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Singular Value Decomposition

SVD

$$\underset{m\times n}{A} = \underset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \underset{n\times n}{V^T}$$

$$A = U \Sigma V^T \quad (m \geq n)$$

① $U^T U = I_m$

② $V^T V = I_n$

③ $$\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ & 0 & \end{pmatrix}_{m \times n}$$

σ_i : singular values

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$; $\underline{r : \text{rank}}$

$$\sigma_{r+1} = \cdots = \sigma_n = 0$$

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}_{n \times n}$$

$$\lambda_i = 0 \qquad i = 1, 2, \ldots, n$$

$$\sigma_i = 1 \qquad i = 1, 2, \ldots, n-1; \quad \sigma_n = 0$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 10^{-6} \end{pmatrix}$$

$\rightarrow \quad A = U \Sigma V^T$

$$AV = U\Sigma \,, \quad V = [\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n]$$

$$A\underset{\sim}{v}_i = \sigma_i \underset{\sim}{u}_i \qquad i = 1, \ldots, n$$

$$A\underset{\sim}{v}_i = \underset{\sim}{0} \qquad i = r+1, \ldots, n$$

$$\tilde{U} = [\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_r] \,, \quad \tilde{V} = [\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_r]$$

$\rightarrow \quad A = \tilde{U} \tilde{\Sigma} \tilde{V}^T \qquad \tilde{U}_{m\times r} \,, \quad \tilde{\Sigma}_{r\times r} \quad \tilde{V}_{n\times r}$

V: right s.v., $\quad U^T A = \Sigma V^T$

$$\underset{m \times m}{A} = U \Sigma V^T$$

$\sigma_m > 0$

$$A^{-1} = (V^T)^{-1} \Sigma^{-1} U^{-1}$$

$$= V \Sigma^{-1} U^T$$

$$\|A\|_2^2 = \max_{x \neq 0} \frac{x^T A^T A x}{x^T x} = \lambda_{max}(A^T A)$$

$$A^T A = V \Sigma^T \underbrace{U^T U} \Sigma V^T = V \Sigma^T \Sigma V^T$$

$$\lambda (A^T A) = \sigma^2 (A^T A)$$

$$\|\underset{\sim}{x}\|_2 = \|Q\underset{\sim}{x}\|_2, \quad Q^T Q = I$$

$$\|A\|_2 = \|U\Sigma V^T\|_2$$

$$= \|\Sigma\|_2 = \sigma_1$$

$$\Pi = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & 0 \\ & & 1 & \\ & 0 & & \ddots \\ & & & & 1 \end{pmatrix}$$

$$\|\Pi A\|_2 = \|A\|_2 = \sigma_1$$

$$\lambda(\Pi A \Pi^T) = \lambda(A)$$

$$\underset{m\times n}{A}\ , \quad \underset{\sim}{b} : m\times 1$$

$$\min \| \underset{\sim}{b} - A\underset{\sim}{\hat{x}} \|_2$$

$$= \min \| \underset{\sim}{b} - U\Sigma V^T \underset{\sim}{\hat{x}} \|_2$$

$$= \min \| U^T \underset{\sim}{b} - \Sigma V^T \underset{\sim}{\hat{x}} \|_2$$

$$U^T \underset{\sim}{b} = \underset{\sim}{c}\ ; \quad V^T \underset{\sim}{x} = y$$

$$= \min \| \underset{\sim}{c} - \Sigma y \|_2$$

$$\min \left\| \underset{\sim}{c} - \Sigma \underset{\sim}{y} \right\|_2 = \left(c_{r+1}^2 + \cdots + c_m^2 \right)^{\frac{1}{2}}$$

$$V^T \underset{\sim}{x} = \underset{\sim}{y}, \quad \underset{\sim}{x} = V \underset{\sim}{y}$$

We want to choose $y_{r+1}, \ldots, y_n$

so that $\|\underset{\sim}{x}\|_2 = \min$.

$$y_{r+1} = \cdots = y_n = 0$$

$$\underset{\sim}{x} = V \underset{\sim}{y}$$

$$= V \Sigma^+ \underset{\sim}{c}, \quad \Sigma^+ = \begin{pmatrix} \sigma_1^{-1} & & 0 \\ & \ddots & \sigma_r^{-1} \\ 0 & 0 & \end{pmatrix}_{n \times m}$$

$$\underset{\sim}{\hat{x}} = A^+ \underset{\sim}{b}$$

$$A^+ = V \Sigma^+ U^T$$

A^+ : pseudo-inverse

$A_{m \times n}$

1) $(AX)^T = AX$ 3) $AXA = A$

2) $(XA)^T = XA$ 4) $XAX = X$

$\boxed{\text{pseudo-inverse}}$

X

$$\mathcal{X} = \{ \underset{\sim}{x} \mid \|\underset{\sim}{b} - A\underset{\sim}{x}\|_2 = \min \}$$

