SEMIALGEBRAIC GEOMETRY OF NONNEGATIVE TENSOR RANK

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Abstract. We study the semialgebraic structure of $D_r$, the set of nonnegative tensors of nonnegative rank not more than $r$, and use the results to infer various properties of nonnegative tensor rank. We determine all nonnegative typical ranks for cubical nonnegative tensors and show that the direct sum conjecture is true for nonnegative tensor rank. Under some mild condition (non-defectivity), we show that nonnegative, real, and complex ranks are all equal for a general nonnegative tensor of nonnegative rank strictly less than the complex generic rank. In addition, such nonnegative tensors always have unique nonnegative rank-$r$ decompositions if the real tensor space is $r$-identifiable. We determine conditions under which a best nonnegative rank-$r$ approximation has a unique nonnegative rank-$r$ decomposition: for $r \leq 3$, this is always the case; for general $r$, this is the case when the best nonnegative rank-$r$ approximation does not lie on the boundary of $D_r$.

Key words. nonnegative tensors, nonnegative tensor rank, nonnegative typical ranks, best nonnegative rank-$r$ approximations, semialgebraic geometry, uniqueness and identifiability

AMS subject classifications. 14P10, 15A69, 41A50, 41A52

1. Introduction. In many applications, notably algebraic statistics [27, 26, 4, 3, 37, 23, 2], one frequently needs to find (i) the nonnegative rank, (ii) a nonnegative rank-$r$ decomposition, or (iii) a best nonnegative rank-$r$ approximation, of a nonnegative third order tensor. Such problems also arise for instance in chemometrics [35] and hyperspectral imaging [42], when quantities like concentration and intensity can only take on nonnegative values. This article addresses questions pertaining to these three problems using tools from semiagebraic geometry.

Questions regarding nonnegative decompositions of a nonnegative tensor are often regarded as being more difficult than the corresponding questions over the complex numbers. One reason is that the tools of classical algebraic geometry are often at one’s disposal in the latter case but not the former. In this article we study nonnegative tensors under the light of semialgebraic geometry. The first main result of our article (cf. Theorem 20) is that for a general nonnegative tensor with nonnegative rank strictly less than the generic rank, its rank over complex numbers, real numbers, and nonnegative real numbers, are all equal under some mild conditions (non-defectivity). Furthermore, for such a nonnegative tensor, its nonnegative rank-$r$ decomposition is unique if the real tensor space is $r$-identifiable. We determine the nonnegative typical ranks in Propositions 27 and 28 and show in Lemma 13 that the nonnegative direct sum conjecture is true, i.e., the nonnegative rank of the direct sum of two nonnegative tensors equals the sum of the respective nonnegative ranks. In our earlier work [38], we showed that almost every nonnegative tensor has a unique best nonnegative rank-$r$ approximation. But it remains to be seen whether this approximation itself has a unique nonnegative rank-$r$ decomposition; we show that this is the case for $r \leq 3$ in Theorem 36, and, for general $r$, we show in Corollary 34 that uniqueness holds for an

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*YQ and PC are supported by the ERC under the European Community’s Seventh Framework Program FP7/2007-2013 Grant 320594. LHL is supported by AFOSR FA9550-13-1-0133, DARPA D15AP00109, NSF IIS 1546413, DMS 1209136, and DMS 1057064.
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open subset of nonnegative tensors under some conditions on the tensor space.

The paper is organized as follows. Section 2 lists some preliminary facts in semialgebraic geometry. The definition of $X$-rank and its basic properties are introduced in Section 3. Lemma 10 is necessary to determine nonnegative typical ranks in Propositions 27 and 28. Our main contributions are then presented in Sections 5, 6, 7.

We begin with a short list of standard definitions. Let $V_1, \ldots, V_d$ be vector spaces over a field $K$, and denote the dual of $V_i$ by $V_i^*$. The tensor space $V_1^* \otimes \cdots \otimes V_d^*$ is the space of multilinear $K$-valued functions on $V_1 \times \cdots \times V_d$. Its elements are called order-$d$ tensors or $d$-tensors or just tensors if the order is implicit. A nonzero tensor in $V_1 \otimes \cdots \otimes V_d$ is said to have rank-one if it is of the form $v_1 \otimes \cdots \otimes v_d$, where $v_i \in V_i$ and $v_1 \otimes \cdots \otimes v_d$ is defined by

$$v_1 \otimes \cdots \otimes v_d(u_1, \ldots, u_d) = v_1(u_1) \cdots v_d(u_d)$$

for all $u_i \in V_i^*$. The rank of a nonzero tensor $T$, denoted by rank($T$), is the minimum number $r$ such that $T$ is a sum of $r$ rank-one tensors. In addition, rank($T$) = 0 iff $T = 0$. An expression of $T$ as a sum of $r = \text{rank}(T)$ rank-one tensors is called a rank-$r$ decomposition. A rank-$r$ decomposition $T = \sum_{i=1}^r T_i$, $T_i = u_i^{(1)} \otimes \cdots \otimes u_i^{(d)}$, is said to be (essentially) unique if the unordered set $\{T_i : i = 1, \ldots, r\}$ is unique [15], i.e., each $u_j^{(i)}$ is unique up to permutation and scaling [29, 31, 34]. The tensor space $V_1 \otimes \cdots \otimes V_d$ is said to be $r$-identifiable if almost every rank-$r$ tensor has a unique rank-$r$ decomposition [13]. There has been intense research on tensor ranks and uniqueness of rank-$r$ decompositions. See [15] for a review.

In this article, the field $K$ will be either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. We will also extend the above to a semiring, denoted by $\mathcal{R}$. Of particular interest to us is the semiring of nonnegative real numbers $\mathbb{R}_+ := [0, \infty)$. It is possible that $\mathcal{R} = \mathbb{R}$ or $\mathbb{C}$, i.e., a result stated for semiring would also apply to a field unless stated otherwise. For convenience of notations, all our results are stated for 3-tensors, i.e., $d = 3$, although most of them can be generalized to tensors of arbitrary order without difficulties.

2. Semialgebraic geometry. In this section we briefly review some well-known facts in semialgebraic geometry, providing in particular a summary of the relevant portions of [10, 17, 36, 24, 18] for our later use.

A semialgebraic subset of $\mathbb{R}^n$ is the union of finitely many subsets of the form

$$\{x \in \mathbb{R}^n : P(x) = 0, \ Q_1(x) > 0, \ldots, Q_m(x) > 0\},$$

where $P, Q_1, \ldots, Q_m \in \mathbb{R}[X_1, \ldots, X_n]$, are polynomials in $n$ variables with real coefficients. Let $S$ and $T$ be semialgebraic sets. A map $f : S \to T$ is called semialgebraic if its graph $G(f) := \{(s, t) \in S \times T : f(s) = t\}$ is semialgebraic. A semialgebraic set is called nonsingular if it is an open subset of the set of nonsingular points of some algebraic set. A Nash manifold is a semialgebraic analytic submanifold of $\mathbb{R}^n$ and a Nash mapping between Nash manifolds is an analytic mapping with a semialgebraic graph.

A point in a semialgebraic set $S$ is said to be general with respect to some property $\mathcal{P}$ if the points in $S$ that do not have the property $\mathcal{P}$ are all contained in a semialgebraic subset $C$ of $S$ with dim $C < \text{dim } S$.

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3An expression of $T$ as a sum of $s$ rank-one tensors where $s$ is not necessarily rank($T$) will just be called an $s$-term decomposition.
Let \( f : M \to N \) be a Nash mapping between Nash manifolds \( M \) and \( N \). The usual semialgebraic version of Sard’s theorem [10] says that the set of critical values of \( f \) is a semialgebraic subset of \( N \) with smaller dimension. As we focus on polynomial maps in this article, we have the following stronger version of Sard’s theorem about critical points of \( f \).

**Theorem 1 (Sard’s theorem).** Let \( f : M \to N \) be a non-constant polynomial map between Nash manifolds. Then the set of critical points of \( f \) is a semialgebraic subset of \( M \), with dimension strictly less than \( \dim M \).

Aside from Sard’s theorem, we also quote the following results and definitions from [10] that we will need later.

**Theorem 2 (Nash Tubular Neighborhood).** Let \( N \subseteq \mathbb{R}^n \) be a Nash submanifold. Then there is an open semialgebraic neighborhood \( U \subseteq \mathbb{R}^n \) and a Nash retraction \( f : U \to N \) such that \( \text{dist}(p, N) = \|p - f(p)\| \) for each \( p \in U \). Here \( \| \cdot \| \) denotes the Euclidean norm in \( \mathbb{R}^n \).

**Definition 3.** A Whitney stratification of a semialgebraic set \( S \subseteq \mathbb{R}^n \) is a finite partition of \( S \) into semialgebraically connected submanifolds \( S = \bigcup_i S_i \) satisfying the following two conditions, known respectively as the ‘frontier condition’ and ‘Whitney condition (a)’:

1. For \( i \neq j \), if \( S_i \cap \text{cl}(S_j) \neq \emptyset \), then \( S_i \subseteq \text{cl}(S_j) \setminus S_j \).
2. For any sequence of points \( (x_k) \) in a stratum \( S_j \), if \( x_k \) converges to a point \( y \) in a stratum \( S_i \), and the sequence of tangent \( (\dim S_j) \)-planes \( T_{x_k}S_j \) converges to a \( (\dim S_j) \)-plane \( T_yS_i \), then \( T \) contains the tangent \( (\dim S_i) \)-plane \( T_yS_i \).

