1. In general, a *semi-iterative method* is one that comprises two steps:

\[ x^{(k+1)} = Mx^{(k)} + b \]  

(Iteration)

and

\[ y^{(m)} = \sum_{k=0}^{m} \alpha_k^{(m)} x^{(k)}. \]  

(Extrapolation)

As in the lectures, we will assume that \( M = I - A \) with \( \rho(M) < 1 \) and that we are interested to solve \( Ax = b \) for some nonsingular matrix \( A \in \mathbb{C}^{n \times n} \). Let

\[ e^{(k)} = x^{(k)} - x \quad \text{and} \quad \epsilon^{(m)} = y^{(m)} - x. \]

(a) By considering what happens when \( x^{(0)} = x \), show that it is natural to impose

\[ \sum_{k=0}^{m} \alpha_k^{(m)} = 1 \quad (1.1) \]

for all \( m \in \mathbb{N} \cup \{0\} \). Henceforth, we will assume that (1.1) is satisfied for all problems in this problem set.

(b) Show that for all \( m \in \mathbb{N} \), we may write

\[ \epsilon^{(m)} = P_m(M)e^{(0)} \]

for some \( P_m(x) = \alpha_0^{(m)} + \alpha_1^{(m)} x + \cdots + \alpha_m^{(m)} x^m \in \mathbb{C}[x] \) with \( \deg(P_m) = m \) and \( P_m(1) = 1 \).

(c) Hence deduce that a necessary condition for \( \epsilon^{(m)} \to 0 \) is that

\[ \lim_{m \to \infty} \|P_m(M)\|_2 < 1 \]

where \( \| \cdot \|_2 \) is the spectral norm. Is this condition also sufficient?

(d) Consider the case when

\[ \alpha_0^{(m)} = \alpha_1^{(m)} = \cdots = \alpha_m^{(m)} = \frac{1}{m+1} \]

for all \( m \in \mathbb{N} \cup \{0\} \). Show that if a sequence (any sequence, not necessarily one generated as in (Iteration)) is convergent and

\[ \lim_{k \to \infty} x^{(k)} = x \]

then

\[ \lim_{m \to \infty} y^{(m)} = x. \]

Is the converse also true?
2. It is clear that in any semi-iterative method defined by some $M \in \mathbb{C}^{n \times n}$ with $\rho(M) < 1$, we would like to solve the problem

$$\min_{P \in \mathbb{C}[x], \deg(P)=m, P(1)=1} \|P(M)\|_2. \tag{2.2}$$

Note that in the lectures, we required the polynomial $P$ to satisfy $P(0) = 1$. Here we use a different condition, $P(1) = 1$, motivated by Problem 1(a).

(a) Show that if $m \geq n$, then a solution to (2.2) is given by

$$P_m(x) = \frac{x^{m-n} \det(xI - M)}{\det(I - M)}.$$

You may assume the Cayley–Hamilton Theorem. How do we know that the denominator is non-zero?

(b) From now on assume that $M$ is Hermitian with minimum and maximum eigenvalues $\lambda_{\min} := a$ and $\lambda_{\max} := b \in \mathbb{R}$. Define

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|.$$

Emulating our discussions in the lectures, show that for $m = 0, 1, \ldots, n - 1$, the solution to the relaxed problem

$$\min_{P \in \mathbb{C}[x], \deg(P)=m, P(1)=1} \|P\|_\infty \tag{2.3}$$

would yield an upper bound to (2.2).

(c) Consider the Chebyshev polynomials defined by

$$C_m(x) = \begin{cases} 
\cos(m \cos^{-1}(x)) & -1 \leq x \leq 1, \\
\cosh(m \cosh^{-1}(x)) & x > 1,
\end{cases}$$

$$(-1)^m \cosh(m \cosh^{-1}(-x)) & x < -1. $$

Suppose $-1 < a < b < +1$. Show that the polynomials defined by

$$P_m(x) = \frac{C_m \left( \frac{2x - (b + a)}{b - a} \right)}{C_m \left( \frac{2 - (b + a)}{b - a} \right)} \tag{2.4}$$

satisfy $\deg(P_m) = m$, $P_m(1) = 1$, and

$$\|P_m\|_\infty = \frac{1}{C_m \left( \frac{2 - (b + a)}{b - a} \right)}.$$

(d) By emulating our discussions in the lectures, show that the solution to (2.3) is given by $P_m$. Note that this solves (2.3) for all $m \in \mathbb{N}$ and not just $m \leq n - 1$.

(e) Show that the solution in (d) is unique.

3. Let $M \in \mathbb{C}^{n \times n}$ be Hermitian with $\rho(M) = \rho < 1$. Moreover, suppose that

$$\lambda_{\min} = -\rho, \quad \lambda_{\max} = \rho.$$

(a) Show that the $P_m$’s in (2.4) satisfy a three-term recurrence relation

$$C_{m+1} \left( \frac{1}{\rho} \right) P_{m+1}(x) = \frac{2x}{\rho} C_m \left( \frac{1}{\rho} \right) P_m(x) - C_{m-1} \left( \frac{1}{\rho} \right) P_{m-1}(x)$$

for all $m \in \mathbb{N}$. 
(b) Show that the semi-iterative method with $\alpha_k^{(m)}$ given by the coefficient of $P_m$ in (2.4) may be written as

$$y^{(m+1)} = \omega_{m+1}(My^{(m)} - y^{(m-1)} + b) + y^{(m-1)}$$

where $y^{(-1)} := 0$, $\omega_1 := 1$, and

$$\omega_{m+1} = \frac{2C_m(1/\rho)}{\rho C_{m+1}(1/\rho)}$$

for $m = 0, 1, 2, \ldots$. This is a slightly different Chebyshev method where we choose the normalization (1.1) instead of $\alpha_k^{(m)} = 1$ in the lecture.

