A Stochastic Competition Model and First-Passage Percolation

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A Stochastic Competition Model

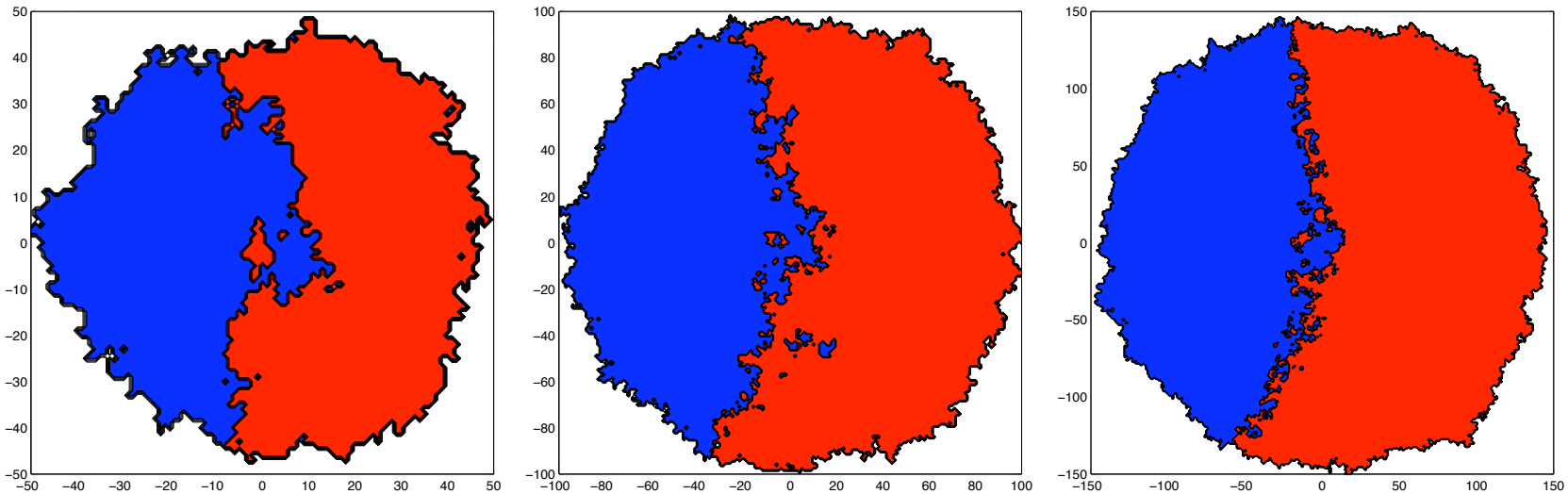
Two species, RED and BLUE, compete for space on the two-dimensional square lattice, in continuous time, according to the following rules:

1. At any time, squares may be RED, BLUE, or WHITE (unoccupied).
2. Red squares colonize neighboring squares at rate λ_R.
3. Blue squares colonize neighboring squares at rate λ_B.
4. At time $t = 0$, finitely many squares are RED and finitely many are BLUE.

Problem 1: Is there positive probability that both species survive forever?

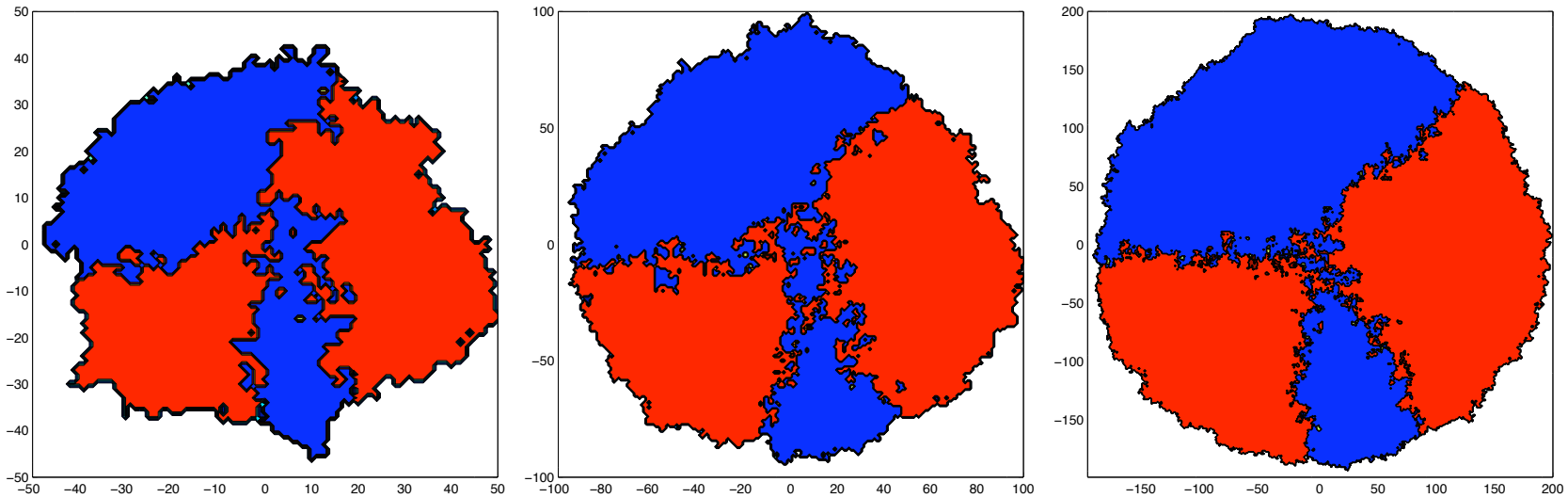
Problem 2: If so, do the regions colonized by RED and BLUE stabliize?