Given two finite families \( \{B_i\} \) and \( \{C_j\} \) of subsets of \( \mathbb{R}^n \), \( \{B_i\} \) is said to be compatible with \( \{C_j\} \) if \( B_i \cap C_j = \emptyset \) or \( B_i \subseteq C_j \) for all \( i \) and \( j \).

**Theorem 4.** For semialgebraic subsets \( S, C_1, \ldots, C_m \) of \( \mathbb{R}^n \), \( S \) admits a Whitney stratification compatible with \( C_1, \ldots, C_m \).

For the reader’s easy reference, we reproduce from [24] the following two propositions, which we will need later.

**Proposition 5.** Let \( f : S \to \mathbb{R}^n \) be a semialgebraic function on a semialgebraic set. Then \( S \) admits a Whitney stratification \( S = \bigcup_i S_i \) such that each graph of \( f|_{S_i} \) is a nonsingular semialgebraic set.

**Proposition 6.** Let \( S \) be a nonsingular semialgebraic set, and \( f : S \to \mathbb{R}^n \) be a function such that \( G(f) \) is nonsingular and semialgebraic. Then the set of points of \( S \) where \( f \) is not differentiable is contained in a closed lower-dimensional semialgebraic subset of \( S \).

**3. \( X \)-ranks.** There has been several attempts to describe tensor ranks in different settings in a unified and general way, e.g. [7, 41] but they do not usually include nonnegative rank as a special case. Here we introduce a generalization of \( X \)-rank [44] to the setting of an arbitrary cone \( X \) and coefficients in a semiring \( \mathcal{R} \) in order to treat nonnegative, real, and complex tensor ranks in a unified setting.

**Definition 7.** Let \( \mathbb{K} \) be a field, and \( \mathcal{R} \subseteq \mathbb{K} \) be a semiring. Given a vector space \( V \) over \( \mathbb{K} \), and a subset \( X \subseteq V \), an \( \mathcal{R} \)-span of \( X \), denoted by \( \text{span}_\mathcal{R}(X) \), is the set of all finite \( \mathcal{R} \)-linear combinations of elements of \( X \), that is,

\[
\text{span}_\mathcal{R}(X) := \left\{ \sum_{i=1}^{k} \alpha_i x_i : k > 0, \ \alpha_i \in \mathcal{R}, \ x_i \in X \right\}.
\]


When $\mathcal{R} = \mathbb{K}$, an $\mathcal{R}$-span is a subspace. When $\mathbb{K} = \mathbb{R}$ and $\mathcal{R} = \mathbb{R}_+$, an $\mathcal{R}$-span is a convex cone. We will denote the $\mathbb{R}_+$-cone of nonnegative vectors in a vector space $V$ by either $V^+$ or $V_+$. Note that in order to specify $V_+$, we will need to first specify a choice of basis on $V$. See [38] for further discussions. With this notation, $V^+_1 \otimes \cdots \otimes V^+_d$ is the cone of nonnegative tensors as defined in [38, Definition 2].

**Definition 8.** We say $X$ is an $\mathcal{R}$-cone, if for $x \in X$ we always have $\lambda x \in X$ for any $\lambda \in \mathcal{R}$. Given an $\mathcal{R}$-cone $X$, for any $p \in \text{span}_\mathcal{R}(X)$, the $X$-rank of $p$, rank$_X(p)$, is defined to be

$$\text{rank}_X(p) := \min \{ r : p = x_1 + \cdots + x_r; \ x_1, \ldots, x_r \in X \}.$$ 

Recall that in algebraic geometry, the affine cone $X \subseteq \mathbb{K}^n$ over a projective variety $Y \subseteq \mathbb{P}^{n-1}$ is defined as $X := \pi^{-1}(Y) \cup \{0\}$ where $\pi : \mathbb{K}^n \setminus \{0\} \to \mathbb{P}^{n-1}, (x_1, \ldots, x_n) \mapsto [x_1 : \cdots : x_n]$ is the canonical projection. Note that an affine cone is a $\mathbb{K}$-cone in the sense of Definition 8.

(i) Let $\mathcal{R} = \mathbb{R}$, $V = V_1 \otimes \cdots \otimes V_d$, and $X$ be the cone of tensors of rank $\leq 1$ (i.e., affine cone over the projective Segre variety). Then rank$_X(p)$ is the real rank of $p$, usually denoted $\text{rank}_\mathbb{R}(p)$.

(ii) Let $\mathcal{R} = \mathbb{R}_+$, $\mathbb{K} = \mathbb{R}$, $V = V_1 \otimes \cdots \otimes V_d$, and $X$ be the $\mathbb{R}_+$-cone of nonnegative tensors of rank $\leq 1$. Then rank$_X(p)$ is the nonnegative rank of $p$, usually denoted $\text{rank}_+(p)$.

(iii) Let $\mathcal{R} = \mathbb{K}$ be an algebraically closed field and $X$ be the affine cone over an irreducible nondegenerate projective variety. Then rank$_X(p)$ is the $X$-rank as defined in [44, 31, 7].

The discussions above are purely algebraic but subsequent discussions will require topological structures on our vector space and field. Recall that a topological vector space over a topological field is one where the vector addition and scalar multiplication are continuous. We will not require any results regarding topological vector space beyond its definition.

**Definition 9.** Let $V$ be a finite-dimensional topological vector space over a topological field $\mathbb{K}$ of characteristic zero, and $\mathcal{R} \subseteq \mathbb{K}$ be a semiring. Let $X \subseteq V$ be an $\mathcal{R}$-cone such that every nonempty open subset of $X$ has the same dimension, and that $\text{span}_\mathcal{R}(X)$ contains a nonempty open subset of $V$. If the set $\{ p \in \text{span}_\mathcal{R}(X) : \text{rank}_X(p) = r \}$ contains a nonempty open subset of $V$, then $r$ is called a typical $X$-rank. The maximum typical $X$-rank is

$$\max \{ r : r \text{ is a typical } X \text{-rank of } \text{span}_\mathcal{R}(X) \},$$

whereas the maximum $X$-rank is

$$\max \{ \text{rank}_X(p) : p \in \text{span}_\mathcal{R}(X) \}.$$ 

We deduce the following lemma using an argument in [25], where it is proved for the case $\mathbb{K} = \mathbb{R}$, $V = V_1 \otimes V_2 \otimes V_3$, and $X = \{ A \in V : \text{rank}_\mathbb{R}(A) \leq 1 \}$. See also [5, Theorem 1.1] for the case where $X$ is the affine cone of a nondegenerate irreducible real projective variety.

**Lemma 10.** Let $\mathbb{K} = \mathbb{R}$ and $X$ be a nonempty semialgebraic $\mathcal{R}$-cone satisfying the conditions in Definition 9. If $m$ and $M$ are two typical $X$-ranks, then any integer between $m$ and $M$ is also a typical $X$-rank.

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2Allowing both superscript and subscript provides notational flexibility when indices or powers are involved.
Proof. For each \( r \in \mathbb{N} \), define the polynomial map \( \varphi_r \) by
\[
\varphi_r : X^r \to \operatorname{span}_\mathbb{R}(X), \quad (x_1, \ldots, x_r) \mapsto x_1 + \cdots + x_r.
\]
Assume \( m \leq M \) and suppose that \( r \in \{m, \ldots, M\} \) is the minimum integer which is not a typical \( X \)-rank. For any fixed \( r \in \mathbb{N} \), every nonempty open subset of \( \operatorname{Im} \varphi_r \) has the same dimension by Theorem 1. Hence \( \varphi_r \setminus \operatorname{Im} \varphi_{r-1} \) does not contain an open subset of \( \operatorname{Im} \varphi_r \), which implies that a general \( p = x_1 + \cdots + x_r \in \operatorname{Im} \varphi_r \) is within \( \operatorname{Im} \varphi_{r-1}, \) i.e., \( p = \bar{x}_1 + \cdots + \bar{x}_{r-1} \). Hence a general \( q = x_1 + \cdots + x_{r+1} \in \operatorname{Im} \varphi_{r+1} \) can be written with \( r \) summands as \( q = \bar{x}_1 + \cdots + \bar{x}_{r-1} + x_{r+1} \), which is in \( \operatorname{Im} \varphi_r \). But we may repeat the same argument to conclude that \( q \) is in \( \operatorname{Im} \varphi_{r-1} \). So by induction, a general point in \( \operatorname{Im} \varphi_M \) is in \( \operatorname{Im} \varphi_{r-1} \), i.e., \( \dim \operatorname{Im} \varphi_M \setminus \operatorname{Im} \varphi_{r-1} < \dim V \), contradicting our assumption that \( M \) is a typical \( X \)-rank.

