Local Limit Theorems For Random Walks On Trees And Groups

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Infinite Homogeneous Tree of Degree $d = 3$
Random Walk on Tree $T_4$

Transition Probabilities:

- Up/Down probability $p_V$
- Right/Left probability $p_H$
- $2p_V + 2p_H = 1$

Problem: What is the probability of return to the root after $2n$ steps?
Asymptotic Behavior of Return Probabilities

More generally, consider homogeneous, nearest-neighbor random walk $X_n$ on the infinite regular tree $T_d$ of degree $d \geq 3$.

**Theorem:** (Gerl-Woess 1986 PTRF) There exist constants $C > 0$ and $R > 1$ (depending on the step distribution) such that as $n \to \infty$,

$$P^1\{X_{2n} = 1\} \sim \frac{C}{R^n n^{3/2}}$$
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What about non-nearest neighbor random walk on $T_d$?

Theorem: (Lalley 1993 Annals) There exist constants $C > 0$ and $R > 1$ (depending on the step distribution) such that as $n \to \infty$,

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The local limit theorem suggests that the event of return to the root at large time $2n$ is a large deviation event.

**Theorem: (SLLN; CLT)** There exist $\mu > 0$ and $\sigma^2 > 0$ depending on the step distribution such that as $n \to \infty$,

\[
\frac{\text{distance}(X_n, \text{root})}{n} \to \mu \quad \text{a.s., and}
\]

\[
\frac{(\text{distance}(X_n, \text{root}) - n\mu)}{\sqrt{n}} \Rightarrow \text{Normal}(0, \sigma^2)
\]

These are less interesting that the return probabilities and can be established by standard techniques, i.e., subadditive ergodic theorem and martingale CLT.
$T_4$ as a Free Group/Regular Language

- Vertices correspond to **finite, reduced words** in the letters $A, A^{-1}, B, B^{-1}$.
- The set $\mathbb{F}_2$ of all such words is a **regular language**.
- $\mathbb{F}_2$ is a **group** under concatenation/reduction.
- $X_n = \xi_1\xi_2\cdots\xi_n$ where $\xi_i$ are i.i.d.
Nonamenable Group: A finitely generated, discrete group $\Gamma$ is nonamenable if its isoperimetric constant $\gamma$ is positive, that is

$$\gamma = \inf_{F \subset \Gamma} \frac{\#(\partial F)}{\#F} > 0$$

Examples: The integer lattices $\mathbb{Z}^d$ are amenable. Free groups and discrete subgroups of matrix groups $GL(n,\mathbb{Z})$ are nonammenable.
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Kesten’s Theorem: For every nontrivial random walk on a nonammenable group there is a constant $R > 1$ such that

$$\lim_{n \to \infty} n^{-1} \log P^1 \{X_{2n} = 1\} = - \log R.$$
Random Walk on $PSL(2, \mathbb{R})$

$PSL(2, \mathbb{R})$ is the group of all $2 \times 2$ real matrices with determinant $\pm 1$. It is of particular interest to mathematicians because it is the isometry group of the hyperbolic plane.

**Theorem:** (Bougerol 1981 Ann. Sci. de l’ENS) Let $X_n = \xi_1 \xi_2 \cdots \xi_n$ be a symmetric, right random walk on $PSL(2, \mathbb{R})$ whose step distribution has a density with respect to Haar measure and rapidly decaying tails. Then the $n$–step transition probability densities $p_n(x, y)$ satisfy

$$p_n(x, y) \sim \frac{C_{x,y}}{R^n n^{3/2}}$$

In fact Bougerol proved that there is a similar asymptotic formula for random walk on any semi-simple Lie group. The exponent $3/2$ is replaced by $k/2$ for some integer $k$ depending only on the structural constants of the Lie group.
The Free Group $\mathbb{F}_k$ as a Subgroup of $PSL(2, \mathbb{R})$

- Isometries of $\mathbb{H}$ are linear fractional transformations.
- Let $A$ be the LFT mapping blue line to blue line and $B$ the LFT mapping red line to red line.
- Let $\Gamma$ be the group generated by $A^{\pm 1}, B^{\pm 1}$.
- Then $\Gamma = \mathbb{F}_2$
The Free Group $\mathbb{F}_k$ as a Subgroup of $PSL(2, \mathbb{R})$

Elements of $\Gamma = \mathbb{F}_2$ are in 1–1 correspondence with the tiles of the tessellation. Each element of $\Gamma$ acts as a permutation on the tiles. The tree $T_4$ is naturally embedded in the tessellation, with each tile containing one vertex of $T_4$.