First Simulation: $\lambda_R = \lambda_B$



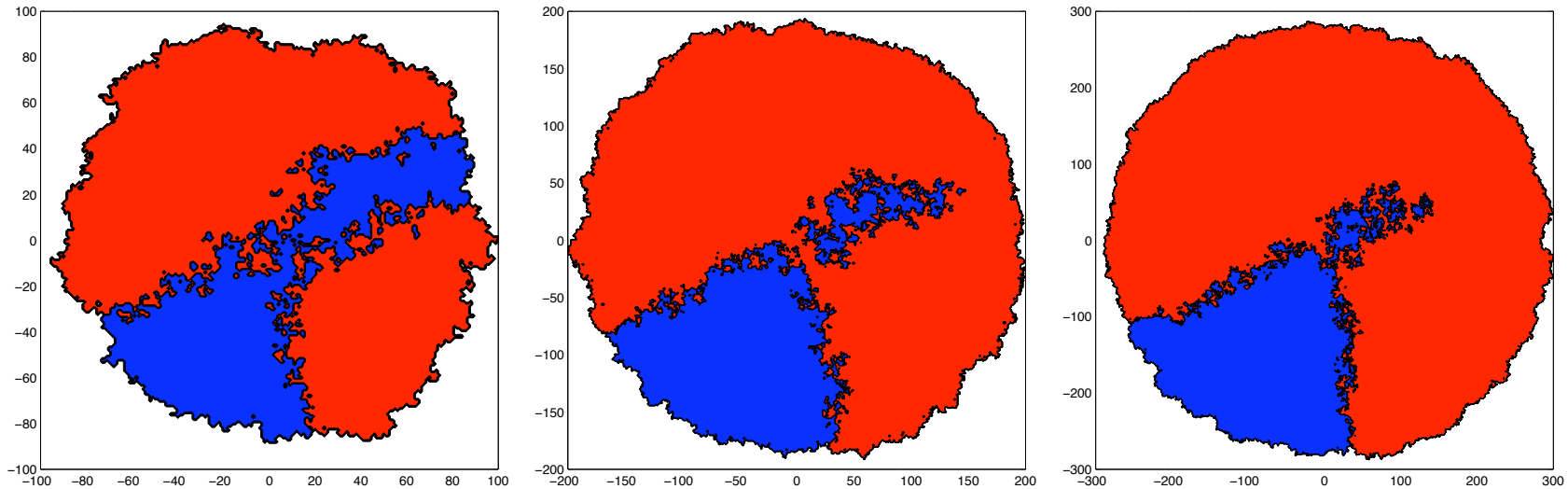
Red and Blue each colonize sectors of a growing disk. The interface is bumpy but roughly piecewise linear.