We will require the use of Lemma 10 in Propositions 27 and 28. This simple lemma is surprisingly potent. As an illustration we provide a short proof for the main result in [6] (see also [5]), that every integer between \( \lfloor (d+2)/2 \rfloor \) and \( d \) is a typical rank of \( S^d(\mathbb{R}^2) \), originally conjectured in [16].

Corollary 11 (Blekherman). Every \( m \) with \( \lfloor (d+2)/2 \rfloor \leq m \leq d \) is a typical rank of \( S^d(\mathbb{R}^2) \).

Proof. The complex generic rank \( \lfloor (d+2)/2 \rfloor \) is necessarily the minimum typical rank by [7]. It has been shown in [11] that \( f \in S^d(\mathbb{R}^2) \) has real rank \( d \) if and only if \( f \) has \( d \) distinct real roots when regarded as a degree-\( d \) homogeneous polynomial in two variables. Since having \( d \) distinct real roots imposes an open condition on \( S^d(\mathbb{R}^2) \), \( d \) is the maximum typical rank. The required result then follows from Lemma 10.

We now introduce a ‘semialgebraic version’ of Terracini’s lemma. First observe that for semialgebraic sets \( X, Y \subseteq V \), if we define the semialgebraic map \( \varphi \) by
\[
\varphi : X \times Y \to V, \quad (x, y) \mapsto x + y,
\]
then \( \operatorname{Im} \varphi \) is semialgebraic by the Tarski–Seidenberg Theorem.

Lemma 12 (Semialgebraic Terracini’s lemma). For general points \( x \in X \) and \( y \in Y \), the tangent space of \( \varphi(X \times Y) \) at \( x + y \) is the span of the tangent spaces \( T_x X \) and \( T_y Y \), i.e.,
\[
T_{x+y} \varphi(X \times Y) = \operatorname{span}(T_x X, T_y Y).
\]

Proof. By taking Whitney stratifications, we may assume without loss of generality \( X \) and \( Y \) are Nash manifolds. By Theorem 1, the rank of \( \varphi \) at a general point \( (x, y) \in X \times Y \) is \( \dim \varphi(X, Y) \). So \( T_{x+y} \varphi(X \times Y) = \varphi^*(T_x X + T_y Y) = \operatorname{span}(T_x X, T_y Y) \).

4. Direct sum conjecture for nonnegative rank. We now show that the direct sum conjecture is true for nonnegative rank. Given vector spaces \( V_1, \ldots, V_d \), and \( W_1, \ldots, W_d \) over \( \mathbb{K} \), for any \( A \in V_1 \otimes \cdots \otimes V_d \) and \( B \in W_1 \otimes \cdots \otimes W_d \), we have the direct sum \( A \oplus B \in (V_1 \oplus W_1) \otimes \cdots \otimes (V_d \oplus W_d) \). For \( d = 2 \), it is obvious that the rank of a block diagonal matrix is the sum of the ranks of the diagonal blocks, i.e., if \( A \) and \( B \) are matrices, then
\[
\operatorname{rank}(A \oplus B) = \operatorname{rank} \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) = \operatorname{rank}(A) + \operatorname{rank}(B).
\]
It has been conjectured by Strassen [39] that the same is true for \( d > 2 \), i.e., \( \operatorname{rank}(A \oplus B) = \operatorname{rank}(A) + \operatorname{rank}(B) \) for any \( d \)-tensors. This has been a long-standing open
problem in algebraic computational complexity. We show here that the analogous statement for nonnegative rank is true. The next two results are true for nonnegative tensors of arbitrary order \( d \) but we will state and prove them for \( d = 3 \) for notational simplicity.

In the following, let \( U_1, V_1, W_1, U_2, V_2, W_2 \) be real vector spaces of dimensions \( m_1, n_1, p_1, m_2, n_2, p_2 \) respectively. Fix a basis for each vector space and choose the bases for \( U_1 \oplus U_2, V_1 \oplus V_2, \) and \( W_1 \oplus W_2 \) so that for \( a = (a_1, \ldots, a_{m_1}) \in U_1 \) and \( b = (b_1, \ldots, b_{m_2}) \in U_2, a \oplus b \) has coordinates \( a \oplus b = (a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}) \) in \( U_1 \oplus U_2; \) likewise for \( V_1 \oplus V_2 \) and \( W_1 \oplus W_2. \)

**Lemma 13** (Nonnegative direct sum conjecture). For \( A \in U_1^+ \otimes V_1^+ \otimes W_1^+ \) and \( B \in U_2^+ \otimes V_2^+ \otimes W_2^+ \),

\[
\text{rank}_+(A \oplus B) = \text{rank}_+(A) + \text{rank}_+(B).
\]

**Proof.** Fix a basis for each vector space and let \( a_{ijk} \) and \( b_{i'j'k'} \) denote the coordinates of \( A \) and \( B \). Note that \((A \oplus B)_{ijk} = a_{ijk}, (A \oplus B)_{i'j'k'} = b_{i'j'k'} \) and other terms are zero. Suppose that \( r \geq \text{rank}_+(A \oplus B) < \text{rank}_+(A) + \text{rank}_+(B) \).

Without loss of generality we may assume that \( A \oplus B = \sum_{i=1}^r u_i \otimes v_i \otimes w_i \) with \( u_1 \in (U_1 \oplus U_2)^+ \setminus (U_1^+ \cup U_2^+)\). Then at least one of the following indices

\[
(i, j', k'), \ (i, j, k'), \ (i', j, k'), \ (i', j', k), \ (i', j', k'),
\]

which we denote by \((\alpha, \beta, \gamma)\), will be such that \((A \oplus B)_{\alpha\beta\gamma}\) is positive, a contradiction. \( \Box \)

We may also deduce the following, clearly also true for \( d > 3 \), from the above proof.

**Corollary 14.** If \( A \) and \( B \) have unique nonnegative rank decompositions in \( U_1^+ \otimes V_1^+ \otimes W_1^+ \) and \( U_2^+ \otimes V_2^+ \otimes W_2^+ \) respectively, then \( A \oplus B \) also has a unique nonnegative rank decomposition.

For a real tensor \( A \in \mathbb{R}^{m_1} \otimes \cdots \otimes \mathbb{R}^{m_d} \subseteq \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_d}, \) the real rank of \( A \) regarded as a tensor in \( \mathbb{R}^{m_1} \otimes \cdots \otimes \mathbb{R}^{m_d} \) equals the real rank of \( A \) regarded as a tensor in \( \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_d} \) [19, Proposition 3.1]. As a corollary of Lemma 13, we see that this also holds for nonnegative rank.

In the following, let \( U_1 \subseteq U_2, \ V_1 \subseteq V_2, \) and \( W_1 \subseteq W_2 \) be inclusions of real vector spaces. Choose bases for \( U_2, V_2, \) and \( W_2 \) such that \( u \in U_1 \) has coordinates \( u = (u_1, \ldots, u_{m_1}, 0, \ldots, 0) \) as a vector in \( U_2; \) likewise for \( V_2 \) and \( W_2. \) Then we have the following corollary, stated for \( d = 3, \) but generalizes easily to arbitrary \( d > 3. \)

**Corollary 15.** Let \( A \in U_1^+ \otimes V_1^+ \otimes W_1^+ \subseteq U_2^+ \otimes V_2^+ \otimes W_2^+ \). Then the nonnegative rank of \( A \) regarded as a nonnegative tensor in \( U_1^+ \otimes V_1^+ \otimes W_1^+ \) is the same as the nonnegative rank of \( A \) regarded as a nonnegative tensor in \( U_2^+ \otimes V_2^+ \otimes W_2^+. \)

**Proof.** Let \( U_1' \subseteq U_2 \) be a complementary subspace of \( U_1, \) i.e., \( U_2 = U_1 \oplus U_1'. \) So \( u' \in U_1' \) has coordinates \( u' = (0, \ldots, 0, u_{m_1+1}', \ldots, u_{m_2}') \) as a vector in \( U_2. \) Likewise, we let \( V_1' \subseteq V_2 \) and \( W_1' \subseteq W_2 \) be complementary subspaces of \( V_1 \) and \( W_1 \). The required statement then follows from applying Lemma 13 to the case \( A \in U_1^+ \otimes V_1^+ \otimes W_1^+ \) and \( B := 0 \in U_1'^+ \otimes V_1'^+ \otimes W_1'^+. \) \( \Box \)

The following simple observation is a nonnegative analogue of [19, Corollary 3.3]. We assume that we fix a basis for each \( V_i \) so that \( V_i^+ \) is defined, \( i = 1, \ldots, d. \)

**Proposition 16.** For any \( k \in \{2, \ldots, d-1\}, \) let \( A \in V_1^+ \otimes \cdots \otimes V_k^+ \) be arbitrary and let \( u_{k+1} \in V_{k+1}^+, \ldots, u_d \in V_d^+ \) be nonzero. Then

\[
\text{rank}_+(A) = \text{rank}_+(A \otimes u_{k+1} \otimes \cdots \otimes u_d).
\]
Proof. The isomorphism of $\mathbb{R}_+ \times \mathbb{R}_+$-cones,
\[ V_1^+ \otimes \cdots \otimes V_k^+ \cong V_1^+ \otimes \cdots \otimes V_k^+ \otimes \text{span}_{\mathbb{R}_+}(u_{k+1}) \otimes \cdots \otimes \text{span}_{\mathbb{R}_+}(u_d), \]
given by $A \mapsto A \otimes u_{k+1} \otimes \cdots \otimes u_d$ implies the required equality.

