STAT 310: MATHEMATICAL COMPUTATIONS II
WINTER 2012
PROBLEM SET 2

1. In general, a *semi-iterative method* is one that comprises two steps:

$$\mathbf{x}^{(k+1)} = M\mathbf{x}^{(k)} + \mathbf{b} \tag{Iteration}$$

and

$$\mathbf{y}^{(m)} = \sum_{k=0}^{m} \alpha_k^{(m)} \mathbf{x}^{(k)}. \tag{Extrapolation}$$

As in the lectures, we will assume that $M = I - A$ with $\rho(M) < 1$ and that we are interested to solve $A\mathbf{x} = \mathbf{b}$ for some nonsingular matrix $A \in \mathbb{C}^{n\times n}$. Let

$$\mathbf{e}^{(k)} = \mathbf{x}^{(k)} - \mathbf{x} \qquad \text{and} \qquad \boldsymbol{\varepsilon}^{(m)} = \mathbf{y}^{(m)} - \mathbf{x}.$$

(a) By considering what happens when $\mathbf{x}^{(0)} = \mathbf{x}$, show that it is natural to impose

$$\sum_{k=0}^{m} \alpha_k^{(m)} = 1 \tag{1.1}$$

for all $m \in \mathbb{N} \cup \{0\}$. Henceforth, we will assume that (1.1) is satisfied for all problems in this problem set.

(b) Show that for all $m \in \mathbb{N}$, we may write

$$\boldsymbol{\varepsilon}^{(m)} = P_m(M)\mathbf{e}^{(0)}$$

for some $P_m \in \mathbb{C}[x]$ with $\deg(P_m) = m$ and $P_m(1) = 1$.

(c) Hence deduce that a necessary condition for convergence is that

$$\limsup_{m\to\infty} \|P_m(M)\|_2 < 1$$

where $\|\cdot\|_2$ is the spectral norm. Is this condition also sufficient?

(d) Consider the case when

$$\alpha_0^{(m)} = \alpha_1^{(m)} = \cdots = \alpha_m^{(m)} = \frac{1}{m+1}$$

for all $m \in \mathbb{N} \cup \{0\}$. Show that if

$$\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{x}$$

then

$$\lim_{m\to\infty} \mathbf{y}^{(m)} = \mathbf{x}.$$

Is the converse also true?

2. It is clear that in any semi-iterative method defined by some $M \in \mathbb{C}^{n\times n}$ with $\rho(M) < 1$, we would like to solve the problem

$$\min_{P\in\mathbb{C}[x],\ \deg(P)=m,\ P(1)=1} \|P(M)\|_2. \tag{2.2}$$

Note that in the lectures, we required the polynomial P to be *monic*. Here we use a different condition, $P(1) = 1$, motivated by Problem **1**(a).

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(a) Show that if $m \geq n$, then a solution to (2.2) is given by

$$P_m(x) = \frac{x^{m-n}\det(xI - M)}{\det(I - M)}.$$

How do we know that the denominator is non-zero?

(b) From now on assume that M is Hermitian with minimum and maximum eigenvalues $\lambda_{\min}, \lambda_{\max} \in \mathbb{R}$. Define

$$\|f\|_\infty = \sup_{x \in [\lambda_{\min}, \lambda_{\max}]} |f(x)|.$$

Emulating our discussions in the lectures, show that for $m = 0, 1, \dots, n-1$, the solution to the relaxed problem

$$\min_{P \in \mathbb{C}[x],\ \deg(P)=m,\ P(1)=1} \|P\|_\infty \tag{2.3}$$

would yield an upper bound to (2.2).

(c) Again by emulating our discussions in the lectures, show that the solution to (2.3) for $\lambda_{\min} = -1$ and $\lambda_{\max} = +1$ is given by the Chebyshev polynomials,

$$C_m(x) = \begin{cases} \cos(m \cos^{-1} x) & |x| \leq 1, \\ \cosh(m \cosh^{-1} x) & |x| \geq 1. \end{cases}$$

(d) Hence deduce that the solution to (2.3) for $\lambda_{\min} = a$ and $\lambda_{\max} = b$ is given

$$P_m(x) = \frac{C_m\left(\dfrac{2x - (b+a)}{b-a}\right)}{C_m\left(\dfrac{2 - (b+a)}{b-a}\right)}. \tag{2.4}$$

Note that this solves (2.3) for all $m \in \mathbb{N}$ and not just $m \leq n-1$.