$\underset{\sim}{\hat{x}} \in \mathcal{X}$ such that

$$\|\underset{\sim}{x}\|_2 = \min.$$

$$\underset{\sim}{\hat{x}} = A^+ \underset{\sim}{b}$$

$$\underset{\sim}{b} = \underset{\sim}{b}_1 + \underset{\sim}{b}_2$$

$$b_1 = AA^+ \underset{\sim}{b}, \quad \underset{\sim}{b}_2 = (I - AA^+) \underset{\sim}{b}$$

$$(AA^+)^2 = AA^+AA^+$$

$$A^+ = X \qquad = (AXA)X$$

$$= AX$$

$$= (AX)^T$$

$$A\underset{\sim}{\hat{x}} = P\underset{\sim}{b} = \underset{\sim}{b}_1$$

$$\underset{\sim}{r} = \underset{\sim}{b} - A\underset{\sim}{\hat{x}}$$: <u>residual</u> <u>v</u>ector

$$= \underset{\sim}{b} - AA^+\underset{\sim}{b} = (I - AA^+)\underset{\sim}{b}$$

$$= P^\perp \underset{\sim}{b}$$

$$A = A^T = U \Lambda U^T, \quad \lambda_1 \geq \cdots \geq \lambda_n \geq 0$$

$$\frac{\underset{\sim}{x}^T A \underset{\sim}{x}}{\underset{\sim}{x}^T \underset{\sim}{x}} = \frac{\underset{\sim}{x}^T U \Lambda U^T \underset{\sim}{x}}{\underset{\sim}{x}^T U U^T \underset{\sim}{x}}$$

$$\equiv \frac{y^T \Lambda y}{y^T y}$$

$$\lambda_n \leq \quad = \frac{\sum_{i=1}^{n} \lambda_i y_i^2}{\sum_{i=1}^{n} y_i^2} \leq \lambda_1$$

$$? \le \frac{\underset{\sim}{x}^T A \underset{\sim}{x}}{\underset{\sim}{x}^T B \underset{\sim}{x}} \le ? \qquad B = B^T, \text{ pd}.$$

$$\underset{m \times n}{A}$$

$$\frac{|\underset{\sim}{u}^T A \underset{\sim}{v}|}{\|\underset{\sim}{u}\|_2 \, \|\underset{\sim}{v}\|_2} = \frac{|\underset{\sim}{u}^T U \Sigma V^T v|}{=\|U^T \underset{\sim}{u}\|_2 \cdot \|V^T \underset{\sim}{v}\|_2}$$

$$= \frac{|\underset{\sim}{x}^T \Sigma \underset{\sim}{y}|}{\|\underset{\sim}{x}\|_2 \, \|\underset{\sim}{y}\|_2}$$

$$A$$

$$\tilde{A} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}_{(m+n)\times(m+n)}$$

$$\tilde{A} = \tilde{A}^T, \quad \tilde{A} = Z \Lambda Z^T$$

$$\tilde{A}^2 = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$$

$$= \begin{pmatrix} AA^T & 0 \\ 0 & A^TA \end{pmatrix}; \quad \lambda(\tilde{A}^2) = \sigma^2(A)$$

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \underset{\sim}{z} = \sigma \underset{\sim}{z} \qquad \sigma \neq 0$$

$$\underset{\sim}{z} = \begin{pmatrix} \underset{\sim}{x} \\ \underset{\sim}{y} \end{pmatrix}$$

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} \underset{\sim}{x} \\ -\underset{\sim}{y} \end{pmatrix} = \begin{pmatrix} -A\underset{\sim}{y} \\ A^T \underset{\sim}{x} \end{pmatrix} = \begin{pmatrix} -\sigma \underset{\sim}{x} \\ \sigma \underset{\sim}{y} \end{pmatrix}$$

$$= -\sigma \begin{pmatrix} \underset{\sim}{x} \\ -\underset{\sim}{y} \end{pmatrix} = -\sigma \underset{\sim}{\zeta}$$

$$\underset{\sim}{z}^T \underset{\sim}{\zeta} = 0$$

$$\underset{\sim}{x}^T \underset{\sim}{x} - \underset{\sim}{y}^T \underset{\sim}{y} = 0 \implies \underset{\sim}{x}^T \underset{\sim}{x} = \frac{1}{2}$$

$$\underset{\sim}{x}^T \underset{\sim}{x} + \underset{\sim}{y}^T \underset{\sim}{y} = 1, \qquad \underset{\sim}{y}^T \underset{\sim}{y} = \frac{1}{2}$$