1. Let \( u \in \mathbb{R}^n, u \neq 0 \). A Householder matrix \( H_u \in \mathbb{R}^{n \times n} \) is defined by
\[
H_u = I - \frac{2uu^T}{\|u\|_2^2}.
\]
(a) Show that \( H_u \) is both symmetric and orthogonal.
(b) Show that for any \( \alpha \in \mathbb{R}, \alpha \neq 0 \),
\[
H_{\alpha u} = H_u.
\]
In other words, \( H_u \) only depends on the ‘direction’ of \( u \) and not on its ‘magnitude’.
(c) In general, given a matrix \( M \in \mathbb{R}^{n \times n} \) and a vector \( x \in \mathbb{R}^n \), computing the matrix-vector product \( Mx \) requires \( n \) inner products — one for each row of \( M \) with \( x \). Show that \( H_u x \) can be computed using only two inner products.
(d) Given \( a, b \in \mathbb{R}^n \) where \( a \neq b \) and \( \|a\|_2 = \|b\|_2 \). Find \( u \in \mathbb{R}^n, u \neq 0 \) such that
\[
H_u a = b.
\]
(e) Show that \( u \) is an eigenvector of \( H_u \). What is the corresponding eigenvalue?
(f) Show that every \( v \in \text{span}\{u\}^\perp \) (cf. orthogonal complement in Homework 2) is an eigenvector of \( H_u \). What are the corresponding eigenvalues? What is \( \dim(\text{span}\{u\}^\perp) \)?
(g) Find the eigenvalue decomposition of \( H_u \), i.e., find an orthogonal matrix \( Q \) and a diagonal matrix \( \Lambda \) such that
\[
H_u = Q \Lambda Q^T.
\]
2. Let \( A \in \mathbb{R}^{m \times n} \) and suppose its complete orthogonal decomposition is given by
\[
A = Q_1 \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} Q_2^T,
\]
where \( Q_1 \) and \( Q_2 \) are orthogonal, and \( L \) is an nonsingular lower triangular matrix. Recall that \( X \in \mathbb{R}^{n \times m} \) is the unique pseudo-inverse of \( A \) if the following Moore–Penrose conditions hold:
(i) \( AXA = A \),
(ii) \( XAX = X \),
(iii) \( (AX)^\dagger = AX \),
(iv) \( (XA)^\dagger = XA \)
and in which case we write \( X = A^\dagger \).
(a) Let
\[
A^\dagger = Q_2 \begin{bmatrix} L^{-1} & Y \\ 0 & 0 \end{bmatrix} Q_1^T, \quad Y \neq 0.
\]
Which of the four conditions (i)–(iv) are satisfied?
(b) Prove that
\[
A^\dagger = Q_2 \begin{bmatrix} L^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q_1^T.
\]
by letting
\[ A^\dagger = Q_2 \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} Q_1^T \]
and by completing the following steps:
- Using (i), prove that \( X_{11} = L^{-1} \).
- Using the symmetry conditions (iii) and (iv), prove that \( X_{12} = 0 \) and \( X_{21} = 0 \).
- Using (ii), prove that \( X_{22} = 0 \).

3. Let \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), and \( c \in \mathbb{R}^n \). We are interested in the least squares problem
\[ \min_{x \in \mathbb{R}^n} \| Ax - b \|_2^2. \] (3.1)

(a) Show that \( x \) is a solution to (3.1) if and only if \( x \) is a solution to the augmented system
\[ \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}. \] (3.2)

(b) Show that the \((m + n) \times (m + n)\) matrix in (3.2) is nonsingular if and only if \( A \) has full column rank.

(c) Suppose \( A \) has full column rank and the QR decomposition of \( A \) is
\[ A = Q \begin{bmatrix} R \\ 0 \end{bmatrix}. \]
Show that the solution to the augmented system
\[ \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} \]
can be computed from
\[ z = R^{-1}c, \quad \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = Q^T b, \]
and
\[ x = R^{-1}(d_1 - z), \quad y = Q \begin{bmatrix} z \\ d_2 \end{bmatrix}. \]

(d) Hence deduce that if \( A \) has full column rank, then
\[ A^\dagger = R^{-1}Q_1^T \]
where \( Q = [Q_1, Q_2] \) with \( Q_1 \in \mathbb{R}^{m \times n} \) and \( Q_2 \in \mathbb{R}^{m \times (m-n)} \). Check that this agrees with the general formula derived for a rank-retaining factorization \( A = GH \) in the lectures.

4. Let \( A \in \mathbb{R}^{m \times n} \). Suppose we apply QR with column pivoting to obtain the decomposition
\[ A = Q \begin{bmatrix} R & S \\ 0 & 0 \end{bmatrix} \Pi^T \]
where \( Q \) is orthogonal and \( R \) is upper triangular and invertible. Let \( x_B \) be the basic solution, i.e.,
\[ x_B = \Pi \begin{bmatrix} R^{-1} \\ 0 \end{bmatrix} Q^T b, \]
and let \( \hat{x} = A^\dagger b \). Show that
\[ \frac{\| x_B - \hat{x} \|_2}{\| \hat{x} \|_2} \leq \| R^{-1} S \|_2. \]

(\textit{Hint: If} \quad \Pi^T x = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{and} \quad Q^T b = \begin{bmatrix} c \\ d \end{bmatrix}, \quad
\pi
consider the associated linearly constrained least-squares problem
\[
\min \|u\|_2^2 + \|v\|_2^2 \quad \text{s.t.} \quad Ru + Sv = c
\]
and write down the augmented system for the constrained problem.)

5. Given a symmetric \( A \in \mathbb{R}^{n \times n} \), \( 0 \neq x \in \mathbb{R}^n \), and \( b \in \mathbb{R}^n \). Let
\[
r = b - Ax
\]
Consider the QR decomposition
\[
[x, r] = QR
\]
and observe that if \( E x = r \), then
\[
(Q^T EQ)(Q^T x) = Q^T r.
\]
Show how to compute a symmetric \( E \in \mathbb{R}^{n \times n} \) so that it attains
\[
\min_{(A+E)x=b} \|E\|_F.
\]