Oct. 3

$$\|\underset{\sim}{x}\| \to 0 \Rightarrow \underset{\sim}{x} \to \underset{\sim}{0}$$

$$\underset{\sim}{x} \neq \underset{\sim}{0} \qquad \underset{\sim}{x} > \underset{\sim}{0}$$

1) $\|\underset{\sim}{x}\| > 0$ iff $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\quad = 0$ if $\underset{\sim}{x} = \underset{\sim}{0}$

2) $\|\alpha \underset{\sim}{x}\| = |\alpha| \, \|\underset{\sim}{x}\|$

3) $\|\underset{\sim}{x} + \underset{\sim}{y}\| \leq \|\underset{\sim}{x}\| + \|\underset{\sim}{y}\|$

Matrix Norms

1) $f(A) > 0$ iff $A \neq 0$

$= 0$ if $A = 0$

2) $f(A+B) \leq f(A) + f(B)$

3) $f(\alpha A) = |\alpha| f(A)$

4) $f(AB) \leq f(A) f(B)$

Frobenius (Euclidean)

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|^2 \right)^{\frac{1}{2}}$$

$$\|A \cdot B\| \leq \|A\| \cdot \|B\|$$

$$\|A^2\| \leq \|A\|^2$$

$$\|A^3\| \leq \|A\|^3$$

$$\|A^n\| \leq \|A\|^n$$

$$\Rightarrow \|A\| < 1, \quad A^n \to 0$$

Natural Norm

$$\|A\| = \max_{\underset{\sim}{x} \neq \underset{\sim}{0}} \frac{\|Ax\|}{\|x\|}$$

induces

1) If $A = 0$, then $\|A\underline{x}\| = 0$.

$A \neq 0$, $a_{ij} > 0$ $\quad \underset{\sim}{e}_j^T = (0, 0, \ldots, 1, 0 \ldots, 0)$

$A e_j = \underset{\sim}{a}_j$, $\|A\underset{\sim}{e}_j\| = \|\underset{\sim}{a}_j\| > 0$

$$\|AB\| \le \|A\| \cdot \|B\|$$

$$\max \frac{\|AB\underset{\sim}{x}\|}{\|\underset{\sim}{x}\|} = \frac{\|A(By_0)\|}{\|y_0\|}$$

$$\le \|A\| \frac{\|By_0\|}{\|y_0\|}$$

$$\le \|A\| \cdot \|B\|.$$

<u>Backup</u>

$$\frac{\|Ay\|}{\|y\|} \le \|A\|$$

$$\|\alpha A\| = \max_{x} \frac{\|\alpha A \underset{\sim}{x}\|}{\|\underset{\sim}{x}\|}$$

$$= \frac{\|\alpha(A \underset{\sim}{y_0})\|}{\|\underset{\sim}{y_0}\|} = |\alpha| \frac{\|A \underset{\sim}{y_0}\|}{\|\underset{\sim}{y_0}\|}$$

$$= |\alpha| \, \|A\|.$$

$\underset{\sim}{z}$: arbitrary, $\underset{\sim}{z} \neq \underset{\sim}{0}$

$$\underline{\|A \underset{\sim}{z}\|} = \left\| A \, \|z\| \cdot \frac{z}{\|z\|} \right\|$$

$$= \|\underset{\sim}{z}\| \cdot \left\| A \frac{\underset{\sim}{z}}{\|\underset{\sim}{z}\|} \right\| \leq \underline{\|\underset{\sim}{z}\| \cdot \|A\|}$$

$$\|A+B\| \overset{?}{\leq} \|A\| + \|B\|$$

$$\max \frac{\|(A+B)\underset{\sim}{x}\|}{\|\underset{\sim}{x}\|} = \frac{\|(A+B)y_0\|}{\|y_0\|}$$

$$\leq \frac{\|A y_0\|}{\|y_0\|} + \frac{\|B y_0\|}{\|y_0\|} \leq \|A\| + \|B\|$$

$$\|A\|_\infty = ?$$

$$\max_{\underline{x} \neq \underset{\sim}{0}} \left(\frac{\|A\underset{\sim}{x}\|_\infty}{\|\underset{\sim}{x}\|_\infty} \right)$$

$$= \max_{\|y\|_\infty = 1} \|A y\|_\infty$$

$$= \max_{\|y\|_\infty = 1} \max_{1 \le i \le m} \left| \sum_{j=1}^{n} a_{ij} y_j \right|$$

$$\le \max_{\|y\|_\infty = 1} \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}| |y_j|$$

$$\max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_\infty \le \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$$

$$= \sum_{j=1}^{n} |a_{Ij}|$$

$\hat{y}_j = +1$ if $a_{Ij} \ge 0$

$= -1$ if $a_{Ij} < 0$

$$\max \|A\hat{y}\| = \sum_{j=1}^{n} |a_{Ij}|$$

$$\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$$

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{8} & 2 \end{pmatrix} \qquad \|A\|_\infty = 2\tfrac{1}{8}$$

$$A = \begin{pmatrix} 0 & \frac{1}{2} & & \\ \frac{1}{4} & 0 & \frac{1}{2} & \\ & \ddots & \ddots & \\ & & \frac{1}{4} & 0 \end{pmatrix} \qquad \|A\|_\infty = 3/4$$

$$A^n \to 0 \text{ as } n \to \infty$$

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 1/2 & 0 & \frac{1}{2} \\ 0 & 1/2 & 0 \end{pmatrix} \qquad \underline{\|A\|_\infty = 1}$$