5. General equivalence of complex, real, and nonnegative ranks. It is well-known that a real tensor may have different real and complex ranks. Likewise a nonnegative tensor may also have different nonnegative and real ranks. In fact, strict inequality can also occur for the nonnegative and real ranks of a nonnegative matrix, a well-known example was provided by H. Robbins [15].

For the case of 3-tensors, two explicit examples are as follows. Let $e_1, e_2 \in \mathbb{R}^2$ be the standard basis vectors, i.e., $e_1 = [1, 0]^T$, $e_2 = [0, 1]^T$. Let
\[ (5.1) \quad A = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_1, \]
\[ B = e_1 \otimes e_1 \otimes e_1 - e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1. \]
Then $A \in \mathbb{R}_+^2 \otimes \mathbb{R}_+^2 \otimes \mathbb{R}_+^2 \subseteq \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ and $B \in \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2 \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. We have
\[ \text{rank}_C(A) = \text{rank}_R(A) = 2 < 4 = \text{rank}_+(A), \]
\[ \text{rank}_C(B) = 2 < 3 = \text{rank}_R(B). \]
See Section 6 for the nonnegative, real, and complex ranks of $A$ and [19] for the real and complex ranks of $B$. We will show in this section that under some mild condition (non-defectivity) on the tensor space, this does not happen for a general nonnegative tensor of nonnegative rank strictly less than the generic rank — its nonnegative, real, and complex ranks will all be equal.

For notational simplicity we focus on 3-tensors, although many of the statements and proofs in this section can be generalized without difficulty to $d$-tensors for any $d > 3$. Let $U$, $V$ and $W$ be real vector spaces of dimensions $n_U$, $n_V$ and $n_W$, respectively. Denote by $V_C$ the complexification of $V$, i.e., $V_C = V \otimes \mathbb{C}$.

We define the polynomial map
\[ \Sigma_r^C : (U_C \times V_C \times W_C)^r \rightarrow U_C \otimes V_C \otimes W_C, \]
\[ (u_1, v_1, w_1, \ldots, u_r, v_r, w_r) \mapsto \sum_{i=1}^r u_i \otimes v_i \otimes w_i, \]
and denote the restriction of $\Sigma_r^C$ to $(U \times V \times W)^r$ by $\Sigma_r$, and the restriction to $(U_+ \times V_+ \times W_+)^r$ by $\Sigma_r$. We have the following commutative diagram:
\[ (5.3) \]
\[ (U_C \times V_C \times W_C)^r \xrightarrow{\Sigma_r^C} U_C \otimes V_C \otimes W_C. \]

Henceforth, we will use the following abbreviated notation when specifying an element of $(U \times V \times W)^r$,
\[ (5.4) \quad (u_1, \ldots, w_r) := (u_1, v_1, w_1, \ldots, u_r, v_r, w_r). \]
The set of nonnegative tensors with nonnegative rank not more than $r$ will be denoted

$$D_r := \text{Im} \Sigma_r$$

By the Tarski–Seidenberg Theorem [10], $D_r$ is a semialgebraic set. By [35, Proposition 6.2], $D_r$ is closed. On the other hand, $\text{Im} \Sigma_r$ and $\text{Im} \Sigma_r^C$ are usually not closed.

Recall that a positive integer $r_g$ is called the generic rank of $U_C \otimes V_C \otimes W_C$ if the set of rank-$r_g$ tensors contains a Zariski open subset [31, 34]. Put in another way, the generic rank is the minimum $r$ such that the morphism $\Sigma_r^C$ is dominant. By [7, Theorem 2], the generic rank is equal to the minimum real typical rank.

**Definition 17.** If $\dim(\text{Im} \Sigma_r) < \min\{r(n_U + n_V + n_W - 2), n_U n_V n_W\}$, then $U \otimes V \otimes W$ is called $r$-defective over $\mathbb{R}$.

The definition of defectivity over $\mathbb{C}$, i.e., identical to Definition 17 but with $U, V, W$ being complex vector spaces, is classical in algebraic geometry [43]. More generally, a complex projective variety $X$ is called $r$-defective [12] if the $r$th secant variety of $X$ does not have the expected dimension. In our context this is equivalent to $\dim_{\mathbb{C}}(\text{Im} \Sigma_r^C) < \min\{r(n_U + n_V + n_W - 2), n_U n_V n_W\}$. Note that if $U \otimes V \otimes W$ is $r$-identifiable, then $U \otimes V \otimes W$ is not $r$-defective.

**Lemma 18.** Let $r < r_g$. Then a general $A \in D_r$ has real rank $r$.

**Proof.** Denote the Jacobian (i.e., the matrix of first-order partial derivatives) of $\Sigma_r$ by $J(\Sigma_r)$. If $\text{rank}(J(\Sigma_{r-1})) = \text{rank}(J(\Sigma_r))$ at general points, then inductively,

$$\text{rank}(J(\Sigma_{r-1})) = \text{rank}(J(\Sigma_r)) = \text{rank}(J(\Sigma_{r+1})) = \cdots$$

at general points, which implies that

$$\dim(\text{Im} \Sigma_{r-1}) = \dim(\text{Im} \Sigma_r) = \cdots = n_U n_V n_W.$$

Hence if $r < r_g$, $\text{rank}(J(\Sigma_{r-1})) < \text{rank}(J(\Sigma_r))$ at general points, implying that

$$\dim(\text{Im} \Sigma_{r-1}) < \dim(\text{Im} \Sigma_r).$$

On the other hand, since $(U_+ \times V_+ \times W_+)^r$ contains an open subset of $(U \times V \times W)^r$, by Theorem 1, $J(\Sigma_r) = J(\Sigma_r^C)$ at a general point, $\text{Im} \Sigma_r$ contains an open subset of $\text{Im} \Sigma_r^C$, i.e.,

$$\dim(D_{r-1}) = \dim(\text{Im} \Sigma_{r-1}) = \dim(\text{Im} \Sigma_{r-1})$$

$$\quad < \dim(\text{Im} \Sigma_r) = \dim(\text{Im} \Sigma_r) = \dim(D_r).$$

Thus a general $A \in D_r$ has nonnegative rank $r$, and the real rank of $A$ is also $r$. □

**Lemma 19.** Let $r < r_g$ and assume that $U \otimes V \otimes W$ is not $r$-defective. Then a general $A \in D_r$ has complex rank $r$.

**Proof.** Let $A = \sum_{i=1}^s u_i \otimes v_i \otimes w_i$. Then $J(\Sigma_r)(u_1, \ldots, w_r) = J(\Sigma_r^C)(u_1, \ldots, w_r)$. Since $r < r_g$, $U \otimes V \otimes W$ is not $r$-defective and $A$ is general in $D_r$,

$$\text{rank} J(\Sigma_r^C)(u_1, \ldots, w_r) = \text{rank} J(\Sigma_r)(u_1, \ldots, w_r) = r(n_U + n_V + n_W - 2),$$

i.e., $U_C \otimes V_C \otimes W_C$ is not $r$-defective. In fact for every $k \leq r$, we have

$$\text{rank} J(\Sigma_r^C) = k(n_U + n_V + n_W - 2)$$
at a general point. Suppose $A$ has complex rank $k$ for some $k < r$. Let $(\tilde{u}_1, \ldots, \tilde{w}_r)$ be general in the fibre $(\Sigma_r^C)^{-1}(A)$. Then

$$\text{rank } J(\Sigma_r^C)(u_1, \ldots, w_r) \leq \text{rank } J(\Sigma_r^C)(\tilde{u}_1, \ldots, \tilde{w}_r)$$

by semicontinuity. Since $\dim(\Sigma_r^C)^{-1}(A) \geq (r-k)(n_U + n_V + n_W - 2)$ by the assumption that $A \in \text{Im } \Sigma_r^k$,

$$\text{rank } J(\Sigma_r^C)(\tilde{u}_1, \ldots, \tilde{w}_r) \leq k(n_U + n_V + n_W - 2).$$

Therefore $\text{rank } J(\Sigma_r^C)(u_1, \ldots, w_r) \leq k(n_U + n_V + n_W - 2)$, contradicting (5.5).

Theorem 20. Let $r < r_g$ and assume that $U \otimes V \otimes W$ is $r$-identifiable. Then a general $A \in D_r$ has both real rank and complex rank equal to $r$ as well as a unique nonnegative rank-$r$ decomposition.

Proof. If $U \otimes V \otimes W$ is $r$-identifiable, then $U \otimes V \otimes W$ is not $r$-defective. The claims about rank are just Lemmas 18 and 19. Since $D_r$ contains an open subset of $\text{Im } \Sigma_r^k$, a general point in $D_r$ has a unique rank-$r$ decomposition. 

There has been a significant amount of work on both defectivity [40, 33, 1] and identifiability [30, 13, 20, 21, 9, 14, 22]. While these focus mainly on complex tensors, some of these methods can be also adapted to real tensors. A notable example is [13, Theorem 1.1], stated below for real tensors.

