

STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2017
PROBLEM SET 4

1. Let $A \in \mathbb{R}^{m \times n}$ where $m \geq n$ and $\text{rank}(A) = n$. Suppose GECP is performed on A to get

$$\Pi_1 A \Pi_2 = LU$$

where $L \in \mathbb{R}^{m \times n}$ is unit lower triangular, $U \in \mathbb{R}^{n \times n}$ is upper triangular, and $\Pi_1 \in \mathbb{R}^{m \times m}$, $\Pi_2 \in \mathbb{R}^{n \times n}$ are permutation matrices.

- (a) Show that U is nonsingular and that L is of the form

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

where $L_1 \in \mathbb{R}^{n \times n}$ is nonsingular.

- (b) We will see how the LU factorization may be used to solve the least squares problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2.$$

- (i) Show that the problem may be solved via

$$U\tilde{\mathbf{x}} = \mathbf{y}, \quad L^T L \mathbf{y} = L^T \tilde{\mathbf{b}},$$

where $\tilde{\mathbf{b}} = \Pi_1 \mathbf{b}$ and $\tilde{\mathbf{x}} = \Pi_2^T \mathbf{x}$.

- (ii) Describe how you would compute the solution $\mathbf{y}$ in

$$L^T L \mathbf{y} = L^T \tilde{\mathbf{b}}.$$

2. Let $\varepsilon > 0$. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 + \varepsilon \\ 1 & 1 - \varepsilon \end{bmatrix}.$$

- (a) Why is it a bad idea to solve the normal equation associated with A , i.e.

$$A^T A \mathbf{x} = A^T \mathbf{b}$$

when ε is small?

- (b) Show that the LU factorization of A is

$$A = LU = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}.$$

- (c) Why is it a much better idea to solve the normal equation associated with L , i.e.

$$L^T L \mathbf{y} = L^T \tilde{\mathbf{b}}?$$

This shows that the method in Problem 1 is a more stable method than using the normal equation in (a) directly.

- (d) Show that the Moore–Penrose pseudoinverse of A is

$$A^\dagger = \frac{1}{6} \begin{bmatrix} 2 & 2 - 3\varepsilon^{-1} & 2 + 3\varepsilon^{-1} \\ 0 & 3\varepsilon^{-1} & -3\varepsilon^{-1} \end{bmatrix}.$$

- (e) Describe a method to compute $A^\dagger$ given L and U . Verify that your method is correct by checking it against the expression in (d).

3. We will now discuss an alternative method to solve the least squares problem in Problem 1 that is more efficient when $m - n < n$.

- (a) Show that the least squares problem in Problem 2 is equivalent to

$$\min_{\mathbf{z} \in \mathbb{R}^n} \left\| \begin{bmatrix} I_n \\ S \end{bmatrix} \mathbf{z} - \tilde{\mathbf{b}} \right\|_2$$

where $S = L_2 L_1^{-1}$ and $L_1 \mathbf{y} = \mathbf{z}$. Here and below, I_n denotes the $n \times n$ identity matrix.

- (b) Write

$$\tilde{\mathbf{b}} = \begin{bmatrix} \tilde{\mathbf{b}}_1 \\ \tilde{\mathbf{b}}_2 \end{bmatrix}$$

where $\tilde{\mathbf{b}}_1 \in \mathbb{R}^n$ and $\tilde{\mathbf{b}}_2 \in \mathbb{R}^{m-n}$. Show that the solution $\mathbf{z}$ is given by

$$\mathbf{z} = \tilde{\mathbf{b}}_1 + S^\top (I_{m-n} + SS^\top)^{-1} (\tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1).$$

- (c) Explain why when $m - n < n$, the method in (a) is much more efficient than the method in Problem 1. For example, what happens when $m = n + 1$?

4. Let $\mathbf{c} \in \mathbb{R}^n$ and consider the linearly constrained least squares problem

$$\begin{aligned} &\text{minimize} && \|\mathbf{w}\|_2 \\ &\text{subject to} && A^\top \mathbf{w} = \mathbf{c}. \end{aligned}$$

- (a) If we write $\tilde{\mathbf{c}} = \Pi_2^\top \mathbf{c}$ and $\tilde{\mathbf{w}} = \Pi_1 \mathbf{w}$, show that

$$\tilde{\mathbf{w}} = L(L^\top L)^{-1} U^{-\top} \tilde{\mathbf{c}}$$

where $U^{-\top} = (U^{-1})^\top = (U^\top)^{-1}$, a standard notation that we will also use below. (*Hint:* You'd need to use something that you've already determined in an earlier part).