(c) Show that

$$\|P_m(M)\|_2 = \frac{1}{C_m(1/\rho)} = \frac{1}{\cosh(m\sigma)}$$

where $\sigma = \cosh^{-1}(1/\rho)$. Deduce that $\|P_m(M)\|_2$ is a strictly decreasing sequence for all $m = 0, 1, 2, \ldots$.

(d) Show that

$$e^{-\sigma} = (\omega - 1)^{1/2}$$

where

$$\omega = \frac{2}{1 + \sqrt{1 - \rho^2}} \quad (3.5)$$

and deduce that

$$\|P_m(M)\|_2 = \frac{2(\omega - 1)^{m/2}}{1 + (\omega - 1)^m}.$$  

(e) Hence show that $(\omega_m)_{m=0}^\infty$ is strictly decreasing for $m \geq 2$ and that

$$\lim_{m \to \infty} \omega_m = \omega.$$

4. Let $M \in \mathbb{C}^{n \times n}$ be nonsingular with $\rho(M) < 1$ and suppose we are interested in solving

$$Mx = b. \quad (4.6)$$

(a) Show that SOR applied to the system

$$\begin{bmatrix} I & -M \\ -M & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix} \quad (4.7)$$

yields the following iterations

$$x^{(m+1)} = \omega(Mz^{(m)} - x^{(m)} + b) + x^{(m)}, \quad z^{(m+1)} = \omega(Mx^{(m+1)} - z^{(m)} + b) + z^{(m)},$$

for $m = 0, 1, 2, \ldots$.

(b) Define the sequence of iterates $y^{(m)}$ by

$$y^{(m)} = \begin{cases} x^{(k)} & \text{if } m = 2k, \\ z^{(k)} & \text{if } m = 2k + 1. \end{cases}$$

Show that the iterations obtained in (a) are exactly the iterations in Problem 3(b). This shows that SOR applied to (4.7) is equivalent to Chebyshev applied to (4.6) but with $\omega_m = \omega$ for all $m \in \mathbb{N}$. Note that if $\omega$ is chosen to be the value in (3.5), then this is in fact the optimal SOR parameter.
5. Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite and $b \in \mathbb{R}^n$. As usual, we write
\[ r_k = b - Ax_k. \] (5.8)

We assume that $x_0$ is initialized in some manner. In the lectures we assumed $x_0 = 0$ and so $r_0 = b$ but we will do it a little more generally here. Consider the quadratic functional
\[ \varphi(x) = x^T A x - 2b^T x. \]

(a) Show that
\[ \nabla \varphi(x_k) = -2r_k \]
and hence if $x_\ast \in \mathbb{R}^n$ is a stationary point of $\varphi$, then
\[ Ax_\ast = b. \]

Show also that $x_\ast$ must be a minimizer of $\varphi$.

(b) Consider an iterative method
\[ x_{k+1} = x_k + \alpha_k p_k \] (5.9)
where $p_0, p_1, p_2, \ldots$ are search directions to be chosen later. Show that if we want $\alpha_k$ so that the function $f : \mathbb{R} \rightarrow \mathbb{R}$,
\[ f(\alpha) = \varphi(x_k + \alpha p_k) \]
is minimized, then we must have
\[ \alpha_k = \frac{r_k^T p_k}{p_k^T A p_k}. \] (5.10)

(c) Deduce that
\[ \varphi(x_{k+1}) - \varphi(x_k) = -\frac{(r_k^T p_k)^2}{p_k^T A p_k} \]
and therefore $\varphi(x_{k+1}) < \varphi(x_k)$ as long as $r_k^T p_k \neq 0$.

(d) Show that if we choose
\[ p_k = r_k, \] (5.11)
we obtain the steepest decent method discussed in the lectures.

(e) Let the eigenvalues of $A$ be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ and $P \in \mathbb{R}[t]$. Show that
\[ \|P(A)x\|_A \leq \max_{1 \leq i \leq n} |P(\lambda_i)|\|x\|_A \]
for every $x \in \mathbb{R}^n$. [Hint: $A \succ 0$ and so has an eigenbasis].

(f) Using (e) and $P_\alpha(t) = 1 - \alpha t$, show that if we have (5.11), then
\[ \|x_k - x_\ast\|_A \leq \max_{1 \leq i \leq n} |P_\alpha(\lambda_i)|\|x_{k-1} - x_\ast\|_A \]
for all $\alpha \in \mathbb{R}$.

(g) Using properties of Chebyshev polynomials, show that
\[ \min_{\alpha \in \mathbb{R}} \max_{\lambda_n \leq t \leq \lambda_1} |1 - \alpha t| = \frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n} \]
and hence deduce that
\[ \|x_k - x_\ast\|_A \leq \frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n} \|x_{k-1} - x_\ast\|_A. \]