Second Simulation: $\lambda_R = \lambda_B$



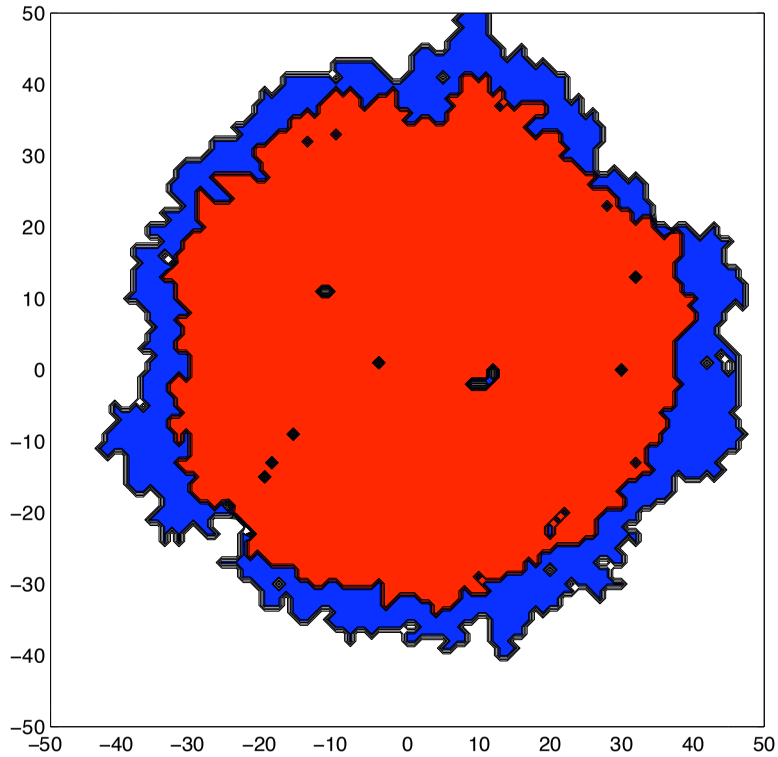
The Red and Blue colonies are each unions of two angular sectors. The shape of the growing disk is the same as in the first simulation.

Third Simulation: $\lambda_R = \lambda_B$



If a colonized region does not maintain contact with the uncolonized exterior, it will slowly disappear.

Unequal Rates $\lambda_R \neq \lambda_B$



Theorem: If $1 < \lambda_B/\lambda_R < C$ for some critical constant C, then mutual survival is possible: Red surrounds Blue but is gradually displaced.

Related Models

Richardson Model: The special case of the competition model in which only one color (Blue) is present at time $t = 0$ is the *Richardson Model* with infection rate λ_B.

Biased Voter Model: The competition model with unequal colonization rates $\lambda_R > \lambda_B$ in which at time $t = 0$ there are finitely many Red sites, infinitely many Blue sites, and *no* White sites is the *Biased Voter Model*.

First-Passage Percolation: I.I.D. random variables, with common distribution F, are attached to the edges bordering adjacent squares: these represent the times needed for infection to cross the borders following the first infection of one of the incident squares. In the special case $F =$ exponential distribution, this is the *Richardson model*.

Haggstrom-Pemantle Model: Is the same as Kordzakhia's 2-species competition model *except* that once a square has first been colonized (by $\textsc{Red}$ or $\textsc{Blue}$), it can never be recolonized.

Limit Shapes

Theorem 1 [1] Let R_t be the infected region at time t in first-passage percolation. Assume that the passage time distribution F is concentrated on $(0, \infty)$, and that it has finite m.g.f. Then there is a compact, convex region R (the limit shape) depending on F, such that with probability one

$$R_t/t \longrightarrow R$$

Theorem 2 [2] Let R_t be the Red region at time t in the biased voter model (with $\lambda_R > \lambda_B$). There exists a nonrandom, compact, convex region U such that, almost surely on the event that Red survives forever,

$$R_t/t \longrightarrow U.$$

Moreover, as $\lambda_R/\lambda_B \downarrow 1$, the limit shape shrinks to a point.

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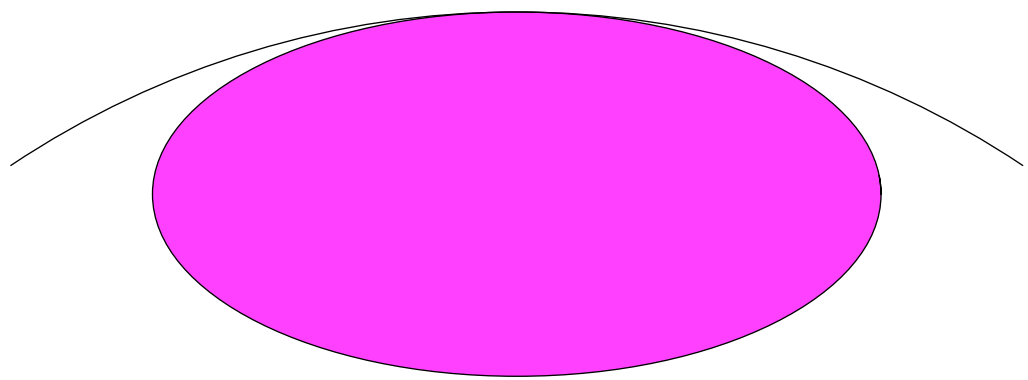
[1] Hammersley & Welsh, 1963

