1. In general, a *semi-iterative method* is one that comprises two steps:

\[ x^{(k+1)} = Mx^{(k)} + b \]  \hspace{1cm} \text{(Iteration)}

and

\[ y^{(m)} = \sum_{k=0}^{m} \alpha_k^{(m)} x^{(k)}. \]  \hspace{1cm} \text{(Extrapolation)}

As in the lectures, we will assume that \( M = I - A \) with \( \rho(M) < 1 \) and that we are interested to solve \( Ax = b \) for some nonsingular matrix \( A \in \mathbb{C}^{n \times n} \). Let

\[ e^{(k)} = x^{(k)} - x \quad \text{and} \quad \varepsilon^{(m)} = y^{(m)} - x. \]

(a) By considering what happens when \( x^{(0)} = x \), show that it is natural to impose

\[ \sum_{k=0}^{m} \alpha_k^{(m)} = 1 \quad \text{(1.1)} \]

for all \( m \in \mathbb{N} \cup \{0\} \). Henceforth, we will assume that (1.1) is satisfied for all problems in this problem set.

(b) Show that for all \( m \in \mathbb{N} \), we may write

\[ \varepsilon^{(m)} = P_m(M)e^{(0)} \]

for some \( P_m(x) = \alpha_0^{(m)} + \alpha_1^{(m)} x + \cdots + \alpha_m^{(m)} x^m \in \mathbb{C}[x] \) with \( \deg(P_m) = m \) and \( P_m(1) = 1 \).

(c) Hence deduce that a necessary condition for \( \varepsilon^{(m)} \to 0 \) is that

\[ \lim_{m \to \infty} \| P_m(M) \|_2 < 1 \]

where \( \| \cdot \|_2 \) is the spectral norm. Is this condition also sufficient?

(d) Consider the case when

\[ \alpha_0^{(m)} = \alpha_1^{(m)} = \cdots = \alpha_m^{(m)} = \frac{1}{m+1} \]

for all \( m \in \mathbb{N} \cup \{0\} \). Show that if a sequence (any sequence, not necessarily one generated as in (Iteration)) is convergent and

\[ \lim_{k \to \infty} x^{(k)} = x \]

then

\[ \lim_{m \to \infty} y^{(m)} = x. \]

Is the converse also true?
2. It is clear that in any semi-iterative method defined by some $M \in \mathbb{C}^{n \times n}$ with $\rho(M) < 1$, we would like to solve the problem

$$\min_{P \in \mathbb{C}[x], \deg(P)=m, P(1)=1} \|P(M)\|_2. \quad (2.2)$$

Note that in the lectures, we required the polynomial $P$ to satisfy $P(0) = 1$. Here we use a different condition, $P(1) = 1$, motivated by Problem 1(a).

(a) Show that if $m \geq n$, then a solution to (2.2) is given by

$$P_m(x) = \frac{x^{m-n} \det(xI - M)}{\det(I - M)}.$$ 

You may assume the Cayley–Hamilton Theorem. How do we know that the denominator is non-zero?

(b) From now on assume that $M$ is Hermitian with minimum and maximum eigenvalues $\lambda_{\text{min}} := a$ and $\lambda_{\text{max}} := b \in \mathbb{R}$. Define

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|.$$ 

Emulating our discussions in the lectures, show that for $m = 0, 1, \ldots, n-1$, the solution to the relaxed problem

$$\min_{P \in \mathbb{C}[x], \deg(P)=m, P(1)=1} \|P\|_\infty \quad (2.3)$$

would yield an upper bound to (2.2).

(c) Consider the Chebyshev polynomials defined by

$$C_m(x) = \begin{cases} 
\cos(m \cos^{-1}(x)) & -1 \leq x \leq 1, \\
\cosh(m \cosh^{-1}(x)) & x > 1, \\
(-1)^m \cosh(m \cosh^{-1}(-x)) & x < -1.
\end{cases}$$

Suppose $-1 < a < b < +1$. Show that the polynomials defined by

$$P_m(x) = \frac{C_m\left(\frac{2x - (b + a)}{b - a}\right)}{C_m\left(\frac{2 - (b + a)}{b - a}\right)} \quad (2.4)$$

satisfy $\deg(P_m) = m$, $P_m(1) = 1$, and

$$\|P_m\|_\infty = \frac{1}{C_m\left(\frac{2 - (b + a)}{b - a}\right)}.$$ 

(d) By emulating our discussions in the lectures, show that the solution to (2.3) is given by $P_m$. Note that this solves (2.3) for all $m \in \mathbb{N}$ and not just $m \leq n - 1$.

(e) Show that the solution in (d) is unique.

3. Let $M \in \mathbb{C}^{n \times n}$ be Hermitian with $\rho(M) = \rho < 1$. Moreover, suppose that $\lambda_{\text{min}} = -\rho$, $\lambda_{\text{max}} = \rho$.

(a) Show that the $P_m$’s in (2.4) satisfy a three-term recurrence relation

$$C_{m+1} \left(\frac{1}{\rho}\right) P_{m+1}(x) = \frac{2x}{\rho} C_m \left(\frac{1}{\rho}\right) P_m(x) - C_{m-1} \left(\frac{1}{\rho}\right) P_{m-1}(x)$$

for all $m \in \mathbb{N}$. 

