April 5, 2006 — Iterative Methods

Note Title 4/5/2006

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

A: large, sparse, possibly structure

1) What is the source of the problem?

PDE's.

Data Fitting (splines)

2) Structure of A.

Is it symmetric? <u>Complex, symmetric?</u>

positive definite? indefinite?

Real, positive $\quad \underset{\sim}{x}^T A \underset{\sim}{x} > 0$

$\equiv \quad A + A^T$ p.d.

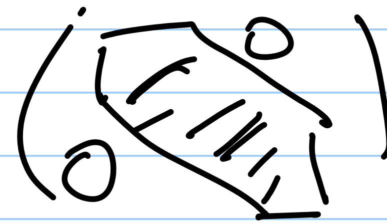
$$A = \begin{pmatrix} 1 & 100 \\ -100 & 1 \end{pmatrix} = H + S$$

$$\underset{\sim}{x}^T S \underset{\sim}{x} = 0$$

$$A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \qquad \begin{aligned} 1 - a^2 &> 0 \\ 1 &> a^2 \end{aligned}$$

3) Matrix Structure

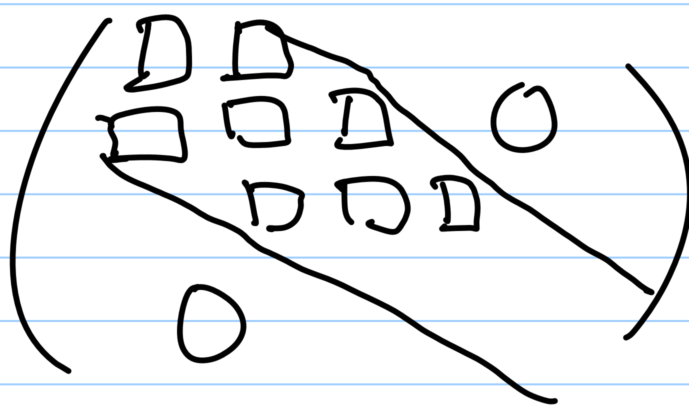
Band matrix



Block matrix

$$\begin{pmatrix} A_1 & B_1 & & 0 \\ B_1^T & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ 0 & & B_{n-1}^T & A_n \end{pmatrix}$$

$A_i : n_i \times n_i$ BLOCK tri-diagonal



4 Special techniques

Is the problem "close" to some other problem that is easy to solve

5. (*) $\| \underset{\sim}{b} - A \underset{\sim}{x} \|_2 = \min.$

i) $A^T A \underset{\sim}{x} = A^T \underset{\sim}{b}$ normal equations

ii) Solve (*) s.t. $C^T \underset{\sim}{x} = \underset{\sim}{d}$

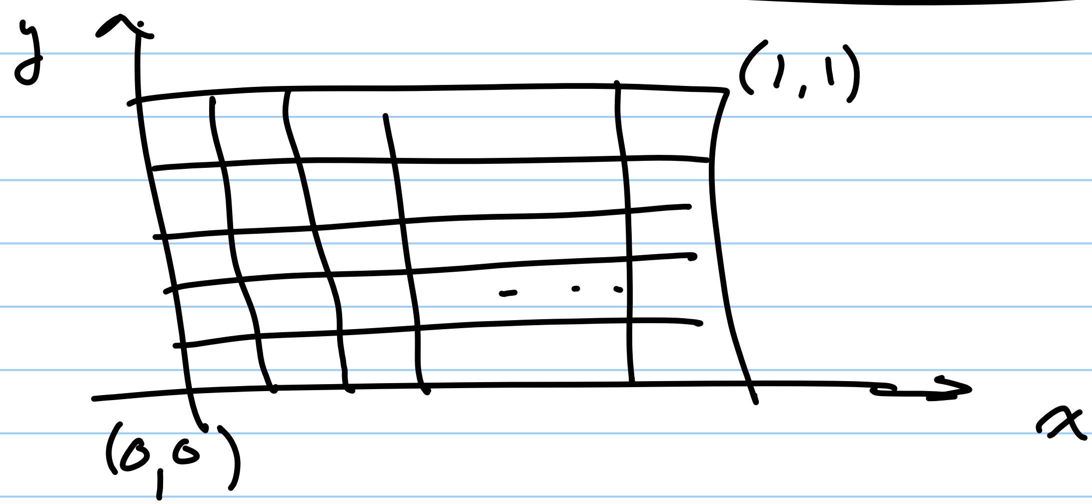
iii) $\| \underset{\sim}{x} \|_2 = \alpha$