[2] Bramson & Griffeath, 1981 Annals of Probability

Limit Shape in First-Passage Percolation

There are no nontrivial distributions F for which the limit shape is explicitly known, or even known to be *strictly* convex. It is widely suspected that at least for distributions F with continuous densities, the limit shapes are not only strictly convex but **uniformly curved** in the following sense (NEWMAN):

Definition 1 A compact region K is *uniformly curved* if there exists $R < \infty$ such that for every boundary point x of K there is a circle of radius R that passes through x and completely encloses K.



Coexistence in the Competition Model

Theorem 3 [3] Assume that the limit shape for the Richardson model is uniformly curved. Then for the competition model with equal colonization rates $\lambda_R = \lambda_B$, there is positive probability that both species will survive forever.

Conjecture 1 On the event of mutual survival, the RED and BLUE regions have limiting shapes, which are the union(s) of finitely many angular wedges of the Richardson limit shape.

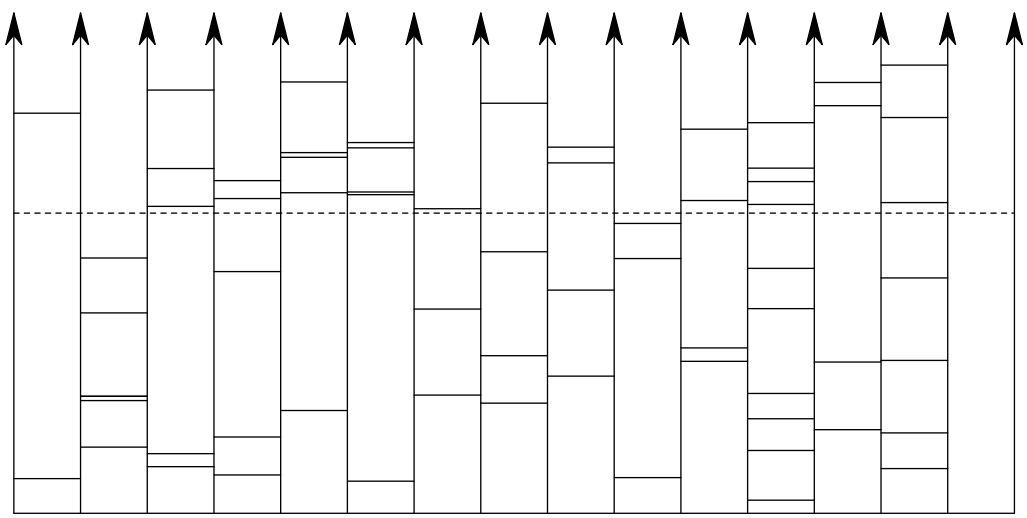
Related Results: (A) For the Haggstrom-Pemantle Model, mutual survival occurs with positive probability. No hypothesis about the Richardson limit shape is required. (B) For first-passage percolation, under suitable hypotheses, in almost every direction geodesic rays are unique.[4].

