A Stochastic Competition Model and First-Passage Percolation

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A Stochastic Competition Model

Two species, RED and BLUE, compete for space on the two-dimensional square lattice, in continuous time, according to the following rules:

1. At any time, squares may be RED, BLUE, or WHITE (unoccupied).
2. Red squares colonize neighboring squares at rate $\lambda_R$.
3. Blue squares colonize neighboring squares at rate $\lambda_B$.
4. At time $t = 0$, finitely many squares are RED and finitely many are BLUE.

**Problem 1:** Is there positive probability that both species survive forever?

**Problem 2:** If so, do the regions colonized by RED and BLUE stabliize?
First Simulation: \( \lambda_R = \lambda_B \)

Red and Blue each colonize sectors of a growing disk. The interface is bumpy but roughly piecewise linear.
Second Simulation: $\lambda_R = \lambda_B$

The Red and Blue colonies are each unions of two angular sectors. The shape of the growing disk is the same as in the first simulation.
Third Simulation: $\lambda_R = \lambda_B$

If a colonized region does not maintain contact with the uncolonized exterior, it will slowly disappear.
**Theorem:** If $1 < \frac{\lambda_B}{\lambda_R} < C$ for some critical constant $C$, then mutual survival is possible: Red surrounds Blue but is gradually displaced.
Related Models

**Richardson Model:** The special case of the competition model in which only one color (Blue) is present at time $t = 0$ is the *Richardson Model* with infection rate $\lambda_B$.

**Biased Voter Model:** The competition model with unequal colonization rates $\lambda_R > \lambda_B$ in which at time $t = 0$ there are finitely many Red sites, infinitely many Blue sites, and no White sites is the *Biased Voter Model*.

**First-Passage Percolation:** I.I.D. random variables, with common distribution $F$, are attached to the edges bordering adjacent squares: these represent the times needed for infection to cross the borders following the first infection of one of the incident squares. In the special case $F = \text{exponential distribution}$, this is the *Richardson model*.

**Haggstrom-Pemantle Model:** Is the same as Kordzakhia’s 2-species competition model except that once a square has first been colonized (by Red or Blue), it can never be recolonized.
Limit Shapes

Theorem 1 ¹ Let $R_t$ be the infected region at time $t$ in first-passage percolation. Assume that the passage time distribution $F$ is concentrated on $(0, \infty)$, and that it has finite m.g.f. Then there is a compact, convex region $R$ (the limit shape) depending on $F$, such that with probability one

$$R_t/t \longrightarrow R$$

Theorem 2 ² Let $R_t$ be the Red region at time $t$ in the biased voter model (with $\lambda_R > \lambda_B$). There exists a nonrandom, compact, convex region $U$ such that, almost surely on the event that Red survives forever,

$$R_t/t \longrightarrow U.$$  

Moreover, as $\lambda_R/\lambda_B \downarrow 1$, the limit shape shrinks to a point.

¹Hammersley & Welsh, 1963  
²Bramson & Griffeath, 1981 Annals of Probability
Limit Shape in First-Passage Percolation

There are no nontrivial distributions $F$ for which the limit shape is explicitly known, or even known to be strictly convex. It is widely suspected that at least for distributions $F$ with continuous densities, the limit shapes are not only strictly convex but uniformly curved in the following sense (Newman):

**Definition 1** A compact region $K$ is uniformly curved if there exists $R < \infty$ such that for every boundary point $x$ of $K$ there is a circle of radius $R$ that passes through $x$ and completely encloses $K$. 
Coexistence in the Competition Model

**Theorem 3** \(^3\) Assume that the limit shape for the Richardson model is uniformly curved. Then for the competition model with equal colonization rates \(\lambda_R = \lambda_B\), there is positive probability that both species will survive forever.

**Conjecture 1** On the event of mutual survival, the RED and BLUE regions have limiting shapes, which are the union(s) of finitely many angular wedges of the Richardson limit shape.

**Related Results:** (A) For the Haggstrom-Pemantle Model, mutual survival occurs with positive probability. No hypothesis about the Richardson limit shape is required. (B) For first-passage percolation, under suitable hypotheses, in almost every direction geodesic rays are unique.\(^4\).

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\(^3\)G. KORDZAKHIA, Ph. D. dissertation

\(^4\)LICEA & NEWMAN, 1996 Annals of Probability
Poisson Plumbing & the Richardson Model

The Richardson model may be constructed with the aid of **Poisson plumbing**. Poisson plumbing consists of an array of timelines over every vertex, together with “pipes” connecting neighboring timelines placed at the points of a Poisson process of rate \( \lambda \). Vertex \( x \) is in the infected set at time \( t \) if and only if there is an upward continuous path through the plumbing that connects \((0, 0)\) to \((x, t)\).

**Corollary:** The probability that vertex \( x \) is in the infected set at time \( t + s \) is the same as the probability that independent Richardson models started at 0 and \( x \), and run for times \( t \) and \( s \), intersect.
Mutual Survival in the Competition Model

Given that \textit{Blue} has begun to colonize an angular sector, it is unlikely (exponentially in the radius of the ball) that \textit{Red} will encroach on the sector.
Oriented Percolation

**Percolation Subgraphs:** Independent, identically distributed Bernoulli-$p$ random variables $Y_e$ are attached to the edges $e$ of the square lattice. Edges $e$ for which $Y_e = 0$ are removed from the lattice; edges $e$ for which $Y_e = 1$ are retained. The resulting random subgraph of the lattice is denoted by $\mathcal{G}_p$.

