} | \mathcal{F}_n]) \]
\[ \leq \max(\widetilde{G}_n, \widetilde{Y}_n) \text{ since } \widetilde{Y}_n \text{ super-mg} \]
\[ \leq \widetilde{Y}_n \text{ since } \widetilde{Y}_n \geq \widetilde{G}_n \]

QED

COROLLARY: IF \( \widetilde{Z}_n \) IS ANOTHER SUPERMARTINGALE, \( \widetilde{Z}_n \geq \widetilde{G}_n \) FOR ALL \( n \), THEN: \( \widetilde{Z}_n \geq \widetilde{V}_n \) FOR ALL \( n \)

PROOF: SET \( \widetilde{Y}_n = \min(\widetilde{V}_n, \widetilde{Z}_n) \) (SO \( \widetilde{V}_n \geq \widetilde{Y}_n \geq \widetilde{G}_n \))

\[ E_\pi[\widetilde{Y}_{n+1} | \mathcal{F}_n] \leq \begin{cases} E_\pi[\widetilde{V}_{n+1} | \mathcal{F}_n] \leq \widetilde{V}_n \\ E_\pi[\widetilde{Z}_{n+1} | \mathcal{F}_n] \leq \widetilde{Z}_n \end{cases} \]

SO \( E_\pi[\widetilde{Y}_{n+1} | \mathcal{F}_n] \leq \min(\widetilde{V}_n, \widetilde{Z}_n) = \widetilde{Y}_n: \widetilde{Y}_n \text{ IS SUPER-MG} \) QED
STopping Times and American Options

\[ \tilde{V}_n = \max(\tilde{G}_n, E_\pi[\tilde{V}_{n+1} \mid \mathcal{F}_n]) \]

Define

\[ \tau = \min\{n : \tilde{V}_n = \tilde{G}_n \} \land T \]

\( \tilde{V}_{n \land \tau} \) is a martingale:

On the set \( A = \{\tau > n\} \) \((\in \mathcal{F}_n)\):

\[ \tilde{V}_{n \land \tau} I_A = \tilde{V}_n I_A = \max(\tilde{G}_n, E_\pi[\tilde{V}_{n+1} \mid \mathcal{F}_n]) I_A = E_\pi[\tilde{V}_{n+1} \mid \mathcal{F}_n] I_A = E_\pi[\tilde{V}_{n+1} I_A \mid \mathcal{F}_n] = E_\pi[\tilde{V}_{(n+1) \land \tau} I_A \mid \mathcal{F}_n] \]

On the complement \( A^c = \{\tau \leq n\} \):

\[ E_\pi[\tilde{V}_{(n+1) \land \tau} I_{A^c} \mid \mathcal{F}_n] = E_\pi[\tilde{V}_{n \land \tau} I_{A^c} \mid \mathcal{F}_n] = \tilde{V}_{n \land \tau} I_{A^c} \]

Add the two terms to show \( \tilde{V}_{n \land \tau} \) is MG

In particular:

\[ \tilde{V}_0 = E_\pi[\tilde{V}_\tau] = E_\pi[\tilde{G}_\tau] \]
STOPPING TIMES AND AMERICAN OPTIONS

$S_n = \text{set of stopping times } \tau \text{ which take values in } \{n, \ldots, T\}$

Define: $\tilde{U}_n = \max_{\tau \in S_n} E_\pi[\tilde{G}_\tau \mid \mathcal{F}_n]$

THEOREM: $\tilde{U}_n = \tilde{V}_n$

PROOF: For optimal stopping time $\tau$:

$\tau(\omega) = n \text{ on } \{\omega : \tilde{G}_n(\omega) > \max_{\tau \in S_{n+1}} E_\pi[\tilde{G}_\tau \mid \mathcal{F}_n](\omega)\}$

$\tau(\omega) > n \text{ on } \{\omega : \tilde{G}_n(\omega) < \max_{\tau \in S_{n+1}} E_\pi[\tilde{G}_\tau \mid \mathcal{F}_n](\omega)\}$

Otherwise redefine $\tau$

Also:

$$\max_{\tau \in S_{n+1}} E_\pi[\tilde{G}_\tau \mid \mathcal{F}_n] = E_\pi[\max_{\tau \in S_{n+1}} E_\pi(\tilde{G}_\tau \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n]$$

$$= E_\pi[\tilde{U}_{n+1} \mid \mathcal{F}_n]$$

Therefore: $\tilde{U}_n = \max(\tilde{G}_n, E_\pi[\tilde{U}_{n+1} \mid \mathcal{F}_n])$

Also: $\tilde{U}_T = \tilde{G}_T \text{ since } S_T = \{T\}$

CONCLUSION:

$\tilde{U}_n$ SATISFIES THE DEFINITION OF $\tilde{V}_n$

QED
\( \tilde{V}_n \) CAN BECOME STRICT SUBMARTINGALE
IF OPTION IS NOT EXERCISED AT \( \tau \)

\[
\tilde{V}_n = \max (\tilde{G}_n, E_\pi [\tilde{V}_{n+1} \mid \mathcal{F}_n])
\]

\[
\tau = \min \{ n : \tilde{V}_n = \tilde{G}_n \} \land T
\]

