STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2020
PROBLEM SET 1

For Problem **4**, use any program you like but present your codes and results in a way that is comprehensible to someone who is unfamiliar with that program, i.e., comment as appropriate.

1. (a) Let $A \in \mathbb{C}^{m\times n}$ and $1 \le p \le \infty$. Find closed-form expressions for

$$\|A\|_{1,p} := \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_1} \qquad \text{and} \qquad \|A\|_{p,\infty} := \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_\infty}{\|\mathbf{x}\|_p}.$$

Your expressions should agree with the matrix 1-norm and matrix ∞-norm when $p = 1$ and ∞ respectively.

(b) Let $\mathbb{S}^n := \{A \in \mathbb{R}^{n\times n} : A^{\mathsf{T}} = A\}$. Recall the Gram matrix from Homework **0**, Problem **4**: For $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^n$, we write

$$G(\mathbf{x}_1, \dots, \mathbf{x}_n) := \begin{bmatrix} \mathbf{x}_1^{\mathsf{T}}\mathbf{x}_1 & \mathbf{x}_1^{\mathsf{T}}\mathbf{x}_2 & \dots & \mathbf{x}_1^{\mathsf{T}}\mathbf{x}_n \\ \mathbf{x}_2^{\mathsf{T}}\mathbf{x}_1 & \mathbf{x}_2^{\mathsf{T}}\mathbf{x}_2 & \dots & \mathbf{x}_2^{\mathsf{T}}\mathbf{x}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_n^{\mathsf{T}}\mathbf{x}_1 & \mathbf{x}_n^{\mathsf{T}}\mathbf{x}_2 & \dots & \mathbf{x}_n^{\mathsf{T}}\mathbf{x}_n \end{bmatrix} \in \mathbb{S}^n.$$

Consider the set $\mathbb{G}^n := \{G(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbb{S}^n : \|\mathbf{x}_1\|_2 \le 1, \dots, \|\mathbf{x}_n\|_2 \le 1\}$. Prove that for any $A \in \mathbb{S}^n$,

$$\|A\|_G := \max\{|\operatorname{tr}(AG)| : G \in \mathbb{G}^n\}$$

defines a norm on $\mathbb{S}^n$. Show that if $A = \operatorname{diag}(a_{11}, \dots, a_{nn}) \in \mathbb{S}^n$, then

$$\|A\|_G = \max\Bigl(\sum_{i=1}^n \max(a_{ii}, 0), -\sum_{i=1}^n \min(a_{ii}, 0)\Bigr).$$

2. Let $A \in \mathbb{C}^{n\times n}$ and $\|\cdot\|_p : \mathbb{C}^{n\times n} \to [0, \infty)$ be the matrix p-norm for some $p \in [1, \infty]$.

(a) Show that if $\|A\|_p < 1$, then $I - A$ is invertible and furthermore,

$$\frac{1}{1 + \|A\|_p} \le \|(I - A)^{-1}\|_p \le \frac{1}{1 - \|A\|_p}.$$

(b) Suppose A is invertible. Show that any $X \in \mathbb{C}^{n\times n}$ with

$$\|X - A\|_p < \frac{1}{\|A^{-1}\|_p}$$

must also be invertible.

(c) Let $\|\cdot\| : \mathbb{C}^{n\times n} \to [0, \infty)$ be an arbitrary norm that may not be submultiplicative. Suppose $\|A\| < 1$, can we conclude that $I - A$ is invertible?

3. Recall that in the lectures, we mentioned that (i) there are matrix norms that are not submultiplicative and an example is the Hölder ∞-norm; (ii) we may always construct a norm that approximates the spectral radius of a given matrix A as closely as we want.

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(a) Let $\|\cdot\| : \mathbb{C}^{m\times n} \to \mathbb{R}$ be a norm, defined for all $m, n \in \mathbb{N}$. Show that there always exists a $c > 0$ such that the constant multiple $\|\cdot\|_c := c\|\cdot\|$ defines a submultiplicative norm, i.e.,

$$\|AB\|_c \leq \|A\|_c\|B\|_c$$

for any $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ (even if $\|\cdot\|$ does not have this property). Find the constant c for the Hölder ∞-norm.

