

Oct. 5, 2005

# Singular Value Decomposition

SVD

$$\cancel{A} = U \Sigma V^T$$

$m \times n \quad m \times m \quad m \times n \quad n \times n$

$$A = U \Sigma V^T \quad (m > n)$$

①  $U^T U = I_m$

②  $V^T V = I_n$

③  $\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix}_{m \times n}$

$\sigma_i$ : singular values

$\sigma_1 > \sigma_2 > \dots > \sigma_r > 0$ ; r: rank

$$\sigma_{r+1} = \dots = \sigma_n = 0$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}_{n \times n}$$

$$\lambda_i = 0 \quad i = 1, 2, \dots, n$$

$$\sigma_i = 1 \quad i = 1, 2, \dots, n-1; \quad \sigma_n = 0$$

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$$\begin{pmatrix} 1 & 0 \\ 0 & 10^{-6} \end{pmatrix}$$

$$\rightarrow A = U \Sigma V^T$$

$$AV = U \Sigma \quad , \quad V = [\underline{v}_1 \dots \underline{v}_n]$$

$$A \underline{v}_i = \sigma_i \underline{u}_i \quad i=1, \dots, n$$

$$A \underline{v}_i = 0 \quad i=r+1, \dots, n$$

$$\tilde{U} = [\underline{u}_1 \dots \underline{u}_r], \quad \tilde{V} = [\underline{v}_1 \dots \underline{v}_r]$$

$$\rightarrow A = \tilde{U} \tilde{\Sigma} \tilde{V}^T \quad \tilde{U}_{m \times r}, \quad \tilde{\Sigma}_{r \times r}, \quad \tilde{V}_{n \times r}$$

$$V: \text{right s.v.}, \quad U^T A = \Sigma V^T$$

$$A = U \Sigma V^T$$

$n \times m$

$$\sigma_m > 0$$

$$A^{-1} = (V^T)^{-1} \Sigma^{-1} U^{-1}$$

$$= U \Sigma^{-1} V^T$$

$$\|A\|_2^2 = \max_{\substack{x \neq 0 \\ x \in \mathbb{R}^n}} \frac{x^T A^T A x}{x^T x} = \lambda_{\max}(A^T A)$$

$$A^T A = V \Sigma^T \underbrace{U^T U}_{\Sigma} \Sigma V^T = V \Sigma^2 V^T$$

$$\lambda_{\max}(A^T A) = \sigma^2(A^T A)$$

$$\|A\|_2 = \|Q\tilde{\Sigma}V^T\|_2, \quad \tilde{Q}\tilde{Q} = I$$

$$\begin{aligned}\|A\|_2 &= \|\tilde{U}\tilde{\Sigma}V^T\|_2 \\ &= \|\tilde{\Sigma}\|_2 = \sigma_1\end{aligned}$$

$$\Pi = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & \ddots & \end{pmatrix}$$

$$\begin{aligned}\|\Pi A\|_2 &= \|A\|_2 = \sigma_1 \\ \lambda(\Pi A \Pi^T) &= \lambda(A)\end{aligned}$$

$$\begin{matrix} A \\ m \times n \end{matrix}, \quad \begin{matrix} b \\ \sim \end{matrix} : m \times 1$$

$$\min \left\| \begin{matrix} b \\ \sim \end{matrix} - A \begin{matrix} x \\ \sim \end{matrix} \right\|_2$$

$$= \min \left\| \begin{matrix} b \\ \sim \end{matrix} - \cancel{\left( \sum V^T \begin{matrix} x \\ \sim \end{matrix} \right)} \right\|_2$$

$$= \min \left\| U^T \begin{matrix} b \\ \sim \end{matrix} - \sum V^T \begin{matrix} x \\ \sim \end{matrix} \right\|_2$$

$$U^T \begin{matrix} b \\ \sim \end{matrix} = \begin{matrix} c \\ \sim \end{matrix}; \quad V^T \begin{matrix} x \\ \sim \end{matrix} = \begin{matrix} y \\ \sim \end{matrix}$$

$$= \min \left\| \begin{matrix} c \\ \sim \end{matrix} - \sum \begin{matrix} y \\ \sim \end{matrix} \right\|_2$$

$$\min \left\| \begin{pmatrix} c_1 & \dots & c_r \\ \vdots & \ddots & \vdots \\ c_{r+1} & \dots & c_m \end{pmatrix} - \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix} \right\|_2 = \left( c_{r+1}^2 + \dots + c_m^2 \right)^{\frac{1}{2}}$$

$$V^T \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} = \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix}, \quad \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} = V \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix}$$

