STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2012
PROBLEM SET 3

1. You are not allowed to use the SVD for this problem, i.e. no arguments should depend on the SVD of A or A^*. Let W be a subspace of $\mathbb{C}^n$. The subspace $W^\perp$ below is called the *orthogonal complement* of W.

$$W^\perp = \{\mathbf{v} \in \mathbb{C}^n \mid \mathbf{v}^*\mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

For any subspace $W \subseteq \mathbb{C}^n$, we write $P_W \in \mathbb{C}^{n\times n}$ for the projection onto W.

(a) Show that $\mathbb{C}^n = W \oplus W^\perp$ and that $W = (W^\perp)^\perp$.

(b) Let $A \in \mathbb{C}^{m\times n}$. Show that

$$\ker(A^*) = \operatorname{im}(A)^\perp \quad \text{and} \quad \operatorname{im}(A^*) = \ker(A)^\perp.$$

(c) Deduce the Fredholm alternative:

$$\mathbb{C}^m = \ker(A^*) \oplus \operatorname{im}(A) \quad \text{and} \quad \mathbb{C}^n = \operatorname{im}(A^*) \oplus \ker(A).$$

In other words any $\mathbf{x} \in \mathbb{C}^m$ and $\mathbf{y} \in \mathbb{C}^n$ can be written uniquely as

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1, \quad \mathbf{x}_0 \in \ker(A),\ \mathbf{x}_1 \in \operatorname{im}(A^*),\ \mathbf{x}_0^*\mathbf{x}_1 = 0,$$
$$\mathbf{y} = \mathbf{y}_0 + \mathbf{y}_1, \quad \mathbf{y}_0 \in \ker(A^*),\ \mathbf{y}_1 \in \operatorname{im}(A),\ \mathbf{y}_0^*\mathbf{y}_1 = 0.$$

(d) Show that

$$\mathbf{x}_0 = P_{\ker(A)}\mathbf{x}, \quad \mathbf{x}_1 = P_{\operatorname{im}(A^*)}\mathbf{x}, \quad \mathbf{y}_0 = P_{\ker(A^*)}\mathbf{y}, \quad \mathbf{y}_1 = P_{\operatorname{im}(A)}\mathbf{y}.$$

(e) Consider the least squares problem for some $\mathbf{b} \in \mathbb{C}^m$,

$$\min_{\mathbf{x}\in\mathbb{C}^n} \|\mathbf{b} - A\mathbf{x}\|_2. \tag{1.1}$$

Show that for any $\mathbf{x} \in \mathbb{C}^n$,

$$\|\mathbf{b} - A\mathbf{x}\|_2 \geq \|\mathbf{b}_0\|_2$$

where $\mathbf{b}_0 = P_{\ker(A^*)}\mathbf{b}$. Deduce that $\mathbf{x} \in \mathbb{C}^n$ is a solution to (1.1) if and only if

$$A\mathbf{x} = \mathbf{b}_1 \quad \text{or, equivalently,} \quad \mathbf{b} - A\mathbf{x} = \mathbf{b}_0. \tag{1.2}$$

Why is $A\mathbf{x} = \mathbf{b}_1$ consistent?

(f) Show that (1.2) is equivalent (i.e. if and only if) to the normal equation

$$A^*A\mathbf{x} = A^*\mathbf{b}. \tag{1.3}$$

Caveat: In numerical analysis, it is an unforgivable sin to solve a least squares problem using its normal equation. Nonetheless (1.3) can be useful in mathematical arguments; just don't ever use it for computations, instead use (1.2).

(g) Show that the pseudoinverse solution

$$\min\left\{\|\mathbf{x}\|_2 : \mathbf{x} \in \operatorname*{argmin}_{\mathbf{x}\in\mathbb{C}^n}\|\mathbf{b} - A\mathbf{x}\|_2\right\}$$

is given by

$$\mathbf{x}_1 = P_{\operatorname{im}(A^*)}\mathbf{x}$$

where $\mathbf{x} \in \mathbb{C}^n$ satisfies (1.2).