[3] G. KORDZAKHIA, Ph. D. dissertation

[4] LICEA & NEWMAN, 1996 Annals of Probability

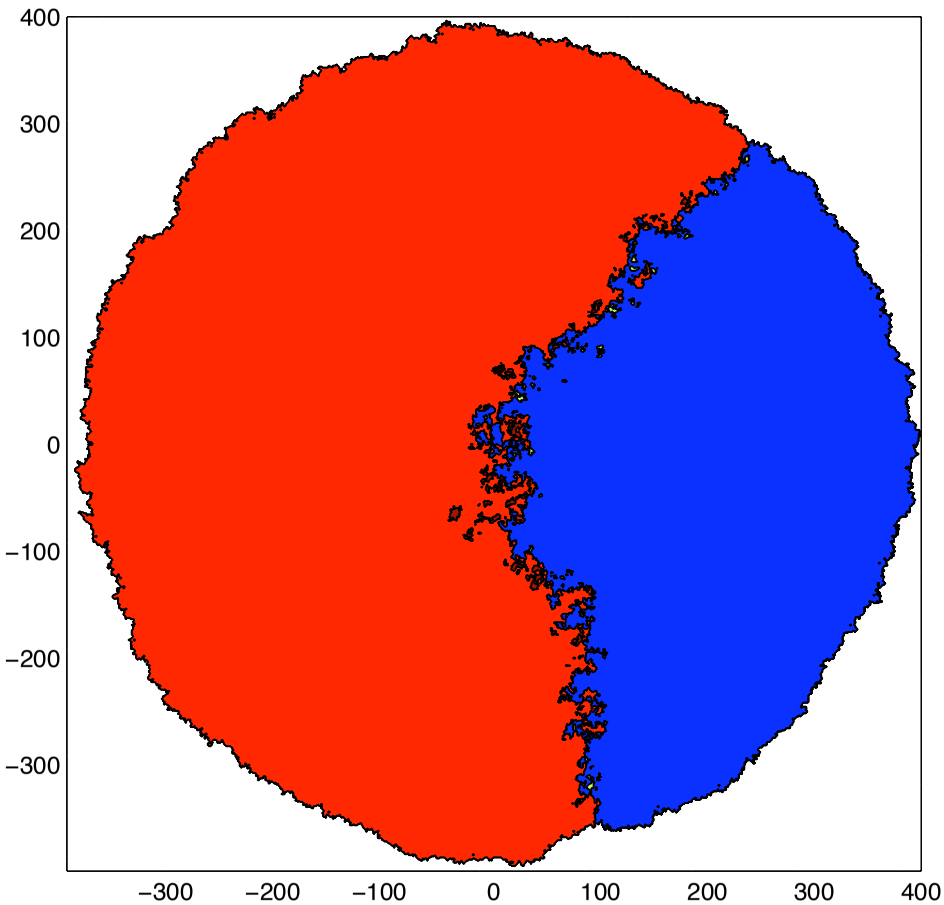
Poisson Plumbing & the Richardson Model

The Richardson model may be constructed with the aid of **Poisson plumbing**. Poisson plumbing consists of an array of timelines over every vertex, together with " pipes" connecting neighboring timelines placed at the points of a Poisson process of rate λ. Vertex x is in the infected set at time t if and only if there is an upward continuous path through the plumbing that connects $(0, 0)$ to (x, t).



Corollary: The probability that vertex x is in the infected set at time $t + s$ is the same as the probability that independent Richardson models started at 0 and x, and run for times t and s, intersect.

Mutual Survival in the Competition Model



Given that BLUE has begun to colonize an angular sector, it is unlikely (exponentially in the radius of the ball) that RED will encroach on the sector.