(b) Show that the semi-iterative method with \( \alpha_k^{(m)} \) given by the coefficient of \( P_m \) in (2.4) may be written as
\[
y^{(m+1)} = \omega_{m+1}(My^{(m)} - y^{(m-1)} + b) + y^{(m-1)}
\]
where \( y^{(-1)} := 0, \omega_1 := 1, \) and
\[
\omega_{m+1} = \frac{2C_m(1/\rho)}{\rho C_{m+1}(1/\rho)}
\]
for \( m = 0, 1, 2, \ldots \). This is a slightly different Chebyshev method where we choose the normalization (1.1) instead of \( \alpha^{(m)} = 1 \) in the lecture.

(c) Show that
\[
\|P_m(M)\|_2 = \frac{1}{C_m(1/\rho)} = \frac{1}{\cosh(m\sigma)}
\]
where \( \sigma = \cosh^{-1}(1/\rho) \). Deduce that \( \|P_m(M)\|_2 \) is a strictly decreasing sequence for all \( m = 0, 1, 2, \ldots \).

(d) Show that
\[
e^{-\sigma} = (\omega - 1)^{1/2}
\]
where
\[
\omega = \frac{2}{1 + \sqrt{1 - \rho^2}}
\]
and deduce that
\[
\|P_m(M)\|_2 = \frac{2(\omega - 1)^{m/2}}{1 + (\omega - 1)^m}.
\]

(e) Hence show that \( (\omega_m)_{m=0}^{\infty} \) is strictly decreasing for \( m \geq 2 \) and that
\[
\lim_{m \to \infty} \omega_m = \omega.
\]

4. Let \( M \in \mathbb{C}^{n \times n} \) be nonsingular with \( \rho(M) < 1 \) and suppose we are interested in solving
\[
Mx = b. \tag{4.6}
\]

(a) Show that SOR applied to the system
\[
\begin{bmatrix} I & -M \\ -M & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix} \tag{4.7}
\]
yields the following iterations
\[
x^{(m+1)} = \omega(Mz^{(m)} - x^{(m)} + b) + x^{(m)},
\]
\[
z^{(m+1)} = \omega(Mx^{(m+1)} - z^{(m)} + b) + z^{(m)},
\]
for \( m = 0, 1, 2, \ldots \).

(b) Define the sequence of iterates \( y^{(m)} \) by
\[
y^{(m)} = \begin{cases} x^{(k)} & \text{if } m = 2k, \\ z^{(k)} & \text{if } m = 2k + 1. \end{cases}
\]
Show that the iterations obtained in (a) are exactly the iterations in Problem 3(b). This shows that SOR applied to (4.7) is equivalent to Chebyshev applied to (4.6) but with \( \omega_m = \omega \) for all \( m \in \mathbb{N} \). Note that if \( \omega \) is chosen to be the value in (3.5), then this is in fact the optimal SOR parameter.
5. Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite and $b \in \mathbb{R}^n$. As usual, we write
\[ r_k = b - Ax_k. \] (5.8)
We assume that $x_0$ is initialized in some manner. In the lectures we assumed $x_0 = 0$ and so $r_0 = b$ but we will do it a little more generally here. Consider the quadratic functional
\[ \varphi(x) = x^T Ax - 2b^T x. \]

(a) Show that
\[ \nabla \varphi(x_k) = -2r_k \]
and hence if $x_* \in \mathbb{R}^n$ is a stationary point of $\varphi$, then
\[ Ax_* = b. \]
Show also that $x_*$ must be a minimizer of $\varphi$.

(b) Consider an iterative method
\[ x_{k+1} = x_k + \alpha_k p_k \] (5.9)
where $p_0, p_1, p_2, \ldots$ are search directions to be chosen later. Show that if we want $\alpha_k$ so that the function $f: \mathbb{R} \to \mathbb{R}$,
\[ f(\alpha) = \varphi(x_k + \alpha p_k) \]
is minimized, then we must have
\[ \alpha_k = \frac{r_k^T p_k}{p_k^T A p_k}. \] (5.10)

(c) Deduce that
\[ \varphi(x_{k+1}) - \varphi(x_k) = -\frac{(r_k^T p_k)^2}{p_k^T A p_k} \]
and therefore $\varphi(x_{k+1}) < \varphi(x_k)$ as long as $r_k^T p_k \neq 0$.

(d) Show that if we choose
\[ p_k = r_k, \] (5.11)
we obtain the steepest decent method discussed in the lectures.

(e) Let the eigenvalues of $A$ be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ and $P \in \mathbb{R}[t]$. Show that
\[ \|P(A)x\|_A \leq \max_{1 \leq i \leq n} |P(\lambda_i)| \|x\|_A \]
for every $x \in \mathbb{R}^n$. \textit{[Hint: $A \succ 0$ and so has an eigenbasis].}

(f) Using (e) and $P_\alpha(t) = 1 - \alpha t$, show that if we have (5.11), then
\[ \|x_k - x_*\|_A \leq \max_{1 \leq i \leq n} |P_\alpha(\lambda_i)| \|x_{k-1} - x_*\|_A \]
for all $\alpha \in \mathbb{R}$.

(g) Using properties of Chebyshev polynomials, show that
\[ \min_{\alpha \in \mathbb{R}} \max_{\lambda_n \leq t \leq \lambda_1} |1 - \alpha t| = \frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n} \]
and hence deduce that
\[ \|x_k - x_*\|_A \leq \frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n} \|x_{k-1} - x_*\|_A. \]