STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2012
PROBLEM SET 1

This homework serves as a linear algebra refresher. We will recall some definitions. The null space or kernel of a matrix $A \in \mathbb{C}^{m\times n}$ is the set

$$\ker(A) = \{\mathbf{x} \in \mathbb{C}^n \mid A\mathbf{x} = \mathbf{0}\}$$

while the range space or image is the set

$$\operatorname{im}(A) = \{\mathbf{y} \in \mathbb{C}^m \mid \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{C}^n\}.$$

Ditto for $\mathbb{R}$ in place of $\mathbb{C}$. The rank and nullity of A are defined as the dimensions of these spaces,

$$\operatorname{rank}(A) = \dim \operatorname{im}(A) \qquad \text{and} \qquad \operatorname{nullity}(A) = \dim \ker(A).$$

1. Let $A \in \mathbb{C}^{n\times n}$. Show that the following statements are equivalent:
(i) $\mathbb{C}^n = \ker(A) + \operatorname{im}(A)$;
(ii) $\ker(A) \cap \operatorname{im}(A) = \{\mathbf{0}\}$;
(iii) $\mathbb{C}^n = \ker(A) \oplus \operatorname{im}(A)$.

2. Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$.
(a) Show that

$$\operatorname{im}(AB) \subseteq \operatorname{im}(A) \qquad \text{and} \qquad \ker(AB) \supseteq \ker(B).$$

(b) Show that B has full row rank (so $n \le p$), then

$$\operatorname{im}(AB) = \operatorname{im}(A).$$

(c) Find a condition on A that guarantees

$$\ker(AB) = \ker(B).$$

3. Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$.
(a) Show that

$$\operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$$

and

$$\operatorname{nullity}(AB) \le \operatorname{nullity}(A) + \operatorname{nullity}(B).$$

(b) Show that

$$\operatorname{rank}(A + B) \le \operatorname{rank}(A) + \operatorname{rank}(B).$$

(c) Show that if $AB = 0$, then

$$\operatorname{rank}(A) + \operatorname{rank}(B) \le n.$$

4. Let $A \in \mathbb{C}^{n\times n}$.
(a) Show that

$$\ker(A^*A) = \ker(A) \qquad \text{and} \qquad \operatorname{im}(A^*A) = \operatorname{im}(A^*).$$

(b) Show that

$$A^*A\mathbf{x} = A^*\mathbf{b}$$

always has a solution (even if $A\mathbf{x} = \mathbf{b}$ has no solution).

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5. Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

(a) Let $\mathbf{x}_0 \in \mathbb{R}^n$ be a solution of $A\mathbf{x} = \mathbf{b}$. Show that every solution of $A\mathbf{x} = \mathbf{b}$ is of the form

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{z}$$

where $\mathbf{z} \in \ker(A)$.

(b) Suppose $\mathbf{b} \neq \mathbf{0}$. Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathbb{R}^n$ be solutions of $A\mathbf{x} = \mathbf{b}$, ie. $A\mathbf{x}_i = \mathbf{b}$ for all $i \in \{1, \dots, k\}$. Show that the linear combination

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_k\mathbf{x}_k$$

is also a solution to $A\mathbf{x} = \mathbf{b}$ if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1.$$

(c) Show that if $A\mathbf{x} = \mathbf{0}$ has a non-zero complex solution, ie. there exists $\mathbf{z} \in \mathbb{C}^m$, $\mathbf{z} \neq \mathbf{0}$, such that $A\mathbf{z} = \mathbf{0}$, then there exists a non-zero real solution.

6. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^n$ and let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be the matrix with

$$a_{ij} = \mathbf{v}_i^\top \mathbf{v}_j$$

for $i, j = 1, \dots, n$. Show that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent if and only if $\operatorname{nullity}(A) = 0$.

7. Show that if $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r \in \mathbb{R}^n$ are pairwise orthogonal unit vectors, ie. $\|\mathbf{u}_i\|_2 = 1$ for all $i = 1, \dots, r$, and $\mathbf{u}_i^\top \mathbf{u}_j = 0$ for all $i \neq j$, then

$$\sum_{i=1}^{r} (\mathbf{v}^\top \mathbf{u}_i)^2 \leq \|\mathbf{v}\|_2^2$$

for all $\mathbf{v} \in \mathbb{R}^n$.