Theorem 21 (Chiantini–Ottaviani). Given real vector spaces $U, V$ and $W$ with dimensions $\dim U \leq \dim V \leq \dim W$, let $\alpha, \beta$ be minimum integers such that $2^\alpha \leq \dim U$ and $2^\beta \leq \dim V$. Then $U \otimes V \otimes W$ is $r$-identifiable if $r \leq 2^{\alpha + \beta - 2}$.

Applying Theorem 21 to Theorem 20, we obtain explicit examples.

Corollary 22. Let $n \geq 4$ and $r \leq \lfloor n^2/16 \rfloor$. A general $A \in \mathbb{R}_+^n \otimes \mathbb{R}_+^n \otimes \mathbb{R}_+^n$ with $\text{rank }^+(A) = r$ has complex rank $r$ (and therefore real rank $r$) and a unique nonnegative rank-$r$ decomposition.

6. Typical and maximum nonnegative ranks. In this section, we investigate typical, maximum, and maximum nonnegative typical ranks, as defined in Definition 9. The following rephrases [35, Proposition 6.2] in the context of this article and may be viewed as a generalization of [8, Theorem 3.1].

Proposition 23. Let $A \in U_+ \otimes V_+ \otimes W_+$ with $\text{rank }^+(A) = r$. Then there is an open ball $B(A, \varepsilon) \subseteq U \otimes V \otimes W$ such that

$$\text{rank }^+(A') \geq r$$

for all $A' \in B(A, \varepsilon) \cap U_+ \otimes V_+ \otimes W_+$.

It follows immediately that the maximum nonnegative typical rank and the maximum nonnegative rank always coincide.

Lemma 24. If $r$ is the maximum nonnegative rank of $U_+ \otimes V_+ \otimes W_+$, then $r$ is the maximum nonnegative typical rank.

What about the minimum nonnegative typical rank then? It turns out that it is always equal to the (complex) generic rank.

Lemma 25. The minimum nonnegative typical rank of $U_+ \otimes V_+ \otimes W_+$ is the complex generic rank $r_g$ of $U_+ \otimes V_+ \otimes W_+$. 

Proof. Since \((U_+ \times V_+ \times W_+)\) contains an open subset of \((U \times V \times W)^r\), by Theorem 1, \(J(\Sigma_r) = J(\Sigma_r)\) at a general point. Hence \(\dim \text{Im} (\Sigma_r) = \dim \text{Im} (\Sigma_r)\), which implies that \(r_g\) is the minimum nonnegative rank.

We will illustrate these with a 2 \times 2 \times 2 example. In this case, the complex generic rank of \(C^2 \otimes C^2 \otimes C^2\) is 2 and the real typical ranks of \(R^2 \otimes R^2 \otimes R^2\) are 2 and 3 [19]. By Lemmas 10, 24, and 25, to completely determine the nonnegative typical ranks of \(R^2_+ \otimes R^2_+ \otimes R^2_+\), it remains to find the maximum nonnegative rank. We will construct a nonnegative tensor with maximum nonnegative rank explicitly. Consider the tensor
\[
(6.1) \quad A = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_1 + e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2
\]
that we saw earlier in (5.1). \(A\) may be represented by a nonnegative hypermatrix
\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in R^2_+ \times 2 \times 2^2.
\]
Now let \(A = \sum_{i=1}^{r'} x_i \otimes y_i \otimes z_i\) be a nonnegative rank-\(r\) decomposition. Then we must be able to write \(A = \sum_{i=1}^{r'} X_i \otimes z_i\) where each \(X_i\) is a nonnegative matrix. By the nonnegativity of each \(z_i\) and \(X_i\), we must have \(z_i = e_1\) or \(e_2\) for all \(i = 1, \ldots, r'\).

Without loss of generality we may assume that \(z_1 = e_1\) and \(z_2 = e_2\). Then \(X_1 = e_1 \otimes e_1 + e_2 \otimes e_2\) and \(X_2 = e_1 \otimes e_2 + e_2 \otimes e_1\). By the uniqueness of the nonnegative decompositions of \(X_1\) and \(X_2\), the nonnegative rank-\(r\) decomposition of \(A\) in (6.1) is unique. Hence \(\text{rank}_+(A) = 4\). Since any \(T \in R^2_+ \otimes R^2_+ \otimes R^2_+\) has the form \(T = Y_1 \otimes e_1 + Y_2 \otimes e_2\) where each \(Y_i\) is a nonnegative matrix, and the nonnegative rank of a nonnegative 2 \times 2 matrix is at most 2, we may conclude that the nonnegative rank of \(T\) is at most 4. Thus the nonnegative typical ranks of \(R^2_+ \otimes R^2_+ \otimes R^2_+\) are 2, 3, and 4.

Both the real and complex ranks of \(A\) are 2 [19]. In fact for any \(A'\) in a sufficiently small open ball \(\mathcal{B}(A, \varepsilon)\), both the real and complex ranks of \(A'\) are also 2. If in addition, \(A' \in \mathcal{B}(A, \varepsilon) \cap (R^2_+ \otimes R^2_+ \otimes R^2_+)\), then the nonnegative rank of \(A'\) is 4. This example can be generalized as follows.

Lemma 26. Let \(P_1, \ldots, P_n \in R^n_+ \times n \cong R_+^n \otimes R_+^n\) be \(n\) permutation matrices such that for each \((i, j) \in \{1, \ldots, n\} \times \{1, \ldots, n\}\), there is one and only one \(P_k\) whose \((i, j)\)th entry is one. Let \(w_1, \ldots, w_n \in R_+^n\) be a basis of \(R^n\). Define
\[
A = P_1 \otimes w_1 + \cdots + P_n \otimes w_n \in R_+^n \otimes R_+^n \otimes R_+^n.
\]
Then \(\text{rank}_+(A) = n^2\) and \(A\) has a unique nonnegative rank-\(n^2\) decomposition.

Proof. It suffices to show that \(A\) has a unique nonnegative rank-\(n^2\) decomposition. Suppose
\[
A = \sum_{i=1}^{n^2} \left[ \sum_{j=1}^{n} \alpha_i^j w_j \right] \otimes \left[ \sum_{j=1}^{n} \beta_i^j w_j \right] \otimes \left[ \sum_{j=1}^{n} \gamma_i^j w_j \right]
\]
for nonnegative \(\alpha_i^j, \beta_i^j, \gamma_i^j\). Without loss of generality, we may assume \(\alpha_1^1, \beta_1^1, \gamma_1^1 \neq 0\). Since there is only one \(P_k\) whose \((1, 1)\)th entry is nonzero, this \(P_k\) must be \(P_1\) and \(\gamma_1^1 = 0\) for all \(j > 1\). Repeating this procedure we may show that when we regard \(A\) as a nonnegative matrix in \(R_+^{n^2 \times n^2} \cong (R_+^n \otimes R_+^n) \otimes R_+^n\), it has a unique nonnegative matrix factorization given by \(A = P_1 \otimes w_1 + \cdots + P_n \otimes w_n\). Since each \(P_k\) has a unique nonnegative matrix factorization [32], \(A\) has a unique nonnegative rank-\(n^2\) decomposition.

A $d$-tensor in $V_1 \otimes \cdots \otimes V_d$ is said to be cubical if $\dim V_1 = \cdots = \dim V_d$. By [33, Theorem 4.4], [40, Theorem 4.6], Lemmas 10, 25, 24, and 26, we completely determine the nonnegative typical ranks of cubical nonnegative tensors.

**Proposition 27.** For $n = 2$, the nonnegative typical ranks of $\mathbb{R}_+^2 \otimes \mathbb{R}_+^2 \otimes \mathbb{R}_+^2$ are given by all integers $m$ where

$$2 \leq m \leq 4.$$

For $n = 3$, the nonnegative typical ranks of $\mathbb{R}_+^3 \otimes \mathbb{R}_+^3 \otimes \mathbb{R}_+^3$ are given by all integers $m$ where

$$5 \leq m \leq 9.$$

For $n \geq 4$, the nonnegative typical ranks of $\mathbb{R}_+^n \otimes \mathbb{R}_+^n \otimes \mathbb{R}_+^n$ are given by all integers $m$ where

$$\left\lceil \frac{n^3}{3n - 2} \right\rceil \leq m \leq n^2.$$

For nonnegative tensors that are not cubical, we may determine the maximum nonnegative typical ranks but since the complex generic ranks for 3-tensors are still not known in some instances, we do not have a complete list of nonnegative typical ranks.

**Proposition 28.** Write $\text{maxrank}_{+}(m, n, p)$ for the maximum nonnegative typical rank of $\mathbb{R}_+^m \otimes \mathbb{R}_+^n \otimes \mathbb{R}_+^p$ and suppose without loss of generality that $m \geq n \geq p$. Then

$$\text{maxrank}_{+}(m, n, p) = \begin{cases} np & \text{if } m = n \geq p, \\ n^2 & \text{if } m \geq n = p, \\ np & \text{if } m > n > p. \end{cases}$$

**Proof.** The required arguments are as in the proof of Lemma 26 but ‘padded with the appropriate number of zeros,’ i.e., applied to matrices of the form

$$\begin{bmatrix} P_k \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} P_k \\ 0 \end{bmatrix}$$

where $P_k$ is a permutation matrix.