- (b) Write

$$\tilde{\mathbf{w}} = \begin{bmatrix} \tilde{\mathbf{w}}_1 \\ \tilde{\mathbf{w}}_2 \end{bmatrix}$$

where $\tilde{\mathbf{w}}_1 \in \mathbb{R}^n$ and $\tilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}$. Show that

$$\tilde{\mathbf{w}}_1 = L_1^{-\top} U^{-\top} \tilde{\mathbf{c}} - S^\top \tilde{\mathbf{w}}_2.$$

- (c) Write $\mathbf{d} = L_1^{-\top} U^{-\top} \tilde{\mathbf{c}}$. Deduce that $\tilde{\mathbf{w}}_2$ may be obtained either as a solution to

$$\min_{\tilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| \begin{bmatrix} S^\top \\ I_{m-n} \end{bmatrix} \tilde{\mathbf{w}}_2 - \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix} \right\|_2$$

or as

$$\tilde{\mathbf{w}}_2 = (I_{m-n} + SS^\top)^{-1} S \mathbf{d}.$$

Note that when $m - n < n$, this method is advantageous for the same reason in Problem 4.

5. So far we have assumed that A has full column rank. Suppose now that $\text{rank}(A) = r < \min\{m, n\}$.

- (a) Show that the LU factorization obtained using GECP is of the form

$$\Pi_1 A \Pi_2 = LU = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

where $L_1, U_1 \in \mathbb{R}^{r \times r}$ are triangular and nonsingular.

(b) Show that the above equation may be rewritten in the form

$$\Pi_1 A \Pi_2 = \begin{bmatrix} I_r \\ S_1 \end{bmatrix} L_1 U_1 \begin{bmatrix} I_r & S_2^\top \end{bmatrix}$$

for some matrices S_1 and S_2 .

(c) Hence show that the Moore–Penrose inverse of A is given by

$$A^\dagger = \Pi_2 \begin{bmatrix} I_r & S_2^\top \end{bmatrix}^\dagger U_1^{-1} L_1^{-1} \begin{bmatrix} I_r \\ S_1 \end{bmatrix}^\dagger \Pi_1.$$

(d) Using the general formula (derived in the lectures) for the Moore–Penrose inverse of a rank-retaining factorization, what do you get for $A^\dagger$?

6. Consider the block matrix

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{n \times n}$$

where $A \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{p \times q}$, $C \in \mathbb{R}^{q \times p}$, $D \in \mathbb{R}^{q \times q}$ and $n = p + q$. The Schur complements of A and D are

$$S = D - CA^\dagger B \quad \text{and} \quad T = A - BD^\dagger C$$

respectively.

(a) Verify that if A and S are nonsingular, then

$$X^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}$$

and if D and T are nonsingular, then

$$X^{-1} = \begin{bmatrix} T^{-1} & -T^{-1}BD^{-1} \\ -D^{-1}CT^{-1} & D^{-1} + D^{-1}CT^{-1}BD^{-1} \end{bmatrix}.$$

(b) Show that

$$\det X = \begin{cases} \det(A) \det(D - CA^{-1}B) & \text{if } A \text{ nonsingular,} \\ \det(D) \det(A - BD^{-1}C) & \text{if } D \text{ nonsingular.} \end{cases}$$

Deduce that

$$\det(A + BC) = \det(A) \det(I + CA^{-1}B)$$

and use it to find the determinants of the following matrices

$$\begin{bmatrix} \frac{1+\lambda_1}{\lambda_1} & 1 & \cdots & 1 \\ 1 & \frac{1+\lambda_2}{\lambda_2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \frac{1+\lambda_n}{\lambda_n} \end{bmatrix}, \quad \begin{bmatrix} 1+\lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1 & 1+\lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \cdots & 1+\lambda_n \end{bmatrix}, \quad \begin{bmatrix} \lambda & \mu & \mu & \cdots & \mu \\ \mu & \lambda & \mu & \cdots & \mu \\ \mu & \mu & \lambda & \cdots & \mu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu & \mu & \mu & \cdots & \lambda \end{bmatrix}.$$

(c) Show that if A has all principal matrices nonsingular so that we may perform Gaussian elimination without pivoting to A , then applying the first p steps of that to X yields

$$X = \begin{bmatrix} L_{11} & 0 \\ L_{21} & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & I_q \end{bmatrix}$$

where $A = L_{11}U_{11}$ is the LU factorization of A . What are L_{21} and U_{12} in terms of L_{11} , U_{11} and the blocks of X ?

- (d) Suppose X is symmetric (so $C = B^\top$) and A is positive definite. Show that applying the first p steps of Cholesky factorization to X yields

$$X = \begin{bmatrix} R_{11}^\top \\ R_{12}^\top \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$$

where $A = R_{11}^\top R_{11}$ is the Cholesky factorization. What is R_{12} in terms of R_{11} and the blocks of X ?