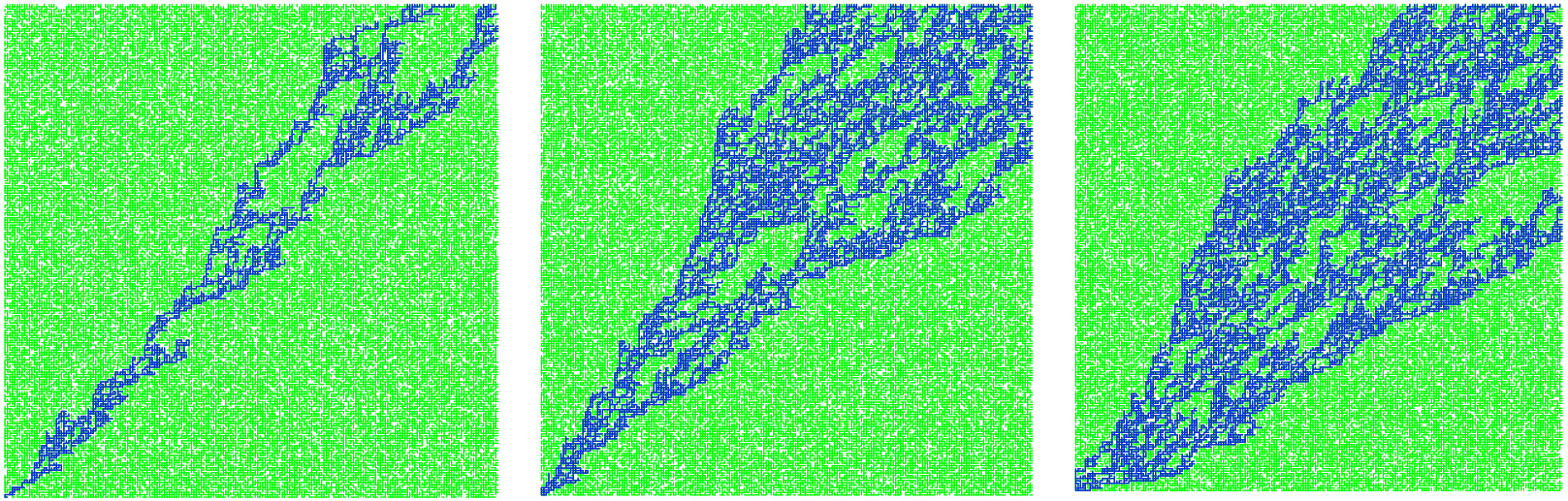
Oriented Percolation

Percolation Subgraphs: Independent, identically distributed Bernoulli-p random variables Y_e are attached to the edges e of the square lattice. Edges e for which $Y_e = 0$ are removed from the lattice; edges e for which $Y_e = 1$ are retained. The resulting random subgraph of the lattice is denoted by $\mathcal{G}_p$.

Oriented Percolation Clusters: The percolation cluster based at a vertex x consists of all vertices and edges that are connected to x by paths in $\mathcal{G}_p$ that make steps only *upward* or *to the right*.

Oriented Percolation is said to occur if the percolation cluster based at the origin is infinite.

Oriented Percolation: Simulations



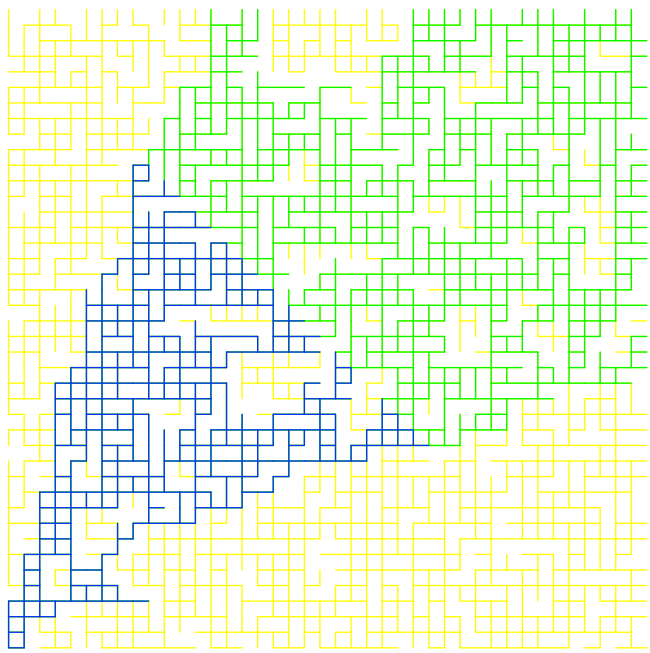
p=.65 p=.66 p=.67

Theorem 4 [5] There is a critical value p_c (known to be between .629 and .667) such that the event of oriented percolation has positive probability for $p > p_c$, but has probability zero for $p \leq p_c$. For $p > p_c$ the right and left edges of the percolation cluster are asymptotically linear, and the percolation cluster intersects the 45° line infinitely often.

[5] Durrett, 1984 Annals of Probability

First-Passage Percolation: Limit Shapes with Flat Spots

Example:[6] Consider the first-passage percolation process whose edge passage times are Bernoulli-p plus 1. If $p > p_c$, the boundary of the limit shape must have a flat spot.



Reason: For any point $x = (u, v)$ on the line $u + v = n$ that is in the oriented percolation cluster of the origin 0, the passage time $T(0, x)$ must satisfy

$$T(0, x) = n.$$

[6] Durrett & Liggett, 1981 Annals of Probability

First-Passage Percolation

Notation:

- ξ_e = traversal time of edge e.
- F = common distribution of the traversal times ξ_e.
- $\tau(\gamma) = \sum_{e\in\gamma} \xi_e$: traversal time of path γ.
- $T(x, y) = \min\ \{\tau(\gamma)\ :\ \text{paths } \gamma \text{ that connect points } x, y\}$.
- $\mu(u) = \lim_{n\to\infty} T(0, nu)/n$ = inverse infection speed in direction u.

