STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2016
PROBLEM SET 0

This homework serves as a linear algebra refresher. We will recall some definitions. The null space or kernel of a matrix $A \in \mathbb{R}^{m \times n}$ is the set

$$\ker(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

while the range space or image is the set

$$\mathrm{im}(A) = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

The rank and nullity of A are defined as the dimensions of these spaces,

$$\mathrm{rank}(A) = \dim \mathrm{im}(A) \quad \text{and} \quad \mathrm{nullity}(A) = \dim \ker(A).$$

By convention we write all vectors in $\mathbb{R}^n$ as column vectors.

1. Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$.
   (a) Show that
$$\mathrm{im}(AB) \subseteq \mathrm{im}(A) \quad \text{and} \quad \ker(AB) \supseteq \ker(B).$$
   (b) Show that B has full row rank (so $n \leq p$), then
$$\mathrm{im}(AB) = \mathrm{im}(A).$$
   (c) Find a condition on A that guarantees
$$\ker(AB) = \ker(B).$$

2. Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$.
   (a) Show that
$$\mathrm{rank}(AB) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$$
   and
$$\mathrm{nullity}(AB) \leq \mathrm{nullity}(A) + \mathrm{nullity}(B).$$
   (b) Show that
$$\mathrm{rank}(A + B) \leq \mathrm{rank}(A) + \mathrm{rank}(B).$$
   (c) Show that if $AB = 0$, then
$$\mathrm{rank}(A) + \mathrm{rank}(B) \leq n.$$

3. (a) Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$. Show that
$$\mathrm{rank}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \mathrm{rank}(A) + \mathrm{rank}(B).$$

We have used the block matrix notation here. For example if $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in \mathbb{R}^{2 \times 3}$ and $B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{R}^{2 \times 1}$, then

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta \end{bmatrix} \in \mathbb{R}^{4 \times 4}.$$

This is sometimes also denoted as $A \oplus B$. It is a direct sum of operators induced by a direct sum of vector spaces.

(b) Let $\mathbf{x} = [x_1, \ldots, x_m]^\mathsf{T} \in \mathbb{R}^m$ and $\mathbf{y} = [y_1, \ldots, y_n]^\mathsf{T} \in \mathbb{R}^n$. Observe that $\mathbf{x}\mathbf{y}^\mathsf{T} \in \mathbb{R}^{m \times n}$. Show that
$$\operatorname{rank}(\mathbf{x}\mathbf{y}^\mathsf{T}) = 1$$
iff $\mathbf{x}$ and $\mathbf{y}$ are both nonzero.

4. Let $A \in \mathbb{R}^{m \times n}$.
  (a) Show that
$$\ker(A^\mathsf{T} A) = \ker(A) \qquad \text{and} \qquad \operatorname{im}(A^\mathsf{T} A) = \operatorname{im}(A^\mathsf{T}).$$
  Give an example to show this is not true over a finite field (e.g. a field of two elements $\mathbb{F}_2 = \{0, 1\}$ with binary arithmetic).
  (b) Show that
$$A^\mathsf{T} A\mathbf{x} = A^\mathsf{T}\mathbf{b}$$
  always has a solution (even if $A\mathbf{x} = \mathbf{b}$ has no solution). Give an example to show that this is not true over a finite field.

5. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ and let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be the matrix with
$$a_{ij} = \mathbf{v}_i^\mathsf{T}\mathbf{v}_j$$
for $i, j = 1, \ldots, n$. Show that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if and only if $\operatorname{nullity}(A) = 0$.

6. Show that if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r \in \mathbb{R}^n$ are pairwise orthogonal unit vectors, i.e., $\|\mathbf{u}_i\|_2 = 1$ for all $i = 1, \ldots, r$, and $\mathbf{u}_i^\mathsf{T}\mathbf{u}_j = 0$ for all $i \neq j$, then
$$\sum_{i=1}^r (\mathbf{v}^\mathsf{T}\mathbf{u}_i)^2 \leq \|\mathbf{v}\|_2^2 \tag{6.1}$$
for all $\mathbf{v} \in \mathbb{R}^n$. What can you say about $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ if equality always holds in (6.1) for all $\mathbf{v} \in \mathbb{R}^n$?

7. Let $A \in \mathbb{C}^{n \times n}$. Recall that A is diagonalizable iff there exists an invertible $X \in \mathbb{C}^{n \times n}$ such that $X^{-1}AX = \Lambda$, a diagonal matrix.
  (a) Show that A is diagonalizable if and only if its minimal polynomial is of the form
$$m_A(x) = (x - \lambda_1) \cdots (x - \lambda_d)$$
  where $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$ are all distinct. Hence deduce for a diagonalizable matrix, the degree of its minimal polynomial equals the number of distinct eigenvalues.
  (b) Let A be diagonalizable. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{C}^n$ be n linearly independent right eigenvectors, i.e., $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$; and $\mathbf{y}_1, \ldots, \mathbf{y}_n \in \mathbb{C}^n$ be n linearly independent left eigenvectors, i.e., $\mathbf{y}_i^\mathsf{T} A = \lambda_i\mathbf{y}_i^\mathsf{T}$. Show that there is a choice of left and right eigenvectors of A such that any vector $\mathbf{v} \in \mathbb{C}^n$ can be expressed as
$$\mathbf{v} = \sum_{i=1}^n (\mathbf{y}_i^\mathsf{T}\mathbf{v})\mathbf{x}_i.$$
  If we write $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{C}^{n \times n}$ and $Y = [\mathbf{y}_1, \ldots, \mathbf{y}_n] \in \mathbb{C}^{n \times n}$. What is the relation between X and Y?