Theory: The fact that $\mathbb{F}_2$ embeds into $PSL(2, \mathbb{R})$ is responsible for the local limit theorem.
Conjecture: (L-1993) The local limit theorem holds for any symmetric, finite-range random walk on any co-compact subgroup of $PSL(2, \mathbb{R})$, and in particular for random walk on any surface group of genus $\geq 2$.

Theorem: (Gouezel-Lalley 2011) It’s true! For any such random walk,

$$P^1\{X_{2n} = 1\} \sim CR^{-n}n^{-3/2}.$$
Random Walk on a Regular Language

A regular language is a set $\mathcal{L}$ of words (finite sequences) over a finite alphabet such that membership in $\mathcal{L}$ can be checked by a finite-state automaton.

Example 1: The group $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$ is the set of all finite words from a 3-letter alphabet $\{a, b, c\}$ in which no letter appears twice consecutively.

Example 2: Any group that has a free group $\mathbb{F}_k$ as a subgroup of finite index (for instance, $PSL(2, \mathbb{Z})$ or its congruence subgroups) has a representation as a regular language.
Random Walk on a Regular Language

A random walk on a regular language $\mathcal{L}$ is a Markov chain $X_n$ on $\mathcal{L}$ whose one-step transitions are such that

- the length of the word can only change by 0, +1, or −1;
- only the last 2 letters are altered; and
- the transition probabilities depend only on the last 2 letters.

Example: LIFO queues with finitely many job types.
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Theorem: (Lalley 2003) Under suitable irreducibility and aperiodicity hypotheses, every transient RWRL satisfies a local limit theorem

$$P^x\{X_n = y\} \sim C_{x,y} R^{-n} n^{-\alpha}$$

for some $R \geq 1$ and $\alpha = 0, \frac{1}{2}, \text{ or } \frac{3}{2}$. For random walk on a virtually free group, $R > 1$ and only $\alpha = \frac{3}{2}$ is possible.
Green’s Function of a Random Walk

The Green’s function of a Markov chain is the matrix of generating functions of transition probabilities: for each pair of states \( x, y \),

\[
G_{x,y}(r) = \sum_{n=0}^{\infty} p_n(x, y) r^n.
\]

If the Markov chain is irreducible then the radius of convergence \( R \geq 1 \) does not depend on \( x, y \). For random walk on any nonamenable group, \( G_{x,y}(R) < \infty \).

The behavior of the Green’s function at the radius of convergence determines and is determined by the asymptotic behavior of the coefficients \( p_n(x, y) \). In particular,

\[
G(R) - G(r) \sim C\sqrt{R - r} + \text{ Tauberian condition}
\]

\[
\Rightarrow p_{2n}(x, x) \sim C_x R^{-n} n^{-3/2}
\]
**Flajolet-Odlysko Tauberian Theorem**

**Theorem:** Let $G(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R$. Suppose that $G$ has an analytic continuation to $\mathbb{C} \setminus ([R, \infty) \cup (-\infty, -R - \varepsilon])$, and suppose that as $z \to R$ in $\Delta_{\rho,\phi}$,

$$G(z) - C \sim K(R - z)^\alpha$$

for some $K \neq 0$ and $\alpha \notin \{0, 1, 2, \ldots \}$. Then as $n \to \infty$,

$$a_n \sim \frac{K}{\Gamma(-\alpha) R^n n^{\alpha+1}}.$$  

**Basic Fact:** The Green’s function $G_{1,1}(r)$ of an aperiodic, symmetric random walk on a discrete group admits an analytic continuation to $\mathbb{C} \setminus ([R, \infty) \cup (-\infty, -R - \varepsilon])$. 

Branching Random Walk: Particles alternately reproduce according to the law of a Galton-Watson process and move according to the law of a symmetric random walk. The default initial configuration has a single particle located at the group identity 1.

Proposition: Let $N_n(x)$ be the number of particles at location $x$ in the $n$th generation. Let $r$ be the mean of the offspring distribution. Then

$$EN_n(x) = r^n p_n(1, x).$$

Consequently, the expected total occupation time of vertex $x$ through the entire history of the BRW is

$$G_{1,x}(r).$$
Weak Survival

Branching random walk on a homogeneous tree has a weak survival phase. If the mean offspring number $r > 1$ then the total number of particles in generation $n$ is a supercritical Galton-Watson process. If $1 < r < R$ then the expected number of particles at the root decays exponentially. So with positive probability, the number of particles grows exponentially, but the particles eventually vacate any finite set of vertices.

Limit Set: Define

$$\Lambda = \{\text{ends in which the BRW survives}\}$$
Branching Brownian Motion: Individual particles execute independent Brownian motions in $\mathbb{H}$ beginning at their birthplaces/times, and independently fission at rate $\lambda > 0$.