On the set \( \{ \omega : \tau(\omega) = N \} \cap \{ \tilde{G}_N > E_\pi [\tilde{V}_{N+1} \mid \mathcal{F}_N] \} \):

\[
E_\pi [\tilde{V}_{N+1} \mid \mathcal{F}_N] < \tilde{V}_N
\]

(A tautology...)
CONVEX EUROPEAN OPTIONS

Let \( s \to g(s) \) be convex, \( \geq 0 \), with \( g(0) = 0 \), say \( g(s) = (s - X)^+ \)

\( S_n \) is a nonnegative security price

\[
e^{-r} g(S_{n+1}) = e^{-r} g(S_{n+1}) + (1 - e^{-r}) g(0) \geq g(e^{-r} S_{n+1} + (1 - e^{-r}) 0) = g(e^{-r} S_{n+1})
\]

, and so

\[
E_\pi[e^{-r} g(S_{n+1}) | \mathcal{F}_n] \geq E_\pi[g(e^{-r} S_{n+1}) | \mathcal{F}_n] \geq g(S_n)
\]

by Jensen’s inequality, since

\[
E_\pi[e^{-r} S_{n+1} | \mathcal{F}_n] = e^{r n} E_\pi[\tilde{S}_{n+1} | \mathcal{F}_n] = e^{r n} \tilde{S}_n = S_n
\]

Thus: \( X_n = e^{-r n} g(S_n) \) is a submartingale

Optional stopping: for any stopping time \( \tau \geq n \):

\[
E_\pi[X_{\tau \wedge n+1}] \geq E_\pi[X_\tau]
\]

By induction: the value of the American option is

\[
\sup_{\tau \in \mathcal{S}_0} E_\pi[e^{-r \tau} g(S_\tau)] = E_\pi[e^{-r T} g(S_T)]
\]

Same as for European option
DOUBLING STRATEGIES

\[ S_n = \sum_{i=1}^{n} Y_i \quad Y_i = \begin{cases} +1 & \text{\$} \\ -1 & \text{\$} \end{cases} \quad \pi = \frac{1}{2} \]

\[ \tau = \inf \{ t : S_t = A \} \quad A > 0 \]

The facts:
1) \( \pi(\tau < \infty) = 1 \)
2) \( A = E_\pi(S_\tau) \neq S_0 = 0 \)

Realistic version: credit constraint

\[ \tau = \inf t : \begin{cases} S_t = A \text{ retire} \\ S_t = -B \text{ go bankrupt} \end{cases} \]

WHY CALLED DOUBLING STRATEGIES?

Take gamble \# i at time \( t_i = 1 - \frac{1}{2^i} \)

\# \$ in play goes to infinity in fixed time interval
THE ODDS

\[ 0 = ES_{\tau \wedge n} \Rightarrow ES_{\tau} \quad \text{as } n \to \infty \]

— use dominated convergence:

\[ |S_{\tau \wedge n}| \leq \max(A, B) \]

— fails for \( B = +\infty \) (no constraint)

\[ 0 = ES_{\tau} \]
\[ = AP(S_{\tau} = A) - BP(S_{\tau} = B) \]

Also: \( P(S_{\tau} = A) + P(S_{\tau} = -B) = 1 \)

like binomial tree:

\[ P(S_{\tau} = A) = \frac{B}{A + B} \]
\[ (\to 1 \text{ for credit } \to +\infty) \]
STopping Time Description of Martingales

\[ \tilde{S}_t \text{ is } (\mathcal{F}_t), \pi - MG \uparrow \]

For all bounded stopping times \( \tau \):

\[ E_{\pi} \tilde{S}_\tau = \tilde{S}_0 \]

\[ \downarrow : \text{optional stopping} \]

\[ \uparrow : \text{for all } A \in \mathcal{F}_t, \text{ set } \tau = \begin{cases} t & \text{if } A^c \\ t+1 & \text{if } A \end{cases} \\ 0 = E\tilde{S}_t = E\tilde{S}_t I_A + E\tilde{S}_t I_{A^c} \]

so

\[ E\tilde{S}_\tau I_{A^c} = E\tilde{S}_t I_{A^c} = -E\tilde{S}_t I_{A} \]

and so

\[ \tilde{S}_0 = 0 : 0 = E\tilde{S}_\tau = E\tilde{S}_t I_A + E\tilde{S}_t I_{A^c} \]

\[ = E\tilde{S}_{t+1} I_A - E\tilde{S}_t I_A \]

\[ = E(\tilde{S}_{t+1} - \tilde{S}_t) I_A = E\Delta \tilde{S}_t I_A \]

Since true for all \( A \in \mathcal{F}_t \): \( E[\Delta \tilde{S}_t \mid \mathcal{F}_t] = 0 \)