We want to choose  $y_{r+1}, \dots, y_n$

so that  $\left\| \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \right\|_2 = \min.$

$$y_{r+1} = \dots = y_n = 0$$

$$\begin{aligned} \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} &= V \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix} \\ &= V \sum_{\sim}^+ \begin{pmatrix} c_1 & \dots & c_r \\ \vdots & \ddots & \vdots \\ c_{r+1} & \dots & c_m \end{pmatrix}; \quad \sum^+ = \begin{pmatrix} \sigma_1^{-1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \sigma_r^{-1} \end{pmatrix}_{n \times m} \end{aligned}$$

$$\hat{\underline{x}} = A^+ \underline{b}$$

$$A^+ = \sqrt{\Sigma}^+ U^T$$

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$A^+$  : pseudo-inverse

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$A_{m \times n}$

$$1) (AX)^T = AX \quad 3) AXA = A$$

$$2) (XA)^T = XA \quad 4) XAX = X$$

pseudo-inverse

+ - 1 V - . + + i

X

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$$X = \left\{ \underline{x} \mid \| \underline{b} - A \underline{x} \|_2 = \min \right\}$$

$\underline{x} \in X$  such that

$$\| \underline{x} \|_2 = \min.$$

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$$\hat{\underline{x}} = A^+ \underline{b}$$

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$$\underline{b} = \underline{b}_1 + \underline{b}_2$$

$$\underline{b}_1 = A A^+ \underline{b}, \quad \underline{b}_2 = (\underline{\underline{I}} - A A^+) \underline{b}$$

$$\begin{aligned}
 (A A^+)^2 &= A A^T A A^+ \\
 A^+ &= x \quad = (A \times A) \times \\
 &\quad = A x \\
 &\quad = (A x)^T
 \end{aligned}$$

$$A \hat{x} = P \tilde{b} = \tilde{b}_1$$

$$\begin{aligned}
 \tilde{r} &= \tilde{b} - A \hat{x} \quad : \text{residual vector} \\
 &= \tilde{b} - A A^+ \hat{x} = (\mathbb{I} - A A^+) \tilde{b} \\
 &\quad = P^\perp \tilde{b}
 \end{aligned}$$

$$A = A^\top = U \Lambda U^\top, \quad \lambda_1 \geq \dots \geq \lambda_n \geq 0$$

$$\frac{\underline{x}^\top A \underline{x}}{\underline{x}^\top \underline{x}} = \frac{\underline{x}^\top U \Lambda U^\top \underline{x}}{\underline{x}^\top U \Lambda U^\top \underline{x}}$$

$$\equiv \frac{\underline{y}^\top \Lambda \underline{y}}{\underline{y}^\top \underline{y}}$$

$$\lambda_n \leq \frac{\sum_{i=1}^n \lambda_i y_i^2}{\sum_{i=1}^n y_i^2} \leq \lambda_1$$

$$\frac{\underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x}}{\underset{\sim}{x}^T \underset{\sim}{B} \underset{\sim}{x}} \leq ? \quad B = B^T, \text{ pd.}$$


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A  
m × n

$$\frac{|\underset{\sim}{U}^T \underset{\sim}{A} \underset{\sim}{V}|}{\|\underset{\sim}{U}\|_2 \|\underset{\sim}{V}\|_2} = \frac{|\underset{\sim}{U}^T \underset{\sim}{U} \Sigma \underset{\sim}{V}^T|}{\|\underset{\sim}{U}^T \underset{\sim}{U}\|_2 \cdot \|\underset{\sim}{V}^T\|_2}$$

$$= \frac{|\underset{\sim}{x}^T \Sigma \underset{\sim}{y}|}{\|\underset{\sim}{x}\|_2 \|\underset{\sim}{y}\|_2}$$

$A$

$$\tilde{A} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}_{(n+n) \times (m+m)}$$

$$\tilde{A} = \tilde{A}^T, \quad \tilde{A} = Z \Lambda \tilde{Z}$$

$$\tilde{A}^2 = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$$

$$= \begin{pmatrix} AA^T & 0 \\ 0 & A^T A \end{pmatrix}; \quad \lambda(\tilde{A}^2) = \sigma^2(A)$$

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \sigma \begin{pmatrix} x \\ y \end{pmatrix} \quad \sigma \neq 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \tilde{x} \\ \tilde{y} \end{pmatrix}$$

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} -A\tilde{y} \\ A^T\tilde{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma \begin{pmatrix} x \\ \tilde{x} \\ \tilde{y} \end{pmatrix} \end{pmatrix}$$

$$= -\sigma \begin{pmatrix} x \\ \tilde{x} \\ \tilde{y} \end{pmatrix} = -\sigma \left\{ \begin{array}{l} x \\ \tilde{x} \\ \tilde{y} \end{array} \right\}$$

$$\tilde{x}^T \tilde{x} = 0$$

$$\begin{aligned} \tilde{x}^T \tilde{x} - \tilde{y}^T \tilde{y} &= 0 & \Rightarrow \tilde{x}^T \tilde{x} &> \frac{1}{2} \\ \tilde{x}^T \tilde{x} + \tilde{y}^T \tilde{y} &= 1, & \tilde{y}^T \tilde{y} &= \frac{1}{2} \end{aligned}$$