(iv) $\underset{\sim}{x} \geq \underset{\sim}{0}$

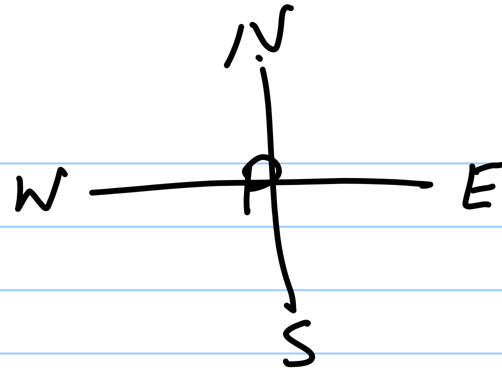
6) Computer Environments. Parallelism?

Model Problem.

$$-\Delta u = f \qquad (x,y) \in R$$
$$u = g \qquad (x,y) \in \partial R$$



$$x_i = ih, \quad h = \frac{1}{n+1}, \quad y_j = jh$$



$$\approx -u_{xx}$$

$$-\Delta_h U(P) = \frac{-U(W)+2U(P)-U(E)}{h^2}$$

$$+\ \frac{-U(N)+2U(P)-U(S)}{h^2}$$

$$\approx -u_{yy}$$

$$-u_{i,j-1}+2u_{ij}-u_{i,j+1}-u_{i-1,j}+2u_{ij}-u_{i+1,j}=h^2 f_{ij}$$

$$i,j = 1,\dots,n$$

$$A \underset{\sim}{u} = \underset{\sim}{f}$$

$$A = \begin{pmatrix} B & -I & & 0 \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ 0 & & -I & B \end{pmatrix} \quad B = \begin{pmatrix} 4 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 4 \end{pmatrix}$$

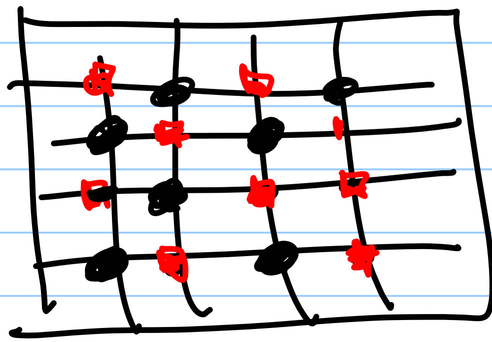
$A = A^T$, p.d.

$$\lambda_i(B) = 4 + 2\cos\frac{i\pi}{n+1} \quad i = 1, 2, \ldots, n$$

$$2 \le \lambda_i(B) \le 6$$

$\|B\|_\infty = 6$, $|4 - \lambda| \le 2$

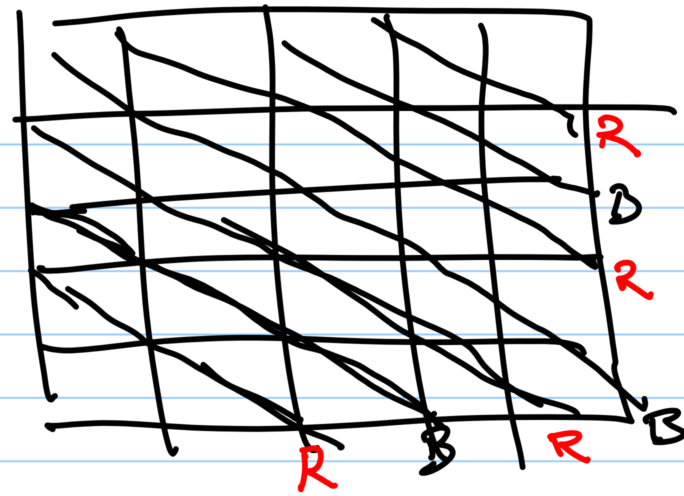
$$B = Q \Lambda Q^T, \quad q_{rs} = c_s \sin \frac{rs\pi}{n+1}, \quad c_s = \sqrt{\frac{2}{n+1}}$$



"natural ordering"

"checker board ordering"

$$\left(\begin{array}{c|c} 4I & //// \\ \hline //// & 4I \end{array}\right) \begin{pmatrix} u_R \\ u_B \end{pmatrix} =$$



$$\pi^T A \pi = \begin{pmatrix} D_1 & C_1 & & \\ C_1^T & D_2 & C_2 & \\ & \ddots & \ddots & \ddots \\ & & C_r^T & \end{pmatrix}$$

