STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2014
PROBLEM SET 2

1. Let $A = [a_{ij}]$ be an $n \times n$ matrix with entries

$$a_{ij} = \begin{cases} n + 1 - \max(i, j) & i \le j + 1, \\ 0 & i > j + 1. \end{cases}$$

This is an example of an *upper Hessenberg* matrix: it is upper triangular except that the entries on the subdiagonal $a_{j+1,j}$ may also be non-zero. For $n = 12$ and $n = 25$, do the following:

(a) Compute $\|A\|_\infty$ and $\|A\|_1$.
(b) Compute $\rho(A)$ and $\|A\|_2$. You may use any built-in functions of your program.
(c) Using Gerschgorin's theorem, describe the domain that contains all of the eigenvalues.
(d) Compute all of the eigenvalues and singular values of A. How many of the eigenvalues are real and how many are complex? You may use any built-in functions of your program.

2. You are not allowed to use the SVD for this problem, i.e., no arguments should depend on the SVD of A or A^*. Let W be a subspace of $\mathbb{C}^n$. The subspace $W^\perp$ below is called the *orthogonal complement* of W.

$$W^\perp = \{\mathbf{v} \in \mathbb{C}^n : \mathbf{v}^*\mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

For any subspace $W \subseteq \mathbb{C}^n$, we write $P_W \in \mathbb{C}^{n\times n}$ for an orthogonal projection onto W.

(a) Show that $\mathbb{C}^n = W \oplus W^\perp$ and that $W = (W^\perp)^\perp$.
(b) Let $A \in \mathbb{C}^{m\times n}$. Show that

$$\ker(A^*) = \operatorname{im}(A)^\perp \quad \text{and} \quad \operatorname{im}(A^*) = \ker(A)^\perp.$$

(c) Deduce the Fredholm alternative:

$$\mathbb{C}^m = \ker(A^*) \oplus \operatorname{im}(A) \quad \text{and} \quad \mathbb{C}^n = \operatorname{im}(A^*) \oplus \ker(A).$$

In other words any $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$ can be written uniquely as

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1, \quad \mathbf{x}_0 \in \ker(A),\ \mathbf{x}_1 \in \operatorname{im}(A^*),\ \mathbf{x}_0^*\mathbf{x}_1 = 0,$$
$$\mathbf{y} = \mathbf{y}_0 + \mathbf{y}_1, \quad \mathbf{y}_0 \in \ker(A^*),\ \mathbf{y}_1 \in \operatorname{im}(A),\ \mathbf{y}_0^*\mathbf{y}_1 = 0.$$

(d) Show that

$$\mathbf{x}_0 = P_{\ker(A)}\mathbf{x}, \quad \mathbf{x}_1 = P_{\operatorname{im}(A^*)}\mathbf{x}, \quad \mathbf{y}_0 = P_{\ker(A^*)}\mathbf{y}, \quad \mathbf{y}_1 = P_{\operatorname{im}(A)}\mathbf{y}.$$

(e) Consider the least squares problem for some $\mathbf{b} \in \mathbb{C}^m$,

$$\min_{\mathbf{x}\in\mathbb{C}^n} \|\mathbf{b} - A\mathbf{x}\|_2. \tag{2.1}$$

Show that for any $\mathbf{x} \in \mathbb{C}^n$,

$$\|\mathbf{b} - A\mathbf{x}\|_2 \ge \|\mathbf{b}_0\|_2$$

where $\mathbf{b}_0 = P_{\ker(A^*)}\mathbf{b}$. Deduce that $\mathbf{x} \in \mathbb{C}^n$ is a solution to (2.1) if and only if

$$A\mathbf{x} = \mathbf{b}_1 \quad \text{or, equivalently,} \quad \mathbf{b} - A\mathbf{x} = \mathbf{b}_0. \tag{2.2}$$

Why is $A\mathbf{x} = \mathbf{b}_1$ consistent?

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(f) Show that (2.2) is equivalent (i.e., if and only if) to the normal equation

$$A^*A\mathbf{x} = A^*\mathbf{b}. \tag{2.3}$$

Caveat: In numerical analysis, it is in general a terrible idea to solve a least squares problem via its normal equation. Nonetheless (2.3) can be useful in mathematical arguments. We will discuss in the lectures the very limited number of scenarios when it makes sense to solve (2.3) via Cholesky decomposition.

(g) Show that the pseudoinverse solution

$$\min\left\{\|\mathbf{x}\|_2 : \mathbf{x} \in \operatorname*{argmin}_{\mathbf{x}\in\mathbb{C}^n} \|\mathbf{b} - A\mathbf{x}\|_2\right\}$$

is given by

$$\mathbf{x}_1 = P_{\operatorname{im}(A^*)}\mathbf{x}$$

where $\mathbf{x} \in \mathbb{C}^n$ satisfies (2.2).