### 7. General uniqueness of decompositions of approximations.

In our previous work [38], we established that the best nonnegative rank-$r$ approximation of a nonnegative tensor is almost always unique. Here we investigate whether this best nonnegative rank-$r$ approximation has a unique nonnegative rank-$r$ decomposition.

Let $U, V, W$ be real vector spaces of dimensions $n_U, n_V, n_W$ respectively. We will assume a choice of basis on these vector spaces, so that $U \cong \mathbb{R}^{n_U}, V \cong \mathbb{R}^{n_V},$ and $W \cong \mathbb{R}^{n_W}$. For a vector $u_i \in U$, we let $u_i, j$ denote the $j$th coordinate of $u_i$. Likewise for $V$ and $W$. For any smooth curve $\gamma(t), t \in [0, 1]$, the right derivative at 0 is denoted by

$$\gamma'(0) := \lim_{t \to 0^+} \frac{\gamma(t) - \gamma(0)}{t - 0}.$$

Recall the map $\Sigma_r : (U_+ \times V_+ \times W_+)^r \to U_+ \otimes V_+ \otimes W_+$ defined in (5.2) and (5.3). The pushforward of $\Sigma_r$ at $\gamma'(0)$ is denoted

$$\Sigma_{r*}(\gamma'(0)) := \lim_{t \to 0^+} \frac{\Sigma_r(\gamma(t)) - \Sigma_r(\gamma(0))}{t - 0}.$$
Let \( S_r \subseteq U_+ \otimes V_+ \otimes W_+ \) denote the set of nonnegative tensors on which the distance function \( \text{dist}(\cdot, D_r) \) is not smooth. Then \( S_r \) contains the nonnegative tensors with non-unique best nonnegative rank-\( r \) approximations and is a nowhere dense semialgebraic subset \([28]\). Let \( \pi_r : U_+ \otimes V_+ \otimes W_+ \setminus S_r \to D_r \) be the map sending a nonnegative tensor to its unique best nonnegative rank-\( r \) approximation. Since the distance function \( \text{dist}(\cdot, D_r) \) is semialgebraic \([17, 28]\), the graph of \( \pi_r \),

\[
G(\pi_r) = \{(p, q) \in (U_+ \otimes V_+ \otimes W_+ \setminus S_r) \times D_r : \text{dist}(p, D_r) = ||p - q||\},
\]
is also semialgebraic. By Proposition 6, \( \pi_r \) is smooth outside a hypersurface \( H_r \). Henceforth we will focus on the restriction of \( \pi_r \) (also denoted \( \pi_r \) with a slight abuse of notation) to a subset of smooth points in \( U_+ \otimes V_+ \otimes W_+ \),

\[
\pi_r : U_+ \otimes V_+ \otimes W_+ \setminus (S_r \cup H_r) \to D_r.
\]

In the following the support of a vector \( u \in U \) is defined to be

\[
\text{supp}(u) := \{ i \in \{1, \ldots, n_U \} : u_i \neq 0 \}.
\]
The next lemma is a slight rephrase of \([38, \text{Lemma 13}]\). We will use it to partition \( D_r \) into a union of semialgebraic sets later.

**Lemma 29.** Let \( p \in U_+ \otimes V_+ \otimes W_+ \setminus (S_r \cup H_r) \) where \( \pi_r(p) \) has a nonnegative rank-\( r \) decomposition

\[
(7.1) \quad \pi_r(p) = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i.
\]

Then for any \( x_i \in U_+, i = 1, \ldots, r \), we have

\[
(7.2) \quad \langle p, x_i \otimes v_i \otimes w_i \rangle \leq \langle \pi_r(p), x_i \otimes v_i \otimes w_i \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product. With respect to the nonnegative vectors \( u_1, \ldots, u_r \) in \( (7.1) \), define the subspaces

\[
(7.3) \quad \widetilde{U}_i := \{ u \in U : \text{supp}(u) \subseteq \text{supp}(u_i) \}
\]

for \( i = 1, \ldots, r \), and define \( \widetilde{V}_i \) and \( \widetilde{W}_i \) similarly. Then for \( x_i \in \widetilde{U}_i, i = 1, \ldots, r \), we have

\[
(7.4) \quad \langle p, x_i \otimes v_i \otimes w_i \rangle = \langle \pi_r(p), x_i \otimes v_i \otimes w_i \rangle.
\]

The analogous statement for \( \widetilde{V}_i \) or \( \widetilde{W}_i \) in place of \( \widetilde{U}_i \) holds true as well.

We first remind the reader of our abbreviated notation in \((5.4)\). Let

\[
T_{\pi_r(p)}(u_1, \ldots, u_r) := \text{span}_\mathbb{R} \left( \bigcup_{i=1}^{r} \widetilde{U}_i \otimes v_i \otimes w_i \cup u_i \otimes \widetilde{V}_i \otimes w_i \cup u_i \otimes v_i \otimes \widetilde{W}_i \right).
\]

Note that this is the tangent space of \( D_r \) at \( \pi_r(p) \) when \( \pi_r(p) \) is a smooth point of \( D_r \). Then \( (7.4) \) implies that\(^3\)

\[
(7.5) \quad \langle T_{\pi_r(p)}(u_1, \ldots, u_r), p - \pi_r(p) \rangle = 0,
\]

\(^3\)Our convention: \( \langle S, u \rangle = \langle u, S \rangle = 0 \) for \( S \subseteq U \) means that every vector in \( S \) is orthogonal to \( u \); \( \langle S, T \rangle = 0 \) for \( S, T \subseteq U \) means that any vector in \( S \) is orthogonal to any vector in \( T \).
i.e., $p - \pi_r(p)$ is orthogonal to the subspace $T_{\pi_r(p)}(u_1, \ldots, w_r)$.

Let $\sigma_r$ denote the Euclidean closure of $\text{Im} \Sigma_r$. Then $D_r \subseteq \sigma_r$. By the Tarski–Seidenberg Theorem, $\sigma_r$ is semialgebraic. By [28, Theorem 3.7], a general $A \in U \otimes V \otimes W \setminus \sigma_r$ has a unique best approximation $\bar{\pi}_r(A)$ in $\sigma_r$. Note that for a nonnegative $A$, $\bar{\pi}_r(A) \in \sigma_r$ may be different from $\pi_r(A) \in D_r$.

In order to study best nonnegative rank approximations, i.e., the image of $\pi_r$, we first partition $D_r$ into a union of special semialgebraic subsets. For any index set $I_i \subseteq \{1, \ldots, n_U\}$, let

$$U_+(I_i) := \{u \in U_+ : \text{supp}(u) = I_i^c\}$$

and likewise define $\partial D_r$ nonnegative $r$ cell $V$ and likewise for $\partial D_r$. We write $D_r(I, J, K) := \{A \in D_r : A = \sum_{i=1}^r u_i \otimes v_i \otimes w_i, u_i \in U_+(I_i), v_i \in V_+(J_i), w_i \in W_+(K_i), i = 1, \ldots, r\}$.

The notion of a cell is important for our study of uniqueness because of the following easy observation.

**Lemma 30.** Let $A \in D_r$. If $A$ belongs to distinct cells, then $A$ has nonunique nonnegative r-term decompositions.

Clearly, if $I_i = J_i = K_i = \emptyset$ for all $i = 1, \ldots, r$, then $\dim D_r(I, J, K) = \dim D_r$ and we call this the trivial cell. The union of all nontrivial cells is called the boundary of $D_r$, and denoted by $\partial D_r$.

**Lemma 31.** If $r < r_g$ and $U \otimes V \otimes W$ is not r-defective, then $\dim \partial D_r < \dim D_r$.

**Proof.** We first describe $\partial D_r$ explicitly. Let $\alpha \in \{1, \ldots, n_U\}$ and $i \in \{1, \ldots, r\}$. Let $\bar{U}_+(\alpha) = \{u \in U_+ : \alpha \notin \text{supp}(u)\}$. Define

$$\partial D_{r,U}^{(i,\alpha)} := \Sigma_r((U_+ \times V_+ \times W_+)^{i-1} \times (\bar{U}_+(\alpha) \times V_+ \times W_+) \times (U_+ \times V_+ \times W_+)^{r-i})$$

We write

$$\partial D_{r,U} := \bigcup_{i=1}^r \bigcup_{\alpha=1}^{n_U} \partial D_{r,U}^{(i,\alpha)}$$

and likewise define $\partial D_{r,V}$ and $\partial D_{r,W}$. The boundary is then the union of these three semialgebraic subsets,

$$\partial D_r = \partial D_{r,U} \cup \partial D_{r,V} \cup \partial D_{r,W}.$$ 

From this description of $\partial D_r$, the required result is evident.
Let $A \in U_+ \otimes V_+ \otimes W_+$ where $\pi_r(A)$ has a nonnegative rank-$r$ decomposition $\pi_r(A) = \sum_{i=1}^r u_i \otimes v_i \otimes w_i$. If there is some $i \in \{1, \ldots, r\}$ such that strict inequality holds in (7.2), i.e., there is some $x_i \in U_+$ with

$$\langle A, x_i \otimes v_i \otimes w_i \rangle < \langle \pi_r(A), x_i \otimes v_i \otimes w_i \rangle,$$

then $\tilde{\pi}_r(A) \neq \pi_r(A)$ and $\pi_r(A) \in \partial D_r$ by Lemma 29. Similarly, if

$$\langle A, u_i \otimes y_i \otimes w_i \rangle < \langle \pi_r(A), u_i \otimes y_i \otimes w_i \rangle$$

