STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2013
PROBLEM SET 0

This homework serves as a linear algebra refresher. We will recall some definitions. The null space or kernel of a matrix $A \in \mathbb{R}^{m\times n}$ is the set

$$\ker(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

while the range space or image is the set

$$\operatorname{im}(A) = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

The rank and nullity of A are defined as the dimensions of these spaces,

$$\operatorname{rank}(A) = \dim \operatorname{im}(A) \qquad \text{and} \qquad \operatorname{nullity}(A) = \dim \ker(A).$$

By convention we write all vectors in $\mathbb{R}^n$ as column vectors.

1. Let $A \in \mathbb{R}^{n\times n}$. Show that the following statements are equivalent:
  (i) $\mathbb{R}^n = \ker(A) + \operatorname{im}(A)$;
  (ii) $\ker(A) \cap \operatorname{im}(A) = \{\mathbf{0}\}$;
  (iii) $\mathbb{R}^n = \ker(A) \oplus \operatorname{im}(A)$.

2. Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$.
  (a) Show that

$$\operatorname{im}(AB) \subseteq \operatorname{im}(A) \qquad \text{and} \qquad \ker(AB) \supseteq \ker(B).$$

  (b) Show that B has full row rank (so $n \le p$), then

$$\operatorname{im}(AB) = \operatorname{im}(A).$$

  (c) Find a condition on A that guarantees

$$\ker(AB) = \ker(B).$$

3. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$.
  (a) Show that

$$\operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$$

  and

$$\operatorname{nullity}(AB) \le \operatorname{nullity}(A) + \operatorname{nullity}(B).$$

  (b) Show that

$$\operatorname{rank}(A + B) \le \operatorname{rank}(A) + \operatorname{rank}(B).$$

  (c) Show that if $AB = 0$, then

$$\operatorname{rank}(A) + \operatorname{rank}(B) \le n.$$

4. (a) Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. Show that

$$\operatorname{rank}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \operatorname{rank}(A) + \operatorname{rank}(B).$$

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We have used the block matrix notation here. For example if $A = \left[\begin{smallmatrix} a & b & c \\ d & e & f \end{smallmatrix}\right] \in \mathbb{R}^{2\times 3}$ and $B = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] \in \mathbb{R}^{2\times 1}$, then

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

This is sometimes also denoted as $A \oplus B$. It is a direct sum of operators induced by a direct sum of vector spaces.

(b) Let $\mathbf{x} = [x_1, \dots, x_m]^\mathsf{T} \in \mathbb{R}^m$ and $\mathbf{y} = [y_1, \dots, y_n]^\mathsf{T} \in \mathbb{R}^n$. Observe that $\mathbf{x}\mathbf{y}^\mathsf{T} \in \mathbb{R}^{m\times n}$. Show that

$$\operatorname{rank}(\mathbf{x}\mathbf{y}^\mathsf{T}) = 1$$

iff $\mathbf{x}$ and $\mathbf{y}$ are both nonzero.

5. Let $A \in \mathbb{R}^{m\times n}$.

(a) Show that

$$\ker(A^\mathsf{T} A) = \ker(A) \qquad \text{and} \qquad \operatorname{im}(A^\mathsf{T} A) = \operatorname{im}(A^\mathsf{T}).$$

Give an example to show this is not true over a finite field (e.g. a field of two elements $\mathbb{F}_2 = \{0, 1\}$ with binary arithmetic).

(b) Show that

$$A^\mathsf{T} A\mathbf{x} = A^\mathsf{T}\mathbf{b}$$

always has a solution (even if $A\mathbf{x} = \mathbf{b}$ has no solution). Give an example to show that this is not true over a finite field.

6. Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

(a) Let $\mathbf{x}_0 \in \mathbb{R}^n$ be a solution of $A\mathbf{x} = \mathbf{b}$. Show that every solution of $A\mathbf{x} = \mathbf{b}$ is of the form

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{z}$$

where $\mathbf{z} \in \ker(A)$.

(b) Suppose $\mathbf{b} \neq \mathbf{0}$. Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathbb{R}^n$ be solutions of $A\mathbf{x} = \mathbf{b}$, i.e., $A\mathbf{x}_i = \mathbf{b}$ for all $i \in \{1, \dots, k\}$. Show that the linear combination

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_k\mathbf{x}_k$$

is also a solution to $A\mathbf{x} = \mathbf{b}$ if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1.$$

(c) Show that if $A\mathbf{x} = \mathbf{0}$ has a non-zero complex solution, i.e., there exists $\mathbf{z} \in \mathbb{C}^m$, $\mathbf{z} \neq \mathbf{0}$, such that $A\mathbf{z} = \mathbf{0}$, then there exists a non-zero real solution.

7. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^n$ and let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be the matrix with

$$a_{ij} = \mathbf{v}_i^\mathsf{T}\mathbf{v}_j$$

for $i, j = 1, \dots, n$. Show that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent if and only if $\operatorname{nullity}(A) = 0$.

8. Show that if $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r \in \mathbb{R}^n$ are pairwise orthogonal unit vectors, i.e., $\|\mathbf{u}_i\|_2 = 1$ for all $i = 1, \dots, r$, and $\mathbf{u}_i^\top\mathbf{u}_j = 0$ for all $i \neq j$, then

$$\sum_{i=1}^{r} (\mathbf{v}^\mathsf{T}\mathbf{u}_i)^2 \leq \|\mathbf{v}\|_2^2 \tag{8.1}$$

for all $\mathbf{v} \in \mathbb{R}^n$. What can you say about $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r$ if equality always holds in (8.1) for all $\mathbf{v} \in \mathbb{R}^n$?