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(h) Let $A \in \mathbb{C}^{n\times n}$ be normal, i.e. $A^*A = AA^*$. Show that

$$\ker(A^*) = \ker(A) \quad \text{and} \quad \operatorname{im}(A^*) = \operatorname{im}(A)$$

and deduce that for a normal matrix,

$$\mathbb{C}^n = \ker(A) \oplus \operatorname{im}(A).$$

2. Let $A, B \in \mathbb{C}^{m\times n}$ with $n \le m$. In the lectures, we claim that the solution $X \in \mathrm{U}(n)$ to

$$\min_{X^*X=I} \|A - BX\|_F$$

is given by $X = UV^*$ where $B^*A = U\Sigma V^*$ is its singular value decomposition. Here we will prove it and consider some variants.

(a) Show that

$$\|A - BX\|_F^2 = \operatorname{tr}(A^*A) + \operatorname{tr}(B^*B) - 2\operatorname{Re}\operatorname{tr}(X^*B^*A)$$

and deduce that the minimization problem is equivalent to

$$\max_{X^*X=I} \operatorname{Re}\operatorname{tr}(X^*B^*A).$$

(b) Show that

$$\operatorname{Re}\operatorname{tr}(X^*B^*A) \le \sum_{i=1}^{n} \sigma_i(B^*A)$$

for any $X \in \mathrm{U}(n)$. When is the upper bound attained?

(c) Show that

$$\min_{X^*X=I} \|A - BX\|_F^2 = \sum_{i=1}^{m} (\sigma_i(A)^2 - 2\sigma_i(B^*A) + \sigma_i(B)^2).$$

(d) Suppose A has full column rank. Show that the following method produces a Hermitian matrix $X \in \mathbb{C}^{n\times n}$ that solves

$$\min_{X^*=X} \|AX - B\|_F. \tag{2.4}$$

(i) Show that the SVD of A takes the form

$$A = U \begin{bmatrix} \Sigma \\ O \end{bmatrix} V^*$$

where $U \in \mathrm{U}(m)$, $V \in \mathrm{U}(n)$, and $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n) \in \mathbb{C}^{n\times n}$ is a diagonal matrix.

(ii) Show that

$$\|AX - B\|_F^2 = \|\Sigma Y - C_1\|_F^2 + \|C_2\|_F^2$$

where $Y = V^*XV$ and $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = U^*BV$.

(iii) Note that Y must be Hermitian if X is. Show that

$$\|\Sigma Y - C_1\|_F^2 = \sum_{i=1}^{n} |\sigma_i y_{ii} - c_{ii}|^2 + \sum_{j>i} |\sigma_i y_{ij} - c_{ij}|^2 + |\sigma_j y_{ij} - \bar{c}_{ji}|^2$$

and deduce that the minimum value of (2.4) is attained when

$$y_{ij} = \frac{\sigma_i c_{ij} + \sigma_j \bar{c}_{ji}}{\sigma_i^2 + \sigma_j^2}$$

for all $i, j = 1, \dots, n$.

(e) Given $A \in \mathbb{C}^{n\times n}$. Describe how you would find $X \in \mathbb{C}^{n\times n}$ that solves
$$\min_{\det(X)=|\det(A)|} \|A - X\|_F.$$
(*Hint*: Consider the SVD of A).

3. Let $\mathbf{x} \in \mathbb{C}^m$, $\mathbf{y} \in \mathbb{C}^n$, and $A = \mathbf{x}\mathbf{y}^* \in \mathbb{C}^{m\times n}$.

(a) Show that $\operatorname{rank}(A) = 1$ iff $\mathbf{x}$ and $\mathbf{y}$ are both non-zero. Such a matrix is usually called a rank-1 matrix.

