Spatial Epidemics: Critical Behavior

Steve Lalley

March 2012
with the collaboration of

Regina Dolgoarshinnykh
Xinghua Zheng
Edwin Perkins
Yuan Shao
Reed-Frost (SIR) Model
  Reed-Frost & Erdös-Renyi
  Branching Envelopes
  Critical Behavior: Scaling Limits

Spatial Epidemic Models
  Spatial SIR Models
  Superprocess Limits
  Spatial Epidemic Models: Critical Scaling
  Branching Random Walk: Local Behavior
  Spatial Extent of SuperBM ($d = 1$)

Measure-Valued Spatial Epidemics

Hybrid Percolation
Reed-Frost (SIR) Model

- Population Size $N < \infty$
- Individuals susceptible (S), infected (I), or recovered (R).
- Recovered individuals immune from further infection.
- Infecteds recover in time 1.
- Infecteds infect susceptibles with probability $p$. 

Critical Case: $p = \frac{1}{N}$
Reed-Frost (SIR) Model

- Population Size $N < \infty$
- Individuals susceptible (S), infected (I), or recovered (R).
- Recovered individuals immune from further infection.
- Infecteds recover in time 1.
- Infecteds infect susceptibles with probability $p$.

Critical Case: $p = 1/N$
Reed-Frost and Erdös-Renyi Random Graphs

Reed-Frost model is equivalent to the Erdös-Renyi random graph model:

- Individuals $\leftrightarrow$ Vertices
- Infections $\leftrightarrow$ Edges
- Epidemic $\leftrightarrow$ Connected Components
Reed-Frost and Random Graphs
Reed-Frost and Random Graphs
Reed-Frost and Random Graphs
Reed-Frost and Random Graphs
Branching Envelope of an Epidemic

- Each epidemic has a branching envelope (GW process)
- Offspring distribution: Binomial- \((N, p) \approx \text{Poisson-1}\)
- Epidemic is dominated by its branching envelope
- When \(I_t \ll S_t\), infected set grows \(\approx\) branching envelope
Critical GW Processes

**Theorem:** (Feller; Jirina) Let $Y_n^N$ be a sequence of Galton-Watson processes, all with offspring distribution Poisson-1, such that $Y_0^N = N$. Then

$$Y_{Nt}/N \xrightarrow{D} Y_t$$

where $Y_t$ is the Feller diffusion started at $Y_0 = 1$:

$$dY_t = \sqrt{Y_t} dB_t.$$
Theorem: (Feller; Jirina) Let $Y_n^N$ be a sequence of Galton-Watson processes, all with offspring distribution Poisson-1, such that $Y_0^N = N$. Then

$$ Y_{Nt}^N / N \xrightarrow{D} Y_t $$

where $Y_t$ is the Feller diffusion started at $Y_0 = 1$:

$$ dY_t = \sqrt{Y_t} \, dB_t. $$

Corollary: (Kolmogorov) If $\tau_N$ is the lifetime of $Y^N$ then

$$ \tau_N / N \xrightarrow{D} \text{Exponential} - 1 $$
Critical Behavior: Reed-Frost Epidemics

- Population size: $N \to \infty$
- # Infected in Generation $n$: $I^N(n)$
- # Recovered in Generation $n$: $R^N(n) = \sum_{j=0}^{n-1} I^N(j)$
- Initial Condition: $I^N(0) \sim bN^\alpha$

**Theorem:** As population size $N \to \infty$, 

$$
\left( \frac{I^N(N^\alpha t)/N^\alpha}{R^N(N^\alpha t)/N^{2\alpha}} \right) \xrightarrow{D} \left( \frac{I(t)}{R(t)} \right)
$$

The limit process satisfies $I(0) = b$ and

$$
dR(t) = I(t) \, dt \\
dl(t) = +\sqrt{l(t)} \, dB_t \quad \text{if } \alpha < 1/3 \\
dl(t) = +\sqrt{l(t)} \, dB_t - l(t)R(t) \, dt \quad \text{if } \alpha = 1/3
$$

Dolgoarshinnykh & Lalley 2005
Critical Behavior: Reed-Frost Epidemics

- Population size: \( N \to \infty \)
- \# Infected in Generation \( n \): \( I^N(n) \)
- \# Recovered in Generation \( n \): \( R^N(n) = \sum_{j=0}^{n-1} I^N(j) \)
- Initial Condition: \( I^N(0) \sim bN^\alpha \)

**Corollary: (Martin-Lof)** If \( \alpha = 1/3 \) then as population size \( N \to \infty \),

\[
R^N(\infty)/N^{2/3} \to \tau(b)
\]

where \( \tau(b) = \) first passage time of \( B(t) + t^2/2 \) to \( b \).
Critical Behavior: Reed-Frost Epidemics

- Population size: $N \to \infty$
- # Infected in Generation $n$: $I^N(n)$
- # Recovered in Generation $n$: $R^N(n) = \sum_{j=0}^{n-1} I^N(j)$
- Initial Condition: $I^N(0) \sim bN^\alpha$

Corollary: (Martin-Lof) If $\alpha = 1/3$ then as population size $N \to \infty$,

$$R^N(\infty)/N^{2/3} \to \tau(b)$$

where $\tau(b) =$ first passage time of $B(t) + t^2/2$ to $b$.