(h) Let $A \in \mathbb{C}^{n\times n}$ be normal, i.e., $A^*A = AA^*$. Show that

$$\ker(A^*) = \ker(A) \quad \text{and} \quad \operatorname{im}(A^*) = \operatorname{im}(A)$$

and deduce that for a normal matrix,

$$\mathbb{C}^n = \ker(A) \oplus \operatorname{im}(A).$$

3. The files required for this problem are in `http://www.stat.uchicago.edu/~lekheng/courses/309/stat309-hw2/`. The matrix in `processed.mat` (Matlab format) or `processed.txt` (comma separated, plain text) is a 49×7 matrix where each row is indexed by a country in `row.txt` and each column is indexed by a demographic variable in `column.txt`, ordered as in the respective files. So for example, if we denote the matrix by $A = [a_{ij}] \in \mathbb{R}^{49\times 7}$, then $a_{23} = -0.2743$ is Austria's population per square kilometers (row index 2 = Austria, column index 3 = population per square kilometers). As you probably notice, this matrix has been slightly preprocessed. If you want to see the raw data, you can find them in `raw.txt` (e.g. the actual value for Austria's population per square kilometers is 84) but you don't need the raw data for this problem.

(a) Find the first two right singular vectors of A, $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^7$. Project the data onto the two-dimensional space spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. Plot this in a graph where the x- and y-axes correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively and where the points correspond to the countries — label each point by the country it corresponds to. Identify the two obvious outliers.

(b) Now do the same with the two left singular vectors of A, $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^{49}$, i.e., project the data onto the two-dimensional space spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$ and plot this in a graph as before. Note that in this case, the points correspond to the demographic variables — label them accordingly.

(c) Overlay the two graphs in (a) and (b). Identify the two demographic variables near the two outlier countries — these explain why the two countries are outliers.

(d) Remove the two outlier countries and redo (a) with this 47×7 matrix. This allows you to see features that were earlier obscured by the outliers. Which two European countries are most alike Japan?

The graphs in (a) and (b) are called *scatter plots* and the overlayed one in (c) is called a *biplot*. See `http://en.wikipedia.org/wiki/Biplot` for more information. The reason we didn't need to adjust the scale of the axes using the singular values of A like in the Wikipedia description is because the preprocessing has taken care of the scaling; if we had started from the raw data, then we would need to deal with this complication.

4. Let $\mathbf{x} \in \mathbb{C}^m$, $\mathbf{y} \in \mathbb{C}^n$, and $A = \mathbf{x}\mathbf{y}^* \in \mathbb{C}^{m\times n}$.

(a) Show that $\operatorname{rank}(A) = 1$ iff $\mathbf{x}$ and $\mathbf{y}$ are both non-zero. Such a matrix is usually called a rank-1 matrix.

(b) Show that

$$\|A\|_F = \|A\|_2 = \|\mathbf{x}\|_2\|\mathbf{y}\|_2 \tag{4.4}$$

and that

$$\|A\|_\infty \le \|\mathbf{x}\|_\infty\|\mathbf{y}\|_1.$$

What can you say about $\|A\|_1$?

(c) Let $\mathbf{x}_1, \dots, \mathbf{x}_r \in \mathbb{C}^m$ be linearly independent and $\mathbf{y}_1, \dots, \mathbf{y}_r \in \mathbb{C}^n$ be linearly independent. Let

$$A = \mathbf{x}_1\mathbf{y}_1^* + \cdots + \mathbf{x}_r\mathbf{y}_r^*.$$

Show that $\operatorname{rank}(A) = r$. Show that this is not necessarily true if we drop either of the linear independence conditions.

(d) Given any $0 \ne A \in \mathbb{C}^{m\times n}$, show that

$$\operatorname{rank}(A) = \min\left\{r \in \mathbb{N} : A = \sum\nolimits_{i=1}^r \mathbf{x}_i\mathbf{y}_i^*\right\}.$$

In other words, the rank of a matrix is the smallest r so that it may be expressed as a sum of r rank-1 matrices.

(e) Show the following generalization of (4.4),

$$\|A\|_F \le \sqrt{\operatorname{rank}(A)}\|A\|_2.$$

Note that $\nu\operatorname{rank}(A) = \|A\|_F^2/\|A\|_2^2$ is one of the three notions of numerical ranks that we discussed. It is often used as a continuous surrogate for matrix rank.

(f) Show that with the nuclear norm we get instead

$$\|A\|_* \le \operatorname{rank}(A)\|A\|_2. \tag{4.5}$$

In other words we could also use $\|A\|_*/\|A\|_2$ as a continuous surrogate for matrix rank.