Fast Poisson Solvers

O. Bunemann / EE

$$-\Delta u + \sigma(x,y)\, u = f$$

$$0 < \sigma \leq \overline{\sigma}$$

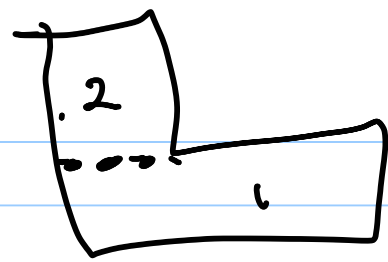
$$B \underset{\sim}{u} = \underset{\sim}{f}$$

$$B = A + h^2 \Sigma$$

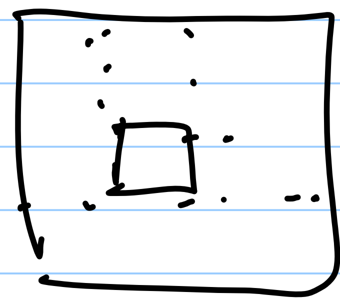
$$\Sigma = \begin{pmatrix} \Sigma_1 & & 0 \\ & \Sigma_2 & \\ 0 & & \Sigma_n \end{pmatrix} \qquad \Sigma_k = \sigma(x_n, y_j)$$

$$A \underset{\sim}{u}^{k+1} = f - h^2 \Sigma \underset{\sim}{u}^k$$

Converges?



L-shaped

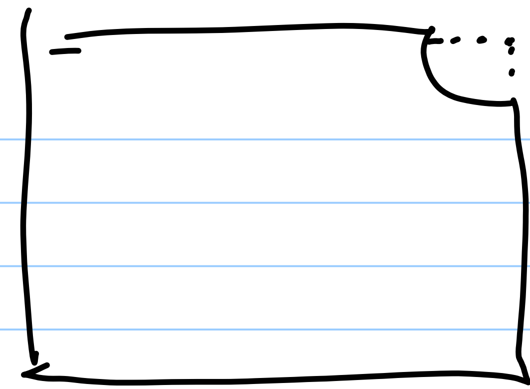


$$\left(\begin{array}{c|c} A_1 & C \\ \hline C^T & A_2 \end{array}\right)$$

$$= \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) + \left(\begin{array}{c|c} 0 & C \\ \hline C^T & 0 \end{array}\right)$$

low rank

"Domain Decomposition"



"embedding" "fictitious domain"

Splitting Methods / Pre-conditioning

Acceleration

$$\begin{pmatrix} D_1 & F \\ G & D_2 \end{pmatrix} \begin{pmatrix} \underset{\sim}{u} \\ \underset{\sim}{v} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{f} \\ \underset{\sim}{g} \end{pmatrix}$$

$$\begin{aligned} D_1 \underset{\sim}{u} + F \underset{\sim}{v} &= \underset{\sim}{f} \\ G \underset{\sim}{u} + D_2 \underset{\sim}{v} &= \underset{\sim}{g} \end{aligned} \quad \Bigg| \quad \underset{\sim}{u} = D_1^{-1} \underset{\sim}{f} - D_1^{-1} F \underset{\sim}{v}$$

$$G (D_1^{-1} f - D_1^{-1} F \underset{\sim}{v}) + D_2 \underset{\sim}{v} = g$$

$$(D_2 - G D_1^{-1} F) \underset{\sim}{v} = \underset{\sim}{k}$$

Schur complement

Block Gaussian Elimination

$$A\underset{\sim}{x} = \underset{\sim}{b};$$

$$A = M - N \quad \text{"splitting"}$$

$$M\underset{\sim}{x}^{k+1} = N\underset{\sim}{x}^{k} + \underset{\sim}{b}.$$

$$M\underset{\sim}{d}^{k} = (\underset{\sim}{b} - A\underset{\sim}{x}^{k}) = \langle\text{residual vector}\rangle$$

$$\boxed{\underset{\sim}{x}^{k+1} = \underset{\sim}{x}^{k} + \underset{\sim}{d}^{k}}$$

$$= \underset{\sim}{x}^{k} + M^{-1}(\underset{\sim}{b} - A\underset{\sim}{x}^{k})$$

$$= \underset{\sim}{x}^{k} + M^{-1}(\underset{\sim}{b} - (M-N)\underset{\sim}{x}^{k})$$

$$= \underset{\sim}{x}^{k} + M^{-1}b - (I - M^{-1}N)\underset{\sim}{x}^{k}$$

$$= M^{-1}\underset{\sim}{f} + M^{-1} N \underset{\sim}{x}^{k}$$

$$M\,\underset{\sim}{\bar{d}}^{(k)} = \underset{\sim}{r}^{(k)} + \underset{\sim}{\xi}^{k}$$