Note: $R^N(\infty)$ is the number of vertices in the connected components of $bN^{1/3}$ randomly chosen vertices of the Erdös-Renyi graph. D. Aldous proved an equivalent result for the size of the maximal connected component.
Critical Behavior: Heuristics

- Critical Epidemic with $I_0 = m$ should last $\approx m$ generations.
- Number $R_t$ recovered should be $\approx m^2$.
- Offspring in branching envelope :: attempted infections.
- **Misfires**: Infections of immunes not allowed.
- **Critical Threshold**: # misfires/generations $\approx O(1)$

**Critical SIR Epidemic:**

$$E(\#\text{misfires in generation } t + 1) \approx I_t R_t / N$$

so there will be observable deviation from branching envelope when

$$I_t \approx N^{1/3} \quad \text{and} \quad R_t \approx N^{2/3}$$
Proof Strategy I

Lemma: Assume that $L_n$ and $L$ are likelihood ratios under $P_n$ and $P$, and define $Q_n$ and $Q$ by

\[ dQ_n = L_n \, dP_n, \]
\[ dQ = L \, dP. \]

Assume that $X_n$ and $X$ are random variables whose distributions under $P_n$ and $P$ satisfy

\[ (X_n, L_n) \Rightarrow (X, L). \]

Then the $Q_n$–distribution of $X_n$ converges weakly to the $Q$–distribution of $X$. 
Proof Strategy II

\[ P_N = \text{law of } Y^N \quad \text{(Galton-Watson)} \]
\[ Q_N = \text{law of } I^N \quad \text{(Reed-Frost)} \]
Proof Strategy II

\[ P_N = \text{law of } Y^N \quad \text{(Galton-Watson)} \]
\[ Q_N = \text{law of } I^N \quad \text{(Reed-Frost)} \]

\[
\frac{dQ_N}{dP_N} \approx \prod_{n=0}^{\infty} (1 - R_n/N)^{Y_{n+1}} \exp\{ Y_nR_n/N \} \\
\approx \exp \left\{ \sum_{n=0}^{\infty} (Y_{n+1} - Y_n)R_n/N - \frac{1}{2} \sum_{n=0}^{\infty} Y_{n+1}R_n^2/N^2 \right\} \\
\approx \text{Girsanov LR}
\]
Critical Behavior: Reed-Frost Epidemics

- Population size: $N \to \infty$
- # Infected in Generation $n := I^N(n)$
- # Recovered in Generation $n := R^N(n) = \sum_{j=0}^{n-1} I^N(j)$
- Initial Condition: $I^N(0) \sim b N^\alpha$

Theorem: As population size $N \to \infty$,

$$\left( \frac{I^N(N^\alpha t)/N^\alpha}{R^N(N^\alpha t)/N^{2\alpha}} \right) \xrightarrow{D} \left( \frac{I(t)}{R(t)} \right)$$

The limit process satisfies $I(0) = b$ and

$$dR(t) = I(t) \, dt$$
$$dl(t) = +\sqrt{l(t)} \, dB_t \quad \text{if } \alpha < 1/3$$
$$dl(t) = +\sqrt{l(t)} \, dB_t - l(t)R(t) \, dt \quad \text{if } \alpha = 1/3$$

Dolgoarshinnykh & Lalley 2005
Spatial SIR Epidemic:

- Villages $V_x$ at Sites $x \in \mathbb{Z}^d$
- Village Size: $N$
- Nearest Neighbor Disease Propagation
- SIR Rules **Locally**:
  - Infected individuals infect susceptibles at same or neighboring site with probability $p_N$
  - Infecteds recover in time 1.
  - Recovered individuals immune from further infection.
Spatial SIR Epidemic:

- Villages \( V_x \) at Sites \( x \in \mathbb{Z}^d \)
- Village Size: \( = N \)
- Nearest Neighbor Disease Propagation
- SIR Rules Locally:
  - Infected individuals infect susceptibles at same or neighboring site with probability \( p_N \)
  - Infecteds recover in time 1.
  - Recovered individuals immune from further infection.

**Critical Case:** Infection probability \( p_N = \frac{1}{(2d + 1)N} \).
Spatial SIR Epidemic:

- Villages $V_x$ at Sites $x \in \mathbb{Z}^d$
- Village Size: $N$
- Nearest Neighbor Disease Propagation
- SIR Rules Locally:
  - Infected individuals infect susceptibles at same or neighboring site with probability $p_N$
  - Infecteds recover in time 1.
  - Recovered individuals immune from further infection.

Critical Case: Infection probability $p_N = 1/((2d + 1)N)$.