(7.7)

or

$$\langle A, u_i \otimes v_i \otimes z_i \rangle < \langle \pi_r(A), u_i \otimes v_i \otimes z_i \rangle$$

(7.8)

for some $y_i \in V_+$ and $z_i \in W_+$, then $\tilde{\pi}_r(A) \neq \pi_r(A)$ and $\pi_r(A) \in \partial D_r$. We define the following sets:

$$\mathcal{L} = \{ \pi_r(A) \in \partial D_r : \pi_r(A) \text{ satisfies } (7.6), (7.7), \text{ or } (7.8) \},$$

(7.9)

$$\mathcal{N} = \{ A \in U_+ \otimes V_+ \otimes W_+ \setminus (S_r \cup H_r) : \pi_r(A) \in \mathcal{L} \}.$$  

(7.10)

We will next show that every positive tensor in $\mathcal{N}$ is an interior point.

**Proposition 32.** If $A \in \mathcal{N}$ is positive, then $A$ has an open neighborhood $\mathcal{V}$ such that $\mathcal{V} \subseteq \mathcal{N}$.

**Proof.** We first describe the structure of an open neighborhood $B(A, \eta)$ of a positive $A \in U_+ \otimes V_+ \otimes W_+$ and its image $\pi_r(B(A, \eta))$. By [38, Proposition 15], $\pi_r(A)$ always has nonnegative rank-$r$. Since $\pi_r$ is smooth, for any $\delta > 0$, there is some $\eta > 0$ such that $\pi_r(B(A, \eta)) \subseteq B(\pi_r(A), \delta) \cap D_r$. Observe that $\Sigma_r^{-1}(B(\pi_r(A), \delta) \cap D_r)$ is a union of at most a countable number of products of open balls, say,

$$\bigcup_{j=1}^s \left( B(u_{1}^{(j)}, \delta_1^{(j)}) \cap U_+ \right) \times \cdots \times \left( B(w_r^{(j)}, \delta_r^{(j)}) \cap W_+ \right) \subseteq (U_+ \times V_+ \times W_+)^r,$$

where $s \in \mathbb{N} \cup \{ \infty \}$, $u_{1}^{(j)} \in U_+$, $v_{i}^{(j)} \in V_+$, $w_r^{(j)} \in W_+$, and $\delta_1^{(j)} > 0$ for $i = 1, \ldots, r$, and $j = 1, \ldots, s$. By dimension count, there exists some $j$ such that the image of

$$U := \left( B(u_{1}^{(j)}, \delta_1^{(j)}) \cap U_+ \right) \times \cdots \times \left( B(w_r^{(j)}, \delta_r^{(j)}) \cap W_+ \right)$$

under $\Sigma_r$ contains $B(\pi_r(A), \delta) \cap D_r$. For notational convenience, we drop the superscript on $u_{1}^{(j)}, v_{i}^{(j)}, w_r^{(j)}$ and write $u_i, v_i, w_i$ below. By decreasing $\delta$ we may choose $\delta_1^{(j)} = \cdots = \delta_r^{(j)} = \varepsilon$ for some $\varepsilon > 0$ small enough. Furthermore, we may assume that $\pi_r(A) = \sum_{i=1}^r u_i \otimes v_i \otimes w_i$ is a nonnegative rank-$r$ decomposition. So for any $p \in B(A, \eta)$, $\pi_r(p)$ has a nonnegative rank-$r$ decomposition $\pi_r(p) = \sum_{i=1}^r u_i(p) \otimes v_i(p) \otimes w_i(p)$ where

$$\|u_i - u_i(p)\| \leq \varepsilon, \quad \|v_i - v_i(p)\| \leq \varepsilon, \quad \|w_i - w_i(p)\| \leq \varepsilon,$$

(7.11)

for $i = 1, \ldots, r$. Thus

$$\text{supp}(u_i) \subseteq \text{supp}(u_i(p)), \quad \text{supp}(v_i) \subseteq \text{supp}(v_i(p)), \quad \text{supp}(w_i) \subseteq \text{supp}(w_i(p)),$$

for $i = 1, \ldots, r$, and all $u_i(p), v_i(p)$ and $w_i(p)$ depend continuously on $p$. The function defined by

$$g(p) := \langle p - \pi_r(p), x_i \otimes v_i(p) \otimes w_i(p) \rangle$$

is therefore continuous on $B(A, \eta)$ for any fixed $x_i \in U_+$. If there is some $x_i \in U_+$ such that $\langle A - \pi_r(A), x_i \otimes v_i \otimes w_i \rangle < 0$, then by the continuity of $g$, there is an open neighborhood $\mathcal{V} \subseteq B(A, \eta)$ such $g(p) < 0$ for all $p \in \mathcal{V}$. Therefore $\mathcal{V} \subseteq \mathcal{N}$. □
The following theorem is the main result of this section. It characterizes the
relation between the image of $\pi_r$ and the cells of $D_r$. Its implication on nonnegative
tensor decomposition and approximation will be given in Corollary 34.

**THEOREM 33.** Let $\pi_r(A) \in D_r(I, J, K)$ for some cell $D_r(I, J, K) \neq \{0\}$. Let $\mathcal{V}$ be an open neighborhood of $A$. Then $\pi_r(\mathcal{V})$ contains an open subset of $D_r(I, J, K)$.

**Proof.** We consider two cases: If $\pi_r(\mathcal{V})$ is zero-dimensional, then we are led to a
contradiction and so this case cannot occur. If $\pi_r(\mathcal{V})$ is positive-dimensional, then we
show that it must have full dimension in $D_r(I, J, K)$ and therefore the required result
follows.

**Case 1.** $\pi_r(\mathcal{V}) = \pi_r(A)$ is a point.

Let $\gamma(t)$ be a curve in $\mathcal{V}$ with $\gamma(0) = A$. Then $\pi_r(\gamma(t)) = \pi_r(A)$ for any $t$. By
(7.5) we have
\[
\langle T_{\pi_r(A)}(u_1, \ldots, w_r), \gamma(t) - \pi_r(A) \rangle = 0,
\]
\[
\langle T_{\pi_r(A)}(u_1, \ldots, w_r), A - \pi_r(A) \rangle = 0,
\]
implying that
\[
\langle T_{\pi_r(A)}(u_1, \ldots, w_r), \gamma(t) - A \rangle = 0.
\]
Since the curve $\gamma(t)$ is arbitrary, we are led to the conclusion that
\[
\langle T_{\pi_r(A)}(u_1, \ldots, w_r), U \otimes V \otimes W \rangle = 0,
\]
contradicting the definition of $T_{\pi_r(A)}(u_1, \ldots, w_r)$.

**Case 2.** $\pi_r(\mathcal{V})$ is of positive dimension.

We will show that $\dim \pi_r(\mathcal{V}) = \dim D_r(I, J, K)$. By (7.11), we may assume that
$\pi_r(A)$ is a smooth point of $\pi_r(\mathcal{V})$ without loss of generality. By giving $\pi_r(\mathcal{V})$ a finer
stratification, we may furthermore assume that $\pi_r(\mathcal{V})$ is a Nash manifold. Suppose
that $\dim \pi_r(\mathcal{V}) < \dim D_r(I, J, K)$. Then by Theorem 2 there is an open semialgebraic
neighborhood $\mathcal{R}$ of $\pi_r(\mathcal{V})$ in $D_r(I, J, K)$ and a Nash retraction $f: \mathcal{R} \to \pi_r(\mathcal{V})$ such
that
\[
\text{dist}(p, \pi_r(\mathcal{V})) = ||p - f(p)||
\]
for any $p \in \mathcal{R}$. So there is a smooth curve $\gamma(t) \subseteq \mathcal{R}$ such that $\gamma(0) = \pi_r(A)$ and
$f(\gamma(t)) = \pi_r(A)$. Let $A(t) := A - \pi_r(A) + \gamma(t)$ and $X(t) := \pi_r(A(t)) \subseteq \pi_r(\mathcal{V})$. Note
that
\[
\gamma(t), X(t) \subseteq D_r(I, J, K), \quad A'(0), X'(0) \in T_{\pi_r(A)}(u_1, \ldots, w_r).
\]
By Lemma 29,
\[
\lim_{t \to 0^+} \frac{d}{dt}(A(t) - X(t), A(t) - X(t)) = 2(A'(0) - X'(0), A - X(0)) = 0.
\]
In fact, for any $s > 0$ small enough, we have
\[
\frac{d}{dt}(A(t) - X(t), A(t) - X(t)) \bigg|_{t=s} = 2(A'(s) - X'(s), A(s) - X(s)) = 0,
\]
implying that $||A(t) - X(t)||$ is constant around $t = 0$. On the other hand,
\[
||A(t) - \gamma(t)|| = ||A - \pi_r(A)||.
\]
So by the uniqueness of $\pi_r(A(t))$, $X(t) = \gamma(t)$, contradicting $\gamma(t) \subseteq \mathcal{R} \setminus \pi_r(\mathcal{V})$ for
t $> 0$. Therefore we must have $\dim \pi_r(\mathcal{V}) = \dim D_r(I, J, K)$. □
**Corollary 34.** Let \( r < r_g \), \( U \otimes V \otimes W \) be \( r \)-identifiable, and \( A \in U_+ \otimes V_+ \otimes W_+ \) be general. If the unique best nonnegative rank-\( r \) approximation \( \pi_r(A) \) of \( A \) is not in the boundary \( \partial D_r \), then \( \pi_r(A) \) has a unique nonnegative rank-\( r \) decomposition.