Inexact sol'n

Iterative Methods – April 6, 2006 II

Note Title 4/5/2006

$$a_{ii}\, x_i + \sum_{j\neq i} a_{ij}\, x_j = b_i$$

$$a_{ii}\, x_i^{(k+1)} = b_i - \sum_{j\neq i} a_{ij}\, x_j^{(k)}$$

Classical <u>Jacobi</u>

$$D = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \qquad E = -\begin{pmatrix} 0 & & 0 \\ & \ddots & \\ & & 0 \end{pmatrix}$$

$$F = -\begin{pmatrix} 0 & & \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

$$D\underset{\sim}{x}^{k+1} = \underset{\sim}{b} + (E+F)\underset{\sim}{x}^{(k)}$$

$M = D$, $N = (E+F)$.

$$D\underset{\sim}{x} = \underset{\sim}{b} + (E+F)\underset{\sim}{x}$$

$$D\underset{\sim}{e}^{k+1} = (E+F)\underset{\sim}{e}^{(k)}, \quad \underset{\sim}{e}^{k} = (\underset{\sim}{x}^{k} - \underset{\sim}{x})$$

$$\underset{\sim}{e}^{k+1} = D^{-1}(E+F)\underset{\sim}{e}^{(k)}, \quad \underset{\sim}{e}^{k} = \left(D^{-1}(E+F)\right)^{k}\underset{\sim}{e}^{0}$$

If I can show $\|D^{-1}(E+F)\| < 1$, then

$\|\underset{\sim}{e}^{k}\| \to 0$ as $k \to \infty$.

$$\|D^{-1}(E+F)\|_\infty = \max_i \sum_{j\neq i} \left|\frac{a_{ij}}{a_{ii}}\right|$$

$$\begin{pmatrix} 4 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 4 \end{pmatrix} \qquad \|D^{-1}(E+F)\|_\infty = \frac{1}{2}$$

$$a_{ii} x_i^{k+1} + \sum_{j<i} a_{ij} x_j^{k+1} + \sum_{j>i} a_{ij} x_j^{k} = b_i$$

Gauss-Seidel

$$(D-F)\,\underset{\sim}{x}^{k+1} = E\,\underset{\sim}{x}^{k} + \underset{\sim}{b}\,, \qquad \underline{M^{-1}N = (D-F)^{-1}E}$$

$$\omega A = (D - \omega E) - (\omega F + (1-\omega) D)$$

$$(D - \omega E)\, x^{k+1} = (\omega F + (1-\omega) D)\, x^{k} + \omega\, b$$

$$M^{-1} N = \mathcal{L}_{\omega} = (D - \omega E)^{-1} (\omega F + (1-\omega) D)$$

$$\boxed{x_i^{k+1} = x_i^{k} + \omega \Big(b_i - \sum_{j<i} a_{ij} x_j^{k+1} - \sum_{j>i} a_{ij} x_j^{k} \Big)}$$

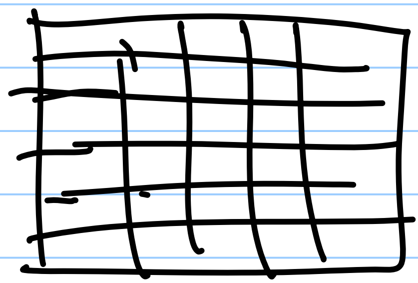
$\omega = 1$; G-S.

$1 < \omega < 2$: Overrelaxation

$0 < \omega < 1$ Underrelax

S.O.R. : succesive overrelaxation.

David Young (Harvard)



Symmetric S.O.R. $\equiv$ S.S.O.R.

$$(D-\omega E)\underset{\sim}{x}^{k+1/2} = (\omega F + (1-\omega)D)\underset{\sim}{x}^{k} + \omega \underset{\sim}{b}$$

$$(D-\omega F)\underset{\sim}{x}^{k+1} = (\omega E + (1-\omega)D)\underset{\sim}{x}^{k+1/2} + \omega \underset{\sim}{b}$$