Note: If F is continuous, then the time-minimizing path ("geodesic") connecting any two vertices is *unique*, and[7] the variance of $T(0, nu)$ diverges as $n \to \infty$.

[7] Newman & Piza, 1995 Annals of Probability; Pemantle & Peres in the exponential case.

FPP : Strict Convexity of the Limit Shape

Theorem 5 [8] Assume that

1. F has a bounded density on $(0, \infty)$ and finite MGF.
2. Long geodesic segments are (to first order) straight line segments.
3. For each direction u there exist a **nontrivial mean-zero** distribution G_u and a scalar sequence $a_n \to \infty$ such that

$$\frac{T(0, nu) - n\mu(u)}{a_n} \longrightarrow G_u \qquad \text{in distribution.}$$

Then the limit shape is strictly convex.

Note: It is suspected that the correct normalizing sequence is $a_n = n^{1/3}$. The limiting distributions G_u, if indeed they exist, may well not be Gaussian. It is now known[9] that for certain atomic distributions F, the variance of $T(0, nu)$ grows *sublinearly* with n.

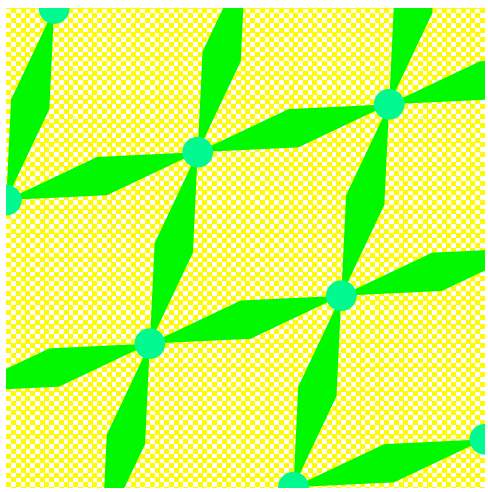
[8] Lalley, 2003 ECP

[9] Benjamini, Kalai, & Schramm, 2003 Annals of Probability

Proof of Theorem 5

It suffices to show that for any two linearly independent vectors (directions) u and v, the speed of propagation is greater in direction $(u+v)/2$ than the average of the propagation speeds in directions u and v. WLOG ,

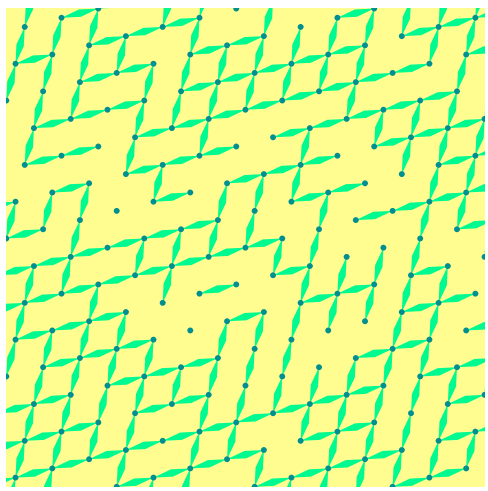
$$\mu(u) = \mu(v).$$



Consider a parallelogram array superimposed on the square lattice. If the parallelograms are large, the travel times between adjacent disks, suitably renormalized, have distributions close to G_u and G_v. Mark a parallelogram as a **Success** if the travel time through it is less than the 70th percentile of G_u (or G_v); otherwise, mark it as a **Failure**.

Proof of Theorem 5

The mean travel time through parallelograms marked **Success** is **strictly** less than $\mu(u)$ (because by Hypothesis 3, the limit distributions G_u and G_v are mean-zero).



Because the critical value p_c for oriented percolation is less than .7, there exist, with positive probability, oriented paths of **Success** parallelograms connecting the origin to points $nu + nv$ with arbitrarily large n. The travel time for such a path is (with probability near one) less than

$$n(\mu(u) + \mu(v) - \varepsilon).$$

First-Passage Percolation: Unsolved Problems

- When is the limit shape strictly convex?
- What is the rate of growth of **var**$(T(0, nu))$?
- To what does the distribution of $(T(0, nu) - n\mu(u))/a_n$ converge?
- Do the geodesic segments behave regularly:
  - Are they (to first order) straight?
  - How do they fluctuate about straight lines?
  - Is there a LLN for their (Euclidean) lengths?
  - Do their empirical distributions behave regularly?