**Theorem:** (Lalley-Sellke 1996) (Lalley-Sellke) If $\lambda < 1/8$ then the process survives weakly. The limit set $\Lambda$ is wp1 a Cantor set of Hausdorff dimension

$$\delta(\lambda) = \frac{1 - \sqrt{1 - 8\lambda}}{2}$$
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The square-root singularity is caused by a corresponding square-root singularity in the Green’s function of hyperbolic Brownian motion.
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There is a similar exact formula for BRW on the homogeneous tree $T_d$, also with square-root singularity at the phase transition point.
Asymptotic Behavior of the Green’s Function at \( r = R \)

**Special Case:** RW on \( T_3 \).

Color all edges Red, Blue, or Green in such a way that every vertex has one incident edge of each color. Let \( X_n \) be the random walk that makes Red, Blue, or Green steps with probabilities \( p_R, p_B, p_G \).

Let \( \tau(i) = \) first time that \( X_n \) visits vertex \( i \) (where \( i = R, B, G \)).
Functional Equations for the Green’s Functions

Define $F_i(r) = E^1 r^{\tau(i)}$. Then

$$G_{1,1}(r) = 1 + \sum_{i=R,B,G} p_i F_i(r) G_{1,1}(r)$$

$$= \left( 1 - \sum_{i=R,B,G} p_i F_i(r) \right)^{-1}$$

and

$$F_i(r) = p_i r + \sum_{j \neq i} p_j r F_j(r) F_i(r)$$

These equations determine an algebraic variety in the variables $F_R, F_G, F_B$ and $r$. Hence, the functions $F_i$ and $G_{1,1}$ are algebraic functions, and thus their only singularities are isolated branch points.
Functional Equations for the Green’s Functions

The algebraic system in vector notation has the form

\[ F = rP + rQ(F) \]

where \( Q \) is a vector of quadratic polynomials with positive coefficients. By the Implicit Function Theorem, analytic continuation of the vector-valued function \( F \) is possible at every point where the linearized system is solvable for \( dF \) in terms of \( dr \):

\[
\begin{align*}
dF &= (dr)P + (dr)Q(F) + r \left( \frac{\partial Q}{\partial F} \right)(dF) \\
&\quad \iff \\
\left( I - r \frac{\partial Q}{\partial F} \right)(dF) &= (dr)(P + rQ(F))
\end{align*}
\]
Thus, the smallest positive singularity of the function \( F(r) \) occurs at the value \( r = R \) where the spectral radius (\( = \) Perron-Frobenius eigenvalue) of \( r(\partial Q/\partial F) \) reaches 1. If \( v^T \) is the left Perron-Frobenius eigenvector then near \( r = R \),

\[
v^T \left( \left( I - r \frac{\partial Q}{\partial F} \right) dF \right) = (dr)(v^T P + rv^T Q(F)) + r(\text{quadratic form in } dF)
\]

The quadratic terms have positive coefficients, hence the singularity at \( r = R \) is a square-root singularity (branch point of order 2).
Life Without Functional Equations

Unfortunately, this analysis depends in an essential way on the recursive structure of a regular language or regular tree. In general, for discrete groups there seems to be no finite system of algebraic equations that determine the Green’s functions.

Conjecture: (P. Sarnak) The Green’s functions of a nearest neighbor random walk on a discrete subgroup $\Gamma$ of $\text{PSL}(2, \mathbb{R})$ are algebraic if and only if $\Gamma$ has a free subgroup of finite index.
The Keys to the Kingdom

The Green’s functions of any Markov chain satisfy a system of quadratic ordinary differential equations:

\[
\frac{d}{dr} G_{x,y}(r) = r^{-1} \sum_z G_{x,z}(r) G_{z,y}(r) - r^{-1} G_{x,y}(r). 
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For random walk on a discrete subgroup of \( \text{PSL}(2, \mathbb{R}) \) the sum blows up as \( r \to R \). The rate of blow-up is determined by the behavior of \( G_{x,z}(r) \) as \( z \) converges to the Martin boundary.

The Bottom Line: As \( r \to R \),

\[
G_{x,y} \sim C_{x,y} \sqrt{R - r}.
\]
What We Still Don’t Know

(1) Is there a local limit theorem for random walks on $T_d$ whose step distributions have infinite support?

(2) Is the square-root singularity at the weak/strong survival transition point for branching random walk on $T_d$ inherited by similar particle systems, e.g., the contact process?

(3) What about $PSL(3, \mathbb{Z})$ and $PSL(4, \mathbb{Z})$ and $\cdots$?