(e) Show that the solution in (d) is unique.

3. Let $M \in \mathbb{C}^{n \times n}$ be Hermitian with $\rho(M) = \rho < 1$. Moreover, suppose that

$$\lambda_{\min} = -\rho, \quad \lambda_{\max} = \rho.$$

(a) Show that the P_m's in (2.4) satisfy a three-term recurrence relation

$$C_{m+1}\left(\frac{1}{\rho}\right) P_{m+1}(x) = \frac{2x}{\rho} C_m\left(\frac{1}{\rho}\right) P_m(x) - C_{m-1}\left(\frac{1}{\rho}\right) P_{m-1}(x)$$

for all $m \in \mathbb{N}$.

(b) Show that the semi-iterative method with $\alpha_k^{(m)}$ given by the coefficient of P_m in (2.4) may be written as

$$\mathbf{y}^{(m+1)} = \omega_{m+1}(M\mathbf{y}^{(m)} - \mathbf{y}^{(m-1)} + \mathbf{b}) + \mathbf{y}^{(m-1)}$$

where $\omega_1 = 1$ and

$$\omega_{m+1} = \frac{2C_m(1/\rho)}{\rho C_{m+1}(1/\rho)}$$

for $m = 0, 1, 2, \dots$. This is a slightly different Chebyshev method where we choose the normalization (1.1) instead of $\alpha_m^{(m)} = 1$ in the lecture.

(c) Show that

$$\|P_m(M)\|_2 = \frac{1}{C_m(1/\rho)} = \frac{1}{\cosh(m\sigma)}$$

where $\sigma = \cosh^{-1}(1/\rho)$. Deduce that $\|P_m(M)\|_2$ is a strictly decreasing sequence for all $m = 0, 1, 2 \dots$.

(d) Show that
$$e^{-\sigma} = (\omega - 1)^{1/2}$$
where
$$\omega = \frac{2}{1+\sqrt{1-\rho^2}} \tag{3.5}$$
and deduce that
$$\|P_m(M)\|_2 = \frac{2(\omega-1)^{m/2}}{1+(\omega-1)^m}.$$

(e) Hence show that $(\omega_m)_{m=0}^{\infty}$ is strictly decreasing for $m \geq 2$ and that
$$\lim_{m\to\infty} \omega_m = \omega.$$

4. Let $M \in \mathbb{C}^{n\times n}$ be nonsingular with $\rho(M) < 1$ and suppose we are interested in solving $M\mathbf{x} = \mathbf{b}$.

(a) Show that SOR applied to the system
$$\begin{bmatrix} I & -M \\ -M & I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix}$$
yields the following iterations
$$\mathbf{x}^{(m+1)} = \omega(M\mathbf{z}^{(m)} - \mathbf{x}^{(m)} + \mathbf{b}) + \mathbf{x}^{(m)},$$
$$\mathbf{z}^{(m+1)} = \omega(M\mathbf{x}^{(m+1)} - \mathbf{z}^{(m)} + \mathbf{b}) + \mathbf{z}^{(m)},$$
for $m = 0, 1, 2, \dots$.

(b) Define the sequence of iterates $\mathbf{y}^{(m)}$ by
$$\mathbf{y}^{(m)} = \begin{cases} \mathbf{x}^{(k)} & \text{if } m = 2k, \\ \mathbf{z}^{(k)} & \text{if } m = 2k+1. \end{cases}$$
Show that the iterations obtained in (a) are exactly the iterations in Problem **3**(b). This shows that SOR is equivalent to Chebyshev but with $\omega_m = \omega$ for all $m \in \mathbb{N}$. Note that if ω is chosen to be the value in (3.5), then this is in fact the optimal SOR parameter (cf. Section 10 in the lecture notes on Stationary Methods).