(b) Show that
$$\|A\|_F = \|A\|_2 = \|\mathbf{x}\|_2\|\mathbf{y}\|_2 \tag{3.5}$$
and that
$$\|A\|_\infty \le \|\mathbf{x}\|_\infty\|\mathbf{y}\|_1.$$
What can you say about $\|A\|_1$?

(c) Let $\mathbf{x}_1, \dots, \mathbf{x}_r \in \mathbb{C}^m$ be linearly independent and $\mathbf{y}_1, \dots, \mathbf{y}_r \in \mathbb{C}^n$ be linearly independent. Let
$$A = \mathbf{x}_1\mathbf{y}_1^* + \cdots + \mathbf{x}_r\mathbf{y}_r^*.$$
Show that $\operatorname{rank}(A) = r$. Show that this is not necessarily true if we drop either of the linear independence conditions.

(d) Given any $0 \ne A \in \mathbb{C}^{m\times n}$, show that
$$\operatorname{rank}(A) = \min\{r \in \mathbb{N} \mid A = \textstyle\sum_{i=1}^r \mathbf{x}_i\mathbf{y}_i^*\}.$$
In other words, the rank of a matrix is the smallest r so that it may be expressed as a sum of r rank-1 matrices.

(e) Show the following generalization of (3.5),
$$\|A\|_F \le \sqrt{\operatorname{rank}(A)}\|A\|_2.$$
Note that $\operatorname{rank}_{\text{CS}}(A) = \|A\|_F^2/\|A\|_2^2$ is the 'computer scientist's numerical rank,' one of the three notions of numerical ranks that we discussed. It is often used as a continuous surrogate for matrix rank.

(f) Show that with the nuclear norm we get instead
$$\|A\|_* \le \operatorname{rank}(A)\|A\|_2. \tag{3.6}$$
In other words we could also use $\|A\|_*/\|A\|_2$ as a continuous surrogate for matrix rank. In fact, this has been quite popular recently.

4. Let $A \in \mathbb{C}^{m\times n}$ and $\mathbf{b} \in \mathbb{C}^m$. We will discuss a variant of $A\mathbf{x} \approx \mathbf{b}$ where the error occurs only in A. Note that in ordinary least squares we assume that the error occurs only in $\mathbf{b}$ while in total least squares we assume that it occurs in both A and $\mathbf{b}$.

(a) Show that if $0 \ne \mathbf{x} \in \mathbb{C}^m$, then
$$\left\| A\left(I - \frac{\mathbf{x}\mathbf{x}^*}{\mathbf{x}^*\mathbf{x}}\right)\right\|_F^2 = \|A\|_F^2 - \frac{\|A\mathbf{x}\|_2^2}{\mathbf{x}^*\mathbf{x}}.$$

(b) Show that the matrix
$$E = \frac{(\mathbf{b} - A\mathbf{x})\mathbf{x}^*}{\mathbf{x}^*\mathbf{x}} \in \mathbb{C}^{m\times n}$$
has the smallest 2-norm of all $m \times n$ matrices E that satisfy
$$(A + E)\mathbf{x} = \mathbf{b}.$$

(c) What are the solutions of
$$\min_{(A+E)\mathbf{x}=\mathbf{b}} \|E\|_2 \qquad \text{and} \qquad \min_{(A+E)\mathbf{x}=\mathbf{b}} \|E\|_F?$$

(d) Given $\mathbf{a} \in \mathbb{C}^n$, $\mathbf{b} \in \mathbb{C}^m$, and $\delta > 0$. Show how to solve the problems

$$\min_{\|E\|_F \le \delta} \|E\mathbf{a} - \mathbf{b}\|_2 \qquad \text{and} \qquad \max_{\|E\|_F \le \delta} \|E\mathbf{a} - \mathbf{b}\|_2$$

over all $E \in \mathbb{C}^{m \times n}$.