5. Let $A, B \in \mathbb{C}^{m\times n}$ with $n \le m$. In the lectures, we claim that the solution to

$$\min_{X^*X=I}\|A - BX\|_F$$

is given by $X = UV^*$ where $B^*A = U\Sigma V^*$ is its singular value decomposition. Here we will prove it and consider some variants.

(a) Show that

$$\|A - BX\|_F^2 = \operatorname{tr}(A^*A) + \operatorname{tr}(B^*B) - 2\operatorname{Re}\operatorname{tr}(X^*B^*A)$$

and deduce that the minimization problem is equivalent to

$$\max_{X^*X=I}\operatorname{Re}\operatorname{tr}(X^*B^*A).$$

(b) Show that

$$\operatorname{Re}\operatorname{tr}(X^*B^*A) \le \sum_{i=1}^n \sigma_i(B^*A)$$

for any unitary X. When is the upper bound attained?

(c) Show that

$$\min_{X^*X=I}\|A - BX\|_F^2 = \sum_{i=1}^n(\sigma_i(A)^2 - 2\sigma_i(B^*A) + \sigma_i(B)^2).$$

(d) Suppose $A \in \mathbb{R}^{m\times n}$ has full column rank. Show that the following method produces a symmetric matrix $X \in \mathbb{R}^{n\times n}$ that solves

$$\min_{X^\mathsf{T}=X}\|AX - B\|_F. \tag{5.6}$$

(i) Show that the SVD of A takes the form

$$A = U \begin{bmatrix} \Sigma \\ O \end{bmatrix} V^{\mathsf{T}}$$

where $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ are unitary matrices and $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n) \in \mathbb{R}^{n\times n}$ is a diagonal matrix.

(ii) Show that

$$\|AX - B\|_F^2 = \|\Sigma Y - C_1\|_F^2 + \|C_2\|_F^2$$

where $Y = V^{\mathsf{T}} X V$ and $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = U^{\mathsf{T}} B V$.

(iii) Note that Y must be symmetric if X is. Show that

$$\|\Sigma Y - C_1\|_F^2 = \sum_{i=1}^{n} |\sigma_i y_{ii} - c_{ii}|^2 + \sum_{j>i} \left(|\sigma_i y_{ij} - c_{ij}|^2 + |\sigma_j y_{ij} - c_{ji}|^2 \right)$$

and deduce that the minimum value of (5.6) is attained when

$$y_{ij} = \frac{\sigma_i c_{ij} + \sigma_j c_{ji}}{\sigma_i^2 + \sigma_j^2}$$

for all $i, j = 1, \dots, n$.

(e) Let $A \in \mathbb{R}^{m\times n}$. Show that

$$\min_{X \in \mathbb{R}^{n\times m}} \|AX - I_m\|_F$$

has a unique solution when A has full column rank. In general, what is the minimum length solution (i.e., where $\|X\|_F$ is minimum) to this problem?

6. In the following, $\kappa(A) := \|A\|\|A^{\dagger}\|$ for $A \in \mathbb{C}^{m\times n}$ where $\|\cdot\|$ denotes a submultiplicative matrix norm. We will write $\kappa_p(A)$ if the norm involved is a matrix p-norm.

(a) Show that for any nonzero $A \in \mathbb{C}^{m\times n}$,

$$\kappa(A) \geq 1.$$

(b) Show that for any $A \in \mathbb{C}^{m\times n}$,

$$\kappa_2(A^*A) = \kappa_2(A)^2$$

but that in general

$$\kappa(A^*A) \neq \kappa(A)^2.$$

(c) Show that for nonsingular $A, B \in \mathbb{C}^{n\times n}$,

$$\kappa(AB) \leq \kappa(A)\kappa(B).$$

Is this true in general without the nonsingular condition?

(d) Let $Q \in \mathbb{C}^{m\times n}$ be a matrix with orthonormal columns. Show that

$$\kappa_2(Q) = 1.$$

Is this true if Q has orthonormal rows instead? Is this true with κ_1 or κ_∞ in place of κ_2?

(e) Let $R \in \mathbb{C}^{n\times n}$ be a nonsingular upper-triangular matrix. Show that

$$\kappa_\infty(R) \geq \frac{\max_{i=1,\dots,n}|r_{ii}|}{\min_{i=1,\dots,n}|r_{ii}|}.$$

(f) Show that for any nonsingular $A \in \mathbb{C}^{n\times n}$,

$$\kappa(A) \geq \max\left\{ \frac{\|AX - I\|}{\|XA - I\|}, \frac{\|XA - I\|}{\|AX - I\|} \right\}.$$

(*Hint:* $AX - I = A(XA - I)A^{-1}$.)