Review

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Result from last time the *LU* factorization
$A = LU$, A, L, U $n \times n$

where $\ell_{ii} = 1$, L lower triangular, $u_{ii} = A^{(i)}_{ii}$
Problems with matrices like

$$\begin{pmatrix} 2 & 1 & 8 \\ 4 & 2 & 7 \\ 3 & 5 & 4 \end{pmatrix} \longmapsto \begin{pmatrix} 2 & 1 & 8 \\ 0 & \underline{0} & \cdot \\ 0 & \sim & - \end{pmatrix}$$

Note also that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$$

$A_{11} = L_{11}U_{11}$, etc

Pivoting

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Consider $\Pi_1 A$ where Π_1 is a permutation matrix; we want to fix Π_1 so that $a_{11} \neq 0$

We could permute, then perform the permutation:

$M_1 \Pi_1 A$
$M_2 \Pi_2 M_1 \Pi_1 A$

etc

Note that each permutation matrix flips the sign of the determinant

This is still called "*LU* factorization"; note that the whole thing could be written

$\Pi A = LU$

A good procedure for looking for which vow to pivot is to choose the row whose diagonal entry has greatest magnitude

Therefore, the multipliers of L look line

$$\frac{a_{i1}}{a_{I1}} \leq 1$$

so all the entries of *L* are *≤ 1*!

This is called "row pivoting" or "partial pivoting"

Example

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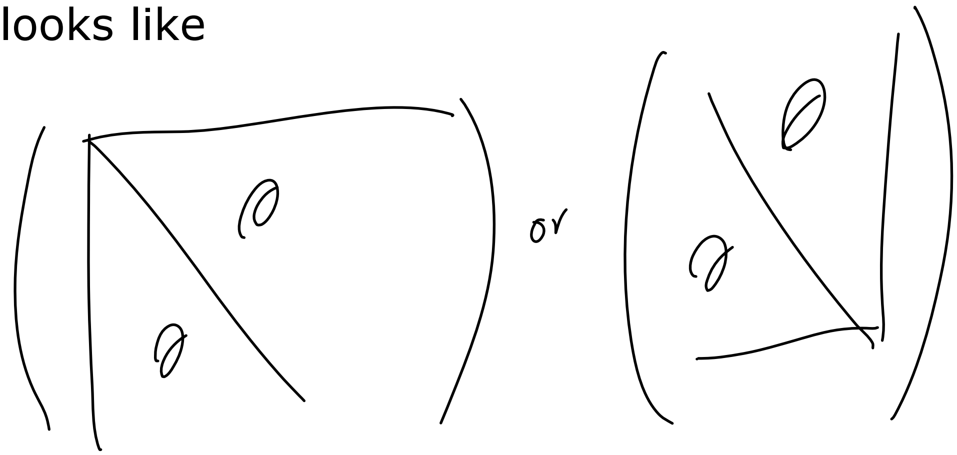
say *A* looks like



called an "Arrow matrix"; the first case requires lots of pivoting, but the second generally does not

Note: Partial pivoting does not give the rank; for that Something fancier is needed

Complete Pivoting

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The dif friend here is that we do
the step
max $|a_{ij}| \to a_{11}$

i.e. searching over all terms

Note that this could be done in parallel,
as the column operations can be performed
independently of each other (really?)

Important disclaimer: "In the absence
of roundoff"

Say we have

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

A_{11}: $p \times p$
A_{11}: $(n\text{-}p) \times (n\text{-}p)$

$$\|$$

$$\begin{pmatrix} L & \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \end{pmatrix} , \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \end{pmatrix} + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & E_{22} \end{array}\right)$$

$A_{11} = L_{11}U_{11}$
$A_{12} = L_{11}U_{12}$
$A_{21} = L_{21}U_{12}$
$A_{22} = L_{21}U_{12} + E_{22}$

We must have

$$U_{12} = L_{11}^{-1} A_{12}$$

$$L_{21} = A_{21} U_{11}^{-1}$$

$$E_{22} = A_{22} - L_{21} U_{12}$$

$$= A_{22} - (A_{21} U_{11}^{-1})(L_{11}^{-1} A_{12})$$

note that since $A_{11} = L_{11}U_{11}$, we have

$$= A_{22} - A_{21} A_{11}^{-1} A_{12}$$

Called the "Schur complement"

Could rewrite as

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

$$D - CA^{-1}B$$

Say after 1 step of Gaussian Elimination
we have

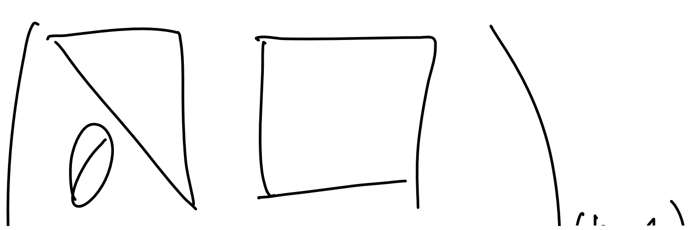
$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ 0 & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ 0 & a_{n2} & a_{nn} \end{pmatrix}$$

Next, we would want to eliminate the elements below a_{22}, but instead, we'll eliminate those above a_{22} as well, etc.

