STAT 310: MATHEMATICAL COMPUTATIONS II
WINTER 2012
PROBLEM SET 1

1. In the following, $\rho(A)$ will denote the spectral radius of a matrix $A \in \mathbb{C}^{n\times n}$ and $\|A\|$ an operator norm, i.e.

$$\|A\| = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p},$$

induced by vector norm $\|\cdot\|_p$ on $\mathbb{C}^n$.

(a) Show that

$$\rho(A) \le \|A\|.$$

(b) Show that

$$\sqrt{\rho(A^*A)} = \|A\|_2.$$

(c) Show that if A is Hermitian, i.e. $A^* = A$, then

$$\rho(A) = \|A\|_2.$$

(d) Let $f(x)$ be a polynomial with real coefficient and A be Hermitian, show that

$$\rho(f(A)) = \|f(A)\|_2.$$

(e) Consider the matrix

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$$

where $\alpha \in \mathbb{C}$. What are $\rho(A)$ and $\rho(A^*A)$? Verify that $\rho(A) < \|A\|$.

2. Recall that a sequence of matrices $(A_k)_{k=1}^{\infty}$ converges to a limt A iff

$$\lim_{k\to\infty} \|A_k - A\| = 0. \tag{2.1}$$

In which case we write

$$\lim_{k\to\infty} A_k = A.$$

Note that since all norms are equivalent on finite-dimensional spaces, this is independent of our choice of $\|\cdot\|$.

(a) Let $A_k = (a_{ij}^{(k)})_{i,j=1}^n$ and $A = (a_{ij})_{i,j=1}^n$. Show that $\lim_{k\to\infty} A_k = A$ if and only if

$$\lim_{k\to\infty} a_{ij}^{(k)} = a_{ij}$$

for all $i, j = 1, \ldots, n$. In other words, (2.1) is the same as entrywise convergence.

(b) Consider the sequence of matrices $(A^k)_{k=1}^{\infty}$. Note that here k is a power, not an index — the kth term of the sequence is

$$A^k = \underbrace{AA\cdots A}_{k \text{ times}}.$$

Show that if $\lim_{k\to\infty} A^k = O$, the $n \times n$ zero matrix, then $\rho(A) < 1$.

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(c) Consider a Jordan block

$$J = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}.$$

Show that for $k \geq n-1$,

$$J^k = \begin{bmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \binom{k}{n-1}\lambda^{k-(n-1)} \\ & \lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \binom{k}{n-2}\lambda^{k-(n-2)} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \binom{k}{1}\lambda^{k-1} \\ & & & & \lambda^k \end{bmatrix}.$$

(d) Using the fact that every $A \in \mathbb{C}^{n\times n}$ has a Jordan canonical form

$$SAS^{-1} = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix},$$

where J_i's are Jordan blocks, show that $\lim_{k\to\infty} A^k = O$ if and only if $\rho(A) < 1$.

(e) Let $A \in \mathbb{C}^{n\times n}$ be nonsingular and $A = M - N$ be a splitting. Show that the iterative method defined by

$$M\mathbf{x}^{(k+1)} = N\mathbf{x}^{(k)} + \mathbf{b}$$

converges for all $\mathbf{x}^{(0)} \in \mathbb{C}^n$ if and only if

$$\rho(M^{-1}N) < 1.$$

(f) Let $A \in \mathbb{C}^{n\times n}$ be arbitrary. Show that the series

$$\sum_{n=0}^{\infty} A^n,$$

i.e. the sequence of partial sums $(\sum_{n=0}^{k} A^n)_{k=1}^{\infty}$, converges if and only if $\rho(A) < 1$. What is the limit?

3. Let $A = (a_{ij})_{i,j=1}^n \in \mathbb{C}^{n\times n}$ be strictly diagonally dominant. Let the eigenvalues of A be $\lambda_1, \ldots, \lambda_n$. [*Hint*: Use Gersgorin's theorem in the notes on basic notions.]

(a) Show that A must be nonsingular. What if A is only diagonally dominant?

(b) Show that if $a_{ii} > 0$ for all $i = 1, \ldots, n$, i.e. A has real and positive diagonal, then

$$\operatorname{Re}\lambda_i > 0$$

for $i = 1, \ldots, n$. What if A is only diagonally dominant and we only have $a_{ii} \geq 0$?

(c) Show that if the matrix A in (b) is also Hermitian, then A is positive definite. What if A is only diagonally dominant?

4. Consider an $n \times n$ tridiagonal matrix of the form

$$T_\alpha = \begin{bmatrix} \alpha & -1 & & & & \\ -1 & \alpha & -1 & & & \\ & -1 & \alpha & -1 & & \\ & & -1 & \alpha & -1 & \\ & & & -1 & \alpha & -1 \\ & & & & -1 & \alpha \end{bmatrix},$$

where $\alpha \in \mathbb{R}$ is a real parameter.

(a) Verify that the eigenvalues of T_α are given by

$$\lambda_j = \alpha - 2\cos(j\theta), \qquad j = 1, \dots, n,$$

where

$$\theta = \frac{\pi}{n+1}$$

and that an eigenvector associated with each λ_j is

$$\mathbf{q}_j = [\sin(j\theta), \sin(2j\theta), \dots, \sin(nj\theta)]^\top.$$

Under what condition on α does this matrix become positive definite?

(b) Now take $\alpha = 2$.

(i) Will the Jacobi iteration converge for this matrix? If so, what will its convergence factor be?

(ii) Will the Gauss-Seidel iteration converge for this matrix? If so, what will its convergence factor be?

(iii) For which values of ω will the SOR iteration converge?