Jordan Canonical Form

$$A = Q J Q^{-1}$$

$$J = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{pmatrix}$$

$$J_r = \begin{pmatrix} \lambda_r & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda_r \end{pmatrix}$$

rank $(J_r - \lambda_r I) = n_r - 1$

$$(J_r - \lambda_r I)\underset{\sim}{x} = 0$$

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}\underset{\sim}{x} = \underset{\sim}{0}$$

$$(J_r - \lambda_r I)\underset{\sim}{e}_1 = \underset{\sim}{0}$$

$$A^2 = QJ\cancel{Q^{-1}} \cdot \cancel{Q}JQ^{-1}$$

$$= QJ^2Q^{-1}; \quad A^k = QJ^kQ^{-1}$$

$$J_r^k = \begin{pmatrix} \lambda_r & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_r \end{pmatrix}^k$$

$$= \left(\lambda_r I + K \right)^k$$

$$K = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

$$= (\lambda_r I + K)^k$$

$$= \sum_{j=0}^{k} \binom{k}{j} \lambda_r^{k-j} K^j$$

$$= \begin{pmatrix} \lambda_r^k & \binom{k}{1}\lambda_r^{k-1} & \cdots & \binom{k}{n_r-1}\lambda_r^{k-(n_r-1)} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \binom{k}{1}\lambda_r^{k-1} \\ 0 & & & \lambda_r^k \end{pmatrix}$$

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^3 = \begin{bmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ & \lambda^3 & 3\lambda^2 \\ 0 & & \lambda^3 \end{bmatrix}$$

$$|\lambda| < 1$$

$$\begin{bmatrix} \lambda & 1 & 0 \\ & \lambda & \\ & & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} \\ & \lambda^k & k\lambda^{k-1} \\ & & \lambda^k \end{bmatrix}$$

$$\longrightarrow 0$$

$$\underset{\sim}{y}' = A\underset{\sim}{y}, \quad \underset{\sim}{y}(t_0) = \underset{\sim}{y}_0$$

$$= QJQ^{-1}\underset{\sim}{y}$$

$$Q^{-1}\underset{\sim}{y}' = JQ^{-1}\underset{\sim}{y}$$

$$\underset{\sim}{z} = Q^{-1}\underset{\sim}{y}$$

$$\underset{\sim}{z}' = J\underset{\sim}{z} \qquad J = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$z_i' = \lambda_i z_i \; ; \quad z_i(t) = e^{\lambda_i t} z(0)$$

$$z_i(t) = e^{\lambda_i t} z_i(0)$$

$$\underset{\sim}{y}(t) = Q \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} Q^{-1} y(0)$$

$$\equiv e^{\underset{\sim}{A} t} \underset{\sim}{y}(0)$$

$$F(A) = Q\, F(\Lambda)\, Q^{-1}$$

$$q(\lambda) = \alpha_0 + \alpha_1 \lambda + \cdots + \alpha_n \lambda^n = 0$$

Characteristic equation

Cayley - Hamilton.

$$q(A) = \alpha_0 I + \alpha_1 A + \cdots + \alpha_n A^n = 0$$

$$\alpha_0 A^{-1} + \alpha_1 I + \alpha_2 A + \cdots + \alpha_n A^{n-1} = 0$$

$$A^{-1} = \beta_1 I + \beta_2 A + \cdots + \beta_{n-1} A^{n-1}$$

$$A^{-1} = P_{n-1}(A)$$

$$F(A) =$$

$$\alpha_1 I + \alpha_1 A + \cdots + \alpha_n A^n = 0$$

$$A^n = P_{n-1}(A)$$

$$A^{n} = \gamma_0 I + \gamma_1 A + \cdots + \gamma_{n-1} A^{n-1}$$

$$A^{n+1} = A \cdot A^n$$

$$A^k = P_{n}(A)$$

SVD: Robust

JCF: Dynamic

$$A = U \Sigma V^T$$

① A, find $\hat{A}$ so that $\hat{A}$ rank r

and $\|A - \hat{A}\|_2 = \min.$

$$A = U \Sigma V^T$$

$$\hat{A} = U \Omega V^T \; ; \; \Omega = \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ 0 & & & 0 \end{pmatrix}$$

$$A = U\Sigma V^T, \quad \tilde{A} = U\Omega_r V^T$$

$$\|A - \tilde{A}\|_F = \|U\Sigma V^T - U\Omega V^T\|_F$$

$$= \|U(\Sigma - \Omega)V^T\|_F$$

$$= \|(\Sigma - \Omega)\|_F$$

$$\Omega = \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{pmatrix} \qquad = \left(\sigma_{r+1}^2 + \cdots + \sigma_n^2\right)^{\frac{1}{2}}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 10^{-6} \end{pmatrix} \qquad \tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

A

$$A = U \Sigma V^T$$

$$\hat{Q} = U I V^T$$

$$\|A - \hat{Q}\|_F = \|U \Sigma V^T - U I V^T\|_F$$

$$= \|\cancel{U}(\Sigma - I)\cancel{V^T}\|_F$$

$$= \left[(\sigma_1 - 1)^2 + (\sigma_2 - 1)^2 + \cdots + (\sigma_n - 1)^2\right]^{\frac{1}{2}}$$

$$\underset{n \times n}{A} \quad , \quad \underset{m \times n}{B}$$

$$\min_{Q^T Q = I_n} \| A - BQ \|_F = \min$$

$$B^T A = U \Sigma V^T$$

$$Q = U I V^T$$

$$x = a + ib$$

$$= \rho e^{i\theta} \qquad \rho \geq 0$$

Polar Decomposition

$$\underset{n\times n}{A} = U \Sigma V^T$$

$$= U V^T \, V \Sigma V^T$$

$$= Q H$$

$$\| b - Ax \|_2 = \min .$$

$$\underset{\sim}{x} = A^+ \underset{\sim}{b}$$

$$\tilde{A} = A + E$$

$$\|E\|_2 \leq \varepsilon$$

$$\hat{A} , \quad \underset{\sim}{\hat{x}} = \hat{A}^+ \underset{\sim}{b}$$

If for some natural norm,

$\|A\| < 1$,

$\Rightarrow$ 1.) $\underline{I - A}$ is non-singular

2) $$\frac{1}{1+\|A\|} \le \|(I-A)^{-1}\| \le \frac{1}{1-\|A\|}$$

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1) Suppose A is singular.

Then there is a vector z so that $Az = 0$

$$(I - A)\underset{\sim}{z} = \underset{\sim}{0}$$

$$z = Az$$

$$\|\underset{\sim}{z}\| \leq \|A\| \, \|\underset{\sim}{z}\|$$

$$\Rightarrow \|A\| \geq 1.$$

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2) $$I = (I - A)(I - A)^{-1}$$

$$1 = \|I\| = \|(I - A) \cdot (I - A)^{-1}\|$$

$$\leq \|(I - A)\| \cdot \|(I - A)^{-1}\|$$

$$\leq (1 + \|A\|) \cdot \|(I - A)^{-1}\|$$

$$\frac{1}{1+\|A\|} \le \|(I-A)^{-1}\|$$

$$(I-A)^{-1}(I-A) = I$$

$$(I-A)^{-1} - (I-A)^{-1}A = I$$

$$(I-A)^{-1} = I + (I-A)^{-1}A$$

$$\|(I-A)^{-1}\| \le 1 + \|(I-A)^{-1}\| \cdot \|A\|$$

$$-\|A\| \cdot \|(I-A)^{-1}\| + \|(I-A)^{-1}\| \le 1$$

$$\|(I-A)^{-1}\| \le \frac{1}{1-\|A\|}$$