*Proof.* Since \( r < r_g \) and \( U \otimes V \otimes W \) is not \( r \)-defective, by Lemma 31,

\[
\dim \partial D_r < \dim D_r < \dim U \otimes V \otimes W.
\]

For any smooth point \( q \in D_r \), there is an open neighborhood \( Q \subseteq D_r \) of \( q \) such that any point in \( Q \) is also smooth. By Theorem 2, there is an open semialgebraic neighborhood \( \mathcal{R} \) of \( Q \) in \( U_+ \otimes V_+ \otimes W_+ \) and a Nash retraction \( f: \mathcal{R} \to Q \) such that \( \text{dist}(p, Q) = \|p - f(p)\| \) for every \( p \in \mathcal{R} \). By shrinking \( \mathcal{R} \) if necessary, we may assume that

\[
\|p - f(p)\| = \text{dist}(p, Q) = \text{dist}(p, D_r)
\]

for every \( p \in \mathcal{R} \), i.e., \( \pi_r(p) = f(p) \). Thus every smooth point of \( D_r \) is contained in \( \text{Im}(\pi_r) \), i.e., \( \text{Im}(\pi_r) \) is a semialgebraic subset of \( D_r \) with

\[
\dim \text{Im}(\pi_r) = \dim D_r > \dim \partial D_r.
\]

The required result then follows from Theorem 20 and Theorem 33 with the trivial cell \( D_r(I, J, K) \supseteq D_r \setminus \partial D_r \).

In the case of real tensors, it is possible that best rank-\( r \) approximations always lie on the boundary of the set of tensors of rank \( \leq r \) [19, Section 8]. So one might perhaps wonder whether Corollary 34 is vacuous. Fortunately this is not the case for nonnegative tensors provided that \( r < r_g \) and \( U \otimes V \otimes W \) is not \( r \)-defective. In fact, the condition (7.12) implies that \( \pi_r(A) \) is not always in \( \partial D_r \).

For the special cases \( r = 2 \) and 3, we can say considerably more than Corollary 34. We will first make an observation regarding the case when \( \pi_r(A) \in \mathcal{L} \) where \( \mathcal{L} \) is as defined in (7.9).

**Lemma 35.** Let \( \pi_r(A) \in \mathcal{L} \). Then

\[
\text{supp}(u_1) \cup \cdots \cup \text{supp}(u_r) = \{1, \ldots, n_U\},
\]

\[
\text{supp}(v_1) \cup \cdots \cup \text{supp}(v_r) = \{1, \ldots, n_V\},
\]

\[
\text{supp}(w_1) \cup \cdots \cup \text{supp}(w_r) = \{1, \ldots, n_W\}.
\]

*Proof.* Suppose \( 1 \notin \bigcup_{i=1}^r \text{supp}(u_i) \). Then by definition

\[
\langle A - \pi_r(A), e_1 \otimes v_1 \otimes w_1 \rangle \leq 0
\]

where \( e_1 = (1, 0, \ldots, 0) \). Since the coordinate \( (\pi_r(A))_{ijk} = 0 \) for any \( j = 1, \ldots, n_V \), \( k = 1, \ldots, n_W \), and \( A \) is positive, we have that \( (A - \pi_r(A))_{ijk} > 0 \). On the other hand, \( (e_1 \otimes v_1 \otimes w_1)_{ijk} = 0 \) for \( i \neq 1 \), and \( (e_1 \otimes v_1 \otimes w_1)_{ijk} \geq 0 \). Hence

\[
\langle A - \pi_r(A), e_1 \otimes v_1 \otimes w_1 \rangle > 0,
\]

a contradiction. \( \square \)

A cell \( D_r(I, J, K) \) is called *admissible* if

\[
\bigcap_{i=1}^r I_i = \bigcap_{i=1}^r J_i = \bigcap_{i=1}^r K_i = \emptyset.
\]

By Proposition 32, Theorem 33, and Lemma 35, if \( A \in \mathcal{N} \), then there is an open neighborhood \( \mathcal{V} \) of \( A \) such that \( \pi_r(\mathcal{V}) \) contains an open subset of some admissible cell
For small values of \( r \), we may check these admissible cells and possibly obtain uniqueness for nonnegative rank-\( r \) decomposition of \( \pi_r(A) \) for a general \( A \). We will do this explicitly for \( r = 2 \) and \( 3 \).

**Theorem 36.** Let \( r = 2 \) or \( 3 \) and let \( n_{U}, n_{V}, n_{W} \geq 3 \). Then for a general \( A \in U_{+} \otimes V_{+} \otimes W_{+} \), its unique best nonnegative rank-\( r \) approximation \( \pi_r(A) \) has a unique nonnegative rank-\( r \) decomposition.

**Proof.** By Corollary 34, it remains to check the case \( \pi_r(A) \in \partial D_r \) for a general \( A \). Theorem 33 and Lemma 35 further restrict the remaining case to checking (i) whether \( \pi_r(A) \) can be contained in an admissible cell, and (ii) whether \( \pi_r(A) \) contained in an admissible cell (if any) has a unique decomposition.

When \( r = 2 \), for a general \( p \) in any admissible cell \( D_r(I, J, K) \), let \( p = u_1 \otimes v_1 \otimes w_1 + u_2 \otimes v_2 \otimes w_2 \) be its nonnegative rank-2 decomposition. Then each set \( \{u_1, u_2\} \), \( \{v_1, v_2\} \), and \( \{w_1, w_2\} \) consists of a pair of linearly independent vectors. By [30], \( p \) has a unique real rank-2 decomposition and thus the nonnegative rank-2 decomposition is unique.

When \( r = 3 \), we may assume without loss of generality [19, Theorem 5.2] that \( n_{U} = n_{V} = n_{W} = 3 \). The only situation where a general point \( p \) of an admissible cell \( D_r(I, J, K) \) does not have a unique nonnegative rank-\( r \) decomposition is if

\[
\begin{align*}
I_1 &= I_2 = \{2, 3\}, \quad I_3 \subseteq \{1\}, \\
J_1 &= J_3 = \{2, 3\}, \quad J_2 \subseteq \{1\}, \\
K_2 &= K_3 = \{2, 3\}, \quad K_1 \subseteq \{1\},
\end{align*}
\]

up to a permutation of the index set \( \{1, 2, 3\} \). We claim that \( \pi_r(A) \) cannot be contained in such a cell \( D_r(I, J, K) \). Suppose not and \( \pi_r(A) \in D_r(I, J, K) \), i.e.,

\[
\begin{align*}
u_1 &= u_2 = (1, 0, \ldots, 0), \quad v_1 = v_3 = (1, 0, \ldots, 0), \\
w_2 &= w_3 = (1, 0, \ldots, 0).
\end{align*}
\]

Then \( (\pi_r(A))_{1jk} = 0 \) for \( j = 2, 3 \), \( k = 2, 3 \). Let

\[
p = u_1 \otimes v_1 \otimes w_1 + u_2 \otimes v_2 \otimes (w_2 + z) + u_3 \otimes v_3 \otimes w_3
\]

for some \( z = (0, \alpha, \beta) \) with \( \alpha, \beta > 0 \) small enough. Then \( \|A - p\| < \|A - \pi_r(A)\| \) for a positive \( A \), contradicting the definition of \( \pi_r(A) \). Therefore \( \pi_r(A) \notin D_r(I, J, K) \), a contradiction.

It is possible that a general point in an admissible cell \( D_r(I, J, K) \) may have non-unique nonnegative rank-\( r \) decompositions. To show uniqueness, we need to exclude such a possibility, i.e., check whether \( \pi_r(A) \) is contained in such a cell for a general \( A \). Evidently, it can be difficult to check all such cells when \( r \) is large. Further results in this direction would require more precise descriptions of \( I_1, \ldots, K_r \) where \( D_r(I, J, K) \cap \text{Im} \pi_r \neq \emptyset \).

**Acknowledgment.** The authors would like to thank G. Blekherman, I. Domanov, P. Eyssidieux, S. Friedland, J. M. Landsberg, B. Mourrain, and N. Vanvieuwenhoven for useful discussions.

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