**Oriented Percolation Clusters:** The percolation cluster based at a vertex $x$ consists of all vertices and edges that are connected to $x$ by paths in $\mathcal{G}_p$ that make steps only *upward* or *to the right*.

**Oriented Percolation** is said to occur if the percolation cluster based at the origin is infinite.
Oriented Percolation: Simulations

\begin{center}
\begin{tabular}{ccc}
\includegraphics[width=0.3\textwidth]{p=0.65} & \includegraphics[width=0.3\textwidth]{p=0.66} & \includegraphics[width=0.3\textwidth]{p=0.67} \\
p=0.65 & p=0.66 & p=0.67 \\
\end{tabular}
\end{center}

**Theorem 4** \(^5\) There is a critical value \(p_c\) (known to be between .629 and .667) such that the event of oriented percolation has positive probability for \(p > p_c\), but has probability zero for \(p \leq p_c\). For \(p > p_c\) the right and left edges of the percolation cluster are asymptotically linear, and the percolation cluster intersects the 45° line infinitely often.

\(^5\)Durrett, 1984 Annals of Probability
First-Passage Percolation: Limit Shapes with Flat Spots

Example: Consider the first-passage percolation process whose edge passage times are Bernoulli-$p$ plus 1. If $p > p_c$, the boundary of the limit shape must have a flat spot.

Reason: For any point $x = (u, v)$ on the line $u + v = n$ that is in the oriented percolation cluster of the origin 0, the passage time $T(0, x)$ must satisfy

\[ T(0, x) = n. \]

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\[ \text{\textsuperscript{6}DURRETT & LIGGETT, 1981 Annals of Probability} \]
First-Passage Percolation

Notation:

- $\xi_e =$ traversal time of edge $e$.
- $F =$ common distribution of the traversal times $\xi_e$.
- $\tau(\gamma) = \sum_{e \in \gamma} \xi_e$: traversal time of path $\gamma$.
- $T(x, y) =$ min $\{\tau(\gamma) : \gamma$ that connect points $x, y\}$.
- $\mu(u) = \lim_{n \to \infty} T(0, nu)/n =$ inverse infection speed in direction $u$.

Note: If $F$ is continuous, then the time-minimizing path ("geodesic") connecting any two vertices is unique, and the variance of $T(0, nu)$ diverges as $n \to \infty$.

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Theorem 5. \(^8\) Assume that

1. \(F\) has a bounded density on \((0, \infty)\) and finite MGF.

2. Long geodesic segments are (to first order) straight line segments.

3. For each direction \(u\), there exist a nontrivial mean-zero distribution \(G_u\) and a scalar sequence \(a_n \to \infty\) such that

\[
T(0, n u) - n \mu(u) \xrightarrow{a_n} G_u
\]

Then the limit shape is strictly convex.

Note: It is suspected that the correct normalizing sequence is \(a_n = n^{1/3}\). The limiting distributions \(G_u\) may or may not be Gaussian. It is now known \(^9\) that for certain atomic distributions \(F\), the variance of \(T(0, n u)\) grows sublinearly with \(n\).

\(^8\) Lalley, 2003 ECP

\(^9\) Benjamini, Kalai, & Schramm, 2003 Annals of Probability
Proof of Theorem 5

It suffices to show that for any two linearly independent vectors (directions) \( u \) and \( v \), the speed of propagation is greater in direction \( (u + v)/2 \) than the average of the propagation speeds in directions \( u \) and \( v \). WLOG,

\[
\mu(u) = \mu(v).
\]

Consider a parallelogram array superimposed on the square lattice. If the parallelograms are large, the travel times between adjacent disks, suitably renormalized, have distributions close to \( G_u \) and \( G_v \). Mark a parallelogram as a Success if the travel time through it is less than the 70th percentile of \( G_u \) (or \( G_v \)); otherwise, mark it as a Failure.
Proof of Theorem 5

The mean travel time through parallelograms marked Success is strictly less than $\mu(u)$ (because by Hypothesis 3, the limit distributions $G_u$ and $G_v$ are mean-zero).

Because the critical value $p_c$ for oriented percolation is less than .7, there exist, with positive probability, oriented paths of Success parallelograms connecting the origin to points $nu + nv$ with arbitrarily large $n$. The travel time for such a path is (with probability near one) less than

$$n(\mu(u) + \mu(v) - \varepsilon).$$
First-Passage Percolation: Unsolved Problems

• When is the limit shape strictly convex?
• What is the rate of growth of $\text{var}(T(0, nu))$?
• To what does the distribution of $(T(0, nu) - n\mu(u))/a_n$ converge?
• Do the geodesic segments behave regularly:
  – Are they (to first order) straight?
  – How do they fluctuate about straight lines?
  – Is there a LLN for their (Euclidean) lengths?
  – Do their empirical distributions behave regularly?