Gauss-Jordan gives as a result a diagonal matrix -- but beware, it only works in special cases

$$\begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{31} & a_{32} & a_{3n} \\ \\ a_{n1} & & a_{nn} \end{pmatrix}$$

The point is, at each subtraction stage, look at the elements above the $(k+1)^{st}$ row, and make an interchange if necessary



The advantage is that at each step we only need too rows in memory, *(k+1)* and the row above it that we're dealing with

Say we want $AX = I$ with
$X = (\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$, $I = (\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n)$
$A\boldsymbol{x}_j = \boldsymbol{e}_j$

From LU, we can easily solve for $\boldsymbol{x}_j$:
$A = LU \Rightarrow A^{-1} = U^{-1}L^{-1}$
Important to note that if U is upper triangular, then so is U^{-1}:
$UZ = I$

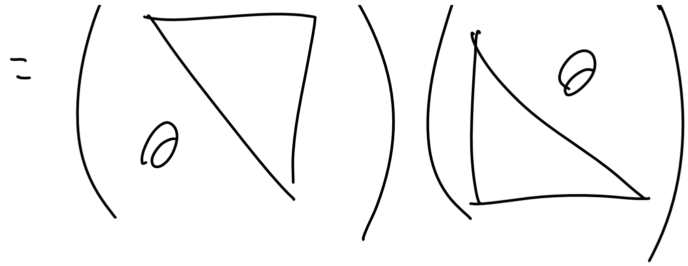
$(\vec{z}_1, \ldots, \vec{z}_n) = (\vec{e}_1, \ldots, \vec{e}_n)$

$U\vec{z}_1 = e_1$

$= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{z}_1 = \begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ etc

So if $A = LU$ then $A^{-1} = U^{-1}L^{-1}$

$= \left(\quad \right) \left(\quad 0 \quad \right)$



Requires the following # of multiplications:

$$\begin{pmatrix} n & (n-1) & \cdots & 1 \\ (n-1) & (n-1) & & \vdots \\ & & & \\ 1 & \cdots & & 1 \end{pmatrix}$$

So the product requires $\sim n^3/3$ therefore
Computing the inverse takes $4n^3/3$,

Note that multiplication only takes n^3 multiplications, and inversion isn't much harder!

Simplex Method

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Say we have A**x** = **b**, A is m × n, m ≤ n. We want a nonnegative solution vector, and we want to maximize $\mathbf{c}^T\mathbf{x}$. First, choose a set of basis vectors *B* from *A*:

$$B = (\vec{a}_{i_1}, \dots, \vec{a}_{i_m})$$

Then solve $B\boldsymbol{x} = \boldsymbol{b}$, $B^T\boldsymbol{w} = \boldsymbol{c}$ (the dual), and $B\boldsymbol{t}^{(r)} = -\boldsymbol{a}_r$

This tells us which of the columns should be thrown out (how to choose which column to put in next? don't worry about it)

$$B_0 = L_0 U_0$$

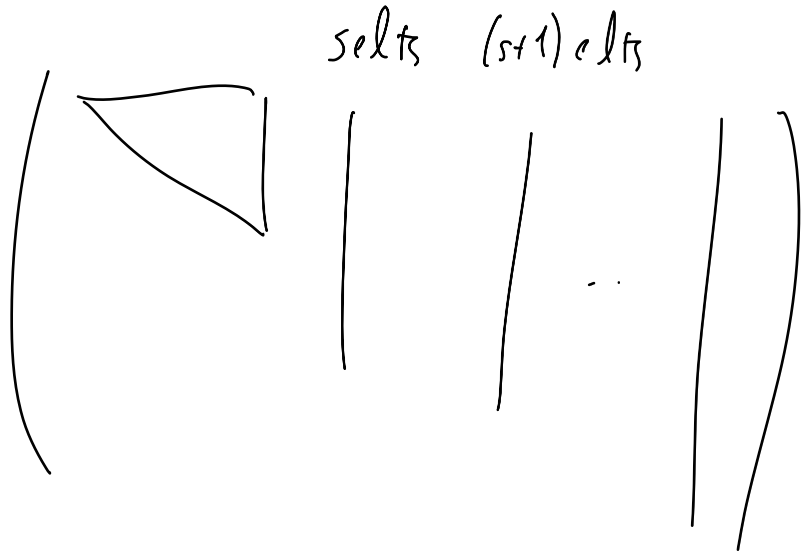
Throw out column *s* from B_0 and introduce a new column *g*:

$$B_1 = (\vec{a}_{i_1}, \vec{a}_{i_2}, \dots, \cancel{\vec{a}_{i_s}}, \vec{a}_{i_{s+1}}, \dots, g)$$

Where *g* is a column of *A*

We had $B_0 = L_0 U_0$

$L_0^{-1} B_1 =$



The effect of throwing out the columns and adding a new one is to put one more element on the diagonal, i.e. a Hessenberg matrix

Hessenberg Matrices

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$$\begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & & \vdots \\ & h_{23} & \ddots & \\ 0 & & & h_{nn} \end{pmatrix} \vec{x} = \vec{b}$$

Converting this into canonical form requires only

$$n + (n+1) + \cdots \approx n^2/2$$

operations

$(H\text{-}\lambda I)\boldsymbol{x} = \boldsymbol{g}$

Uniqueness

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Is the *LU* decomposition unique?

Say $A = L_1U_1 = L_2U_2$

$$L_2^{-1}L_1 = U_2U_1^{-1}$$

Lower triangular   Upper triangular

So, they're both diagonal -- but L_1 and L_2^{-1} both have 1's on the

diagonal, so they're the same, subject to no interchanges