(b) Let $J \in \mathbb{C}^{n\times n}$ be in Jordan form, i.e.,

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix}$$

where each block J_r, for $r = 1, \dots, k$, has the form

$$J_r = \begin{bmatrix} \lambda_r & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_r \end{bmatrix}.$$

Let $\varepsilon > 0$ and $D_\varepsilon = \operatorname{diag}(1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1})$. Verify that

$$D_\varepsilon^{-1} J D_\varepsilon = \begin{bmatrix} J_{1,\varepsilon} & & \\ & \ddots & \\ & & J_{k,\varepsilon} \end{bmatrix}$$

where $J_{r,\varepsilon}$ is the matrix you obtain by replacing the 1's on the superdiagonal of J_r by ε's,

$$J_{r,\varepsilon} = \begin{bmatrix} \lambda_r & \varepsilon & & \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon \\ & & & \lambda_r \end{bmatrix}$$

(c) Show that

$$\|D_\varepsilon^{-1} J D_\varepsilon\|_\infty \leq \rho(J) + \varepsilon.$$

(d) Hence, or otherwise, show that for any given $A \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$, there exists a norm $\|\cdot\|_\star$ on $\mathbb{C}^n$ such that

$$\|A\| = \max_{x\neq 0} \frac{\|Ax\|_\star}{\|x\|_\star}$$

has the property that

$$\rho(A) \leq \|A\| \leq \rho(A) + \varepsilon.$$

(*Hint*: Transform A into Jordan form).

4. Let $A = [a_{ij}]$ be an $n \times n$ matrix with entries

$$a_{ij} = \begin{cases} n + 1 - \max(i, j) & i \leq j + 1, \\ 0 & i > j + 1. \end{cases}$$

This is an example of an *upper Hessenberg* matrix: it is upper triangular except that the entries on the subdiagonal $a_{j+1,j}$ may also be non-zero. For $n = 12$ and $n = 25$, do the following:

(a) Compute $\|A\|_\infty$ and $\|A\|_1$.
(b) Compute $\rho(A)$ and $\|A\|_2$. You may use any built-in functions of your program.
(c) Using Gerschgorin's theorem, describe the domain that contains all of the eigenvalues.

(d) Compute all of the eigenvalues and singular values of A. How many of the eigenvalues are real and how many are complex? You may use any built-in functions of your program.

5. You are not allowed to use the SVD for this problem, i.e., no arguments should depend on the SVD of A or A^*. Let W be a subspace of $\mathbb{C}^n$. The subspace $W^\perp$ below is called the *orthogonal complement* of W.

$$W^\perp = \{\mathbf{v} \in \mathbb{C}^n : \mathbf{v}^*\mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

For any subspace $W \subseteq \mathbb{C}^n$, we write $P_W \in \mathbb{C}^{n\times n}$ for an orthogonal projection onto W.

(a) Show that $\mathbb{C}^n = W \oplus W^\perp$ and that $W = (W^\perp)^\perp$.

(b) Let $A \in \mathbb{C}^{m\times n}$. Show that

$$\ker(A^*) = \operatorname{im}(A)^\perp \quad \text{and} \quad \operatorname{im}(A^*) = \ker(A)^\perp.$$

(c) Deduce the Fredholm alternative:

$$\mathbb{C}^m = \ker(A^*) \oplus \operatorname{im}(A) \quad \text{and} \quad \mathbb{C}^n = \operatorname{im}(A^*) \oplus \ker(A).$$

In other words any $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$ can be written uniquely as

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_0 + \mathbf{x}_1, \quad \mathbf{x}_0 \in \ker(A),\ \mathbf{x}_1 \in \operatorname{im}(A^*),\ \mathbf{x}_0^*\mathbf{x}_1 = 0,\\ \mathbf{y} &= \mathbf{y}_0 + \mathbf{y}_1, \quad \mathbf{y}_0 \in \ker(A^*),\ \mathbf{y}_1 \in \operatorname{im}(A),\ \mathbf{y}_0^*\mathbf{y}_1 = 0. \end{aligned}$$

(d) Show that

$$\mathbf{x}_0 = P_{\ker(A)}\mathbf{x}, \quad \mathbf{x}_1 = P_{\operatorname{im}(A^*)}\mathbf{x}, \quad \mathbf{y}_0 = P_{\ker(A^*)}\mathbf{y}, \quad \mathbf{y}_1 = P_{\operatorname{im}(A)}\mathbf{y}.$$