We can show by eigenanalysis that $A^n \to 0$.

$$\|\underset{\sim}{x}\|_2 = \left(\sum |x_i|^2\right)^{\frac{1}{2}} = \left(\underset{\sim}{x}^T \underset{\sim}{x}\right)^{\frac{1}{2}}$$

$$\|A\|_2 = \max_{\underset{\sim}{x} \neq \underset{\sim}{0}} \frac{\|A\underset{\sim}{x}\|_2}{\|\underset{\sim}{x}\|_2} = \max_{\underset{\sim}{x} \neq \underset{\sim}{0}} \left(\frac{\underset{\sim}{x}^T A^T A \underset{\sim}{x}}{\underset{\sim}{x}^T \underset{\sim}{x}}\right)^{\frac{1}{2}}$$

How large can

$$\frac{\underset{\sim}{x}^T A^T A \underline{x}}{\underset{\sim}{x}^T \underset{\sim}{x}} \; ?$$

$$A^T A = (A^T A)^T, \quad \underline{\text{positive}\overset{\text{semi-}}{\;}\text{def}}.$$

$$A^T A = U \Sigma U^T$$

$$U^T U = I, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{pmatrix}$$

σ_i^2: eigenvalues of $A^T A$

$$\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_n^2 \geq 0 \qquad U^T U = I$$

$$\frac{\underline{x}^T A^T A \underline{x}}{\underline{x}^T \underline{x}} = \frac{\underline{x}^T U \Sigma U^T \underline{x}}{\underline{x}^T U U^T \underline{x}}$$

$$(\underline{w} = U^T \underline{x}) \qquad = \frac{\underline{w}^T \Sigma \underline{w}}{\underline{w}^T \underline{w}}$$

$$= \frac{\sum_{i=1}^{n} \sigma_i^2 w_i^2}{\sum_{i=1}^{n} w_i^2} \leq \sigma_1^2 \frac{\sum w_i^2}{\sum w_i^2} = \sigma_1^2$$

$$\frac{\underset{\sim}{x}^T A^T A \underset{\sim}{x}}{\underset{\sim}{x}^T \underset{\sim}{x}} \leq \sigma_1^2$$

$$\|A\|_2^2 \leq \sigma_1^2$$

$$A^T A = U \Sigma U^T$$

$$A^T A U = U \Sigma \quad , \quad U = (\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n)$$

$$A^T A \underset{\sim}{u}_i = \sigma_i^2 \underset{\sim}{u}_i$$

$$A^T A \underset{\sim}{u}_1 = \sigma_1^2 \underset{\sim}{u}_1$$

$$\frac{\underset{\sim}{x}^T A^T A \underset{\sim}{x}}{\underset{\sim}{x}^T \underset{\sim}{x}} = \frac{\underset{\sim}{x}^T U \Sigma U^T \underset{\sim}{x}}{\underset{\sim}{x}^T \underset{\sim}{x}}$$

$$\underset{\sim}{x} = \underset{\sim}{e}_1 \qquad = \sigma_1^2$$

$$\|A\|_2 = \sigma_1$$

$$\sigma_1 = \left[\lambda_{max}(A^T A)\right]^{\frac{1}{2}}$$

$\|A\|_2$ – 2-norm – spectral norm

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 3/4 & 10 \end{pmatrix}, \quad \|A\|_\infty = 10\tfrac{3}{4}$$

$$\rho(A) \le 10\tfrac{3}{4}$$

$$\begin{pmatrix} 0 & \frac{1}{4} & & 0 \\ 1/4 & \ddots & \ddots & \\ & \ddots & \ddots & 1/4 \\ 0 & & 1/4 & 0 \end{pmatrix} \quad \rho(A) \le \frac{1}{2}$$

$$\|A\|_\infty = \frac{1}{2}$$

$$\|A\|_{\cancel{\infty}} \le \rho(A) + \varepsilon$$

$$\|A\|_\alpha \;:\; \text{depends on } A.$$

$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

$$\|A\|_\infty = 4$$

$$\rho(A) = 2 + 2\cos\frac{\pi}{n+1} \le 4.$$

$$c_1 \|A\|_\alpha \le \|A\|_\beta \le c_2 \|A\|_\alpha$$

$$\frac{1}{\sqrt{n}} \|A\|_\infty \le \|A\|_2 \le \sqrt{n} \|A\|_\infty$$

Gerschgorin Theorem

$$A \underset{\sim}{x}_k = \lambda_k \underset{\sim}{x}_k$$

$$\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i$$

$$(a_{ii} - \lambda) x_i = -\sum_{j \neq i} a_{ij} x_j$$

$$|a_{ii} - \lambda| \, |x_i| \leq \sum_{j \neq i} |a_{ij}| \, |x_j| \qquad i = 1, 2, \ldots, n$$

$$|x_j| \leq |x_I|$$

$$r_i = \sum_{j \neq i} |a_{ij}|$$

$$|\lambda - a_{ii}| \leq r_i$$