5. Let $A \in \mathbb{C}^{m \times n}$ be a matrix with missing entries. More precisely we let $\Omega \subseteq \{1, \dots, m\} \times \{1, \dots, n\}$ be a subset of the row and column indices. We know the value of a_{ij} if $(i, j) \in \Omega$ but not otherwise. Now one way to recover the matrix A is to find an $X \in \mathbb{C}^{m \times n}$ whereby some loss function f is minimized, subjected to the constraint that x_{ij} agrees with all known entries of A:

$$\begin{array}{ll} \text{minimize} & f(X) \\ \text{subject to} & x_{ij} = a_{ij} \text{ for } (i, j) \in \Omega. \end{array}$$

One could argue that the most natural candidate for f is

$$f(X) = \operatorname{rank}(X), \tag{5.7}$$

but matrix rank is a discrete valued function and techniques of continuous optimization cannot be applied. A popular alternative is to instead use

$$f(X) = \|X\|_*$$

because nuclear norm is the largest convex function that satisfies (3.6). Here we will see how we may nonetheless solve the rank-mimization problem (in principle)

$$\begin{array}{ll} \text{minimize} & \operatorname{rank}(X) \\ \text{subject to} & x_{ij} = a_{ij} \text{ for } (i, j) \in \Omega, \end{array} \tag{5.8}$$

(a) For $1 \le r \le \min(m, n)$, let $f_r : \mathbb{C}^{m \times n} \to [0, \infty)$ be the function[1]

$$f_r(X) = \sum_{i=r+1}^{\min(m,n)} \sigma_i(X)^2.$$

and consider the minimization problem

$$\begin{array}{ll} \text{minimize} & f_r(X) \\ \text{subject to} & x_{ij} = a_{ij} \text{ for } (i, j) \in \Omega. \end{array} \tag{5.9}$$

Let X_r be a minimizer of (5.9) and X_* be a minimizer of (5.8). Show that

$$f_r(X_r) = 0 \quad \text{if and only if} \quad r \ge \operatorname{rank}(X_*).$$

(b) Deduce that the smallest $r \in \{1, \dots, \min(m, n)\}$ such that the minimum value of (5.9) is 0 would have the property that

$$X_r = X_*.$$

(c) Implement this strategy in MATLAB. Start with $r = \min(m, n)$ and solve (5.9) using any means you know. If the minimum is 0, reduce r by 1 and repeat. Keep doing this until you get to a value of r where the minimum is non-zero. Then the previous value of r and the corresponding X_r is the solution to (5.8).

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[1] Motivated by the 'optimization theorist's numerical rank' that we discussed in lectures:

$$\operatorname{rank}_{\text{OT}}(A) := \min\left\{ r \in \mathbb{N} \;\middle|\; \frac{\sum_{i \ge r+1} \sigma_i(A)^2}{\sum_{i \ge 1} \sigma_i(A)^2} \le \tau \right\}.$$

(d) Test how well your algorithm works by generating a random matrix $A \in \mathbb{R}^{20\times 10}$ of rank 5, removing 50% of its entries at random (so $\#\Omega = 100$), and then use your algorithm to find X_*. Now check how well X_* agrees with your original A by computing

$$\frac{\sum_{(i,j)\notin\Omega}(a_{ij} - x_{ij})^2}{\sum_{(i,j)\notin\Omega} a_{ij}^2}. \tag{5.10}$$

Repeat this experiment 40 times by generating 20 random A's with standard normal entries and another 20 with standard uniform $(0, 1)$ entries (i.e. use `randn` and `rand` respectively). Record the value of (5.10) and $\operatorname{rank}(X_*)$ each time.

(e) Modify your algorithm so that it now works for $A \in \{1, 2, 3, 4, 5\}^{m\times n}$, i.e. a matrix whose entries are random integers between 1 and 5. Now you need to find some way to round off the entries of your output so that your algorithm yields $X_* \in \{1, 2, 3, 4, 5\}^{m\times n}$. Repeat (d) for 40 random $A \in \{1, 2, 3, 4, 5\}^{20\times 10}$ (use `randi` to generate your A).