(e) Consider the least squares problem for some $\mathbf{b} \in \mathbb{C}^m$,

$$\min_{\mathbf{x}\in\mathbb{C}^n} \|\mathbf{b} - A\mathbf{x}\|_2. \tag{5.1}$$

Show that for any $\mathbf{x} \in \mathbb{C}^n$,

$$\|\mathbf{b} - A\mathbf{x}\|_2 \ge \|\mathbf{b}_0\|_2$$

where $\mathbf{b}_0 = P_{\ker(A^*)}\mathbf{b}$. Deduce that $\mathbf{x} \in \mathbb{C}^n$ is a solution to (5.1) if and only if

$$A\mathbf{x} = \mathbf{b}_1 \quad \text{or, equivalently,} \quad \mathbf{b} - A\mathbf{x} = \mathbf{b}_0. \tag{5.2}$$

Why is $A\mathbf{x} = \mathbf{b}_1$ consistent?

(f) Show that (5.2) is equivalent (i.e., if and only if) to the normal equation

$$A^*A\mathbf{x} = A^*\mathbf{b}. \tag{5.3}$$

Caveat: In numerical analysis, it is in general a terrible idea to solve a least squares problem via its normal equation. Nonetheless (5.3) can be useful in mathematical arguments. We will discuss in the lectures the very limited number of scenarios when it makes sense to solve (5.3) via Cholesky decomposition.

(g) Show that the pseudoinverse solution

$$\min\left\{\|\mathbf{x}\|_2 : \mathbf{x} \in \operatorname*{argmin}_{\mathbf{x}\in\mathbb{C}^n} \|\mathbf{b} - A\mathbf{x}\|_2\right\}$$

is given by

$$\mathbf{x}_1 = P_{\operatorname{im}(A^*)}\mathbf{x}$$

where $\mathbf{x} \in \mathbb{C}^n$ satisfies (5.2).

(h) Let $A \in \mathbb{C}^{n\times n}$ be normal, i.e., $A^*A = AA^*$. Show that

$$\ker(A^*) = \ker(A) \quad \text{and} \quad \operatorname{im}(A^*) = \operatorname{im}(A)$$

and deduce that for a normal matrix,

$$\mathbb{C}^n = \ker(A) \oplus \operatorname{im}(A).$$

6. Let $\mathbf{x} \in \mathbb{C}^m$, $\mathbf{y} \in \mathbb{C}^n$, and $A = \mathbf{x}\mathbf{y}^* \in \mathbb{C}^{m\times n}$.

(a) Show that

$$\|A\|_F = \|A\|_2 = \|\mathbf{x}\|_2\|\mathbf{y}\|_2 \tag{6.4}$$

and that

$$\|A\|_\infty = \|\mathbf{x}\|_\infty\|\mathbf{y}\|_1.$$

What can you say about $\|A\|_1$?

(b) Let $\mathbf{x}_1, \dots, \mathbf{x}_r \in \mathbb{C}^m$ be linearly independent and $\mathbf{y}_1, \dots, \mathbf{y}_r \in \mathbb{C}^n$ be linearly independent. Let

$$A = \mathbf{x}_1\mathbf{y}_1^* + \cdots + \mathbf{x}_r\mathbf{y}_r^*.$$

Show that $\operatorname{rank}(A) = r$. Show that this is not necessarily true if we drop either of the linear independence conditions.

(c) Given any $0 \neq A \in \mathbb{C}^{m\times n}$, show that

$$\operatorname{rank}(A) = \min\left\{r \in \mathbb{N} : A = \sum\nolimits_{i=1}^r \mathbf{x}_i\mathbf{y}_i^*\right\}.$$

In other words, the rank of a matrix is the smallest r so that it may be expressed as a sum of r rank-1 matrices.

(d) Show the following generalization of (6.4),

$$\|A\|_F \le \sqrt{\operatorname{rank}(A)}\|A\|_2.$$

Note that $\nu\operatorname{rank}(A) = \|A\|_F^2/\|A\|_2^2$ is one of the three notions of numerical ranks in the lecture notes. It is often used as a continuous surrogate for matrix rank.

(e) Show that with the nuclear norm we get instead

$$\|A\|_* \le \operatorname{rank}(A)\|A\|_2. \tag{6.5}$$

In other words we could also use $\|A\|_*/\|A\|_2$ as a continuous surrogate for matrix rank.