Problem: What are the temporal and spatial extents of the epidemic under various initial conditions?
Percolation Representation

Spatial SIR epidemic is equivalent to critical bond percolation on the graph \( G_N := K_N \times \mathbb{Z}^d \) with nearest neighbor connections:

- Vertex set \([N] \times \mathbb{Z}^d\)
- Edges connect vertices \((i, x)\) and \((j, y)\) if \(\text{dist}(x, y) \leq 1\)
Percolation Representation

Spatial SIR epidemic is equivalent to critical bond percolation on the graph $G_N := K_N \times \mathbb{Z}^d$ with nearest neighbor connections:

- Vertex set $[N] \times \mathbb{Z}^d$
- Edges connect vertices $(i, x)$ and $(j, y)$ if $\text{dist}(x, y) \leq 1$

Problem: At critical point $p = 1/(2d + 1)N$,

- How does connectivity probability decay?
- How does size of largest connected cluster scale with $N$?
- Joint distribution of largest, 2nd largest, \ldots?
Critical Spatial SIS Epidemic: Simulation

Village Size: 20224
Initial State: 2048 infected at 0
Infection Probability: \( p = \frac{1}{20224} \)
Branching Envelope

The branching envelope of a spatial SIR epidemic is a branching random walk: In each generation,

- A particle at $x$ puts offspring at $x$ or neighbors $x + e$.
- #Offspring are independent Binomial $- (N, p_N)$ or Poisson-$Np_N$
The branching envelope of a spatial SIR epidemic is a branching random walk: In each generation,

- A particle at $x$ puts offspring at $x$ or neighbors $x + e$.
- #Offspring are independent Binomial $\text{Binomial}(N, p_N)$ or Poisson-$Np_N$.
- Critical BRW: $p = p_N = 1/(2d + 1)N$.

Associated Measure-Valued Processes

$$X_t^M = \text{measure that puts mass} \frac{1}{M} \text{ at } \frac{x}{\sqrt{M}} \text{ for each particle at site } x \text{ at time } t.$$
Watanabe’s Theorem

Let $X_t^M$ be the measure-valued process associated to a critical nearest neighbor branching random walk. If

$$X_0^M \implies X_0$$

then

$$X_{Mt}^M \implies X_t$$

where $X_t$ is the Dawson-Watanabe process (superBM). The DW process is a measure-valued diffusion.

Note 1: The total mass $\|X_t\|$ is a Feller diffusion.
Note 2: In 1D, $X_t$ has a continuous density $X(t, x)$. 
Let $X_{t}^{M, N}$ be the measure that puts mass $1/M$ at $x/\sqrt{M}$ for each infected individual at site $x$ at time $t$.

**Theorem:** Assume that the epidemic is critical and that $M = N^{\alpha}$. If $X_{0}^{M, N} \Rightarrow X_{0}$ then

$$X_{Mt}^{M, N} \Rightarrow X_{t}$$

where

- If $\alpha < 2/5$ then $X_{t}$ is the Dawson-Watanabe process.
- If $\alpha = 2/5$ then $X_{t}$ is the Dawson-Watanabe process with killing at rate

$$\int_{0}^{t} X(s, x) \, ds$$

Lalley 2008
Scaling Limits: SIR Spatial Epidemics in $d = 2, 3$

Recall: $X_t^{M,N}$ is the measure that puts mass $1/M$ at $x/\sqrt{M}$ for each infected individual at site $x$ at time $t$.

Theorem: Assume that the epidemic is critical and that $M = N^\alpha$. If $X_0^{M,N} \Rightarrow X_0$ and $X_0$ satisfies a smoothness condition then

$$X_{Mt}^{M,N} \Rightarrow X_t$$

where

- If $\alpha < 2/(6 - d)$ then $X_t$ is the Dawson-Watanabe process.
- If $\alpha = 2/(6 - d)$ then $X_t$ is the Dawson-Watanabe process with killing rate $L(t, x) = \text{Sugitani local time density}$.

Lalley-Zheng 2010
Theorem: Assume that $d = 2$ or $3$ and that the initial configuration $X_0 = \mu$ of the super-BM $X_t$ satisfies

Smoothness Condition:

$$\int_0^t \int_{x \in \mathbb{R}^d} \phi_t(x - y) \, d\mu(y)$$

is jointly continuous in $t, x$, where $\phi_t(x)$ is the heat kernel (Gaussian density). Then for each $t \geq 0$ the occupation measure

$$L_t := \int_0^t X_s \, ds$$

is absolutely continuous with jointly continuous density $L(t, x)$. 
Critical Scaling: Heuristics \((\text{SIR Epidemics, } d = 1)\)

- Duration: \(\approx M\) generations.
- # Infected Per Generation: \(\approx M\)
- # Infected Per Site: \(\approx \sqrt{M}\)
- # Recovered Per Site: \(\approx M\sqrt{M}\)
- # Misfires Per Site: \(\approx M^2/N\)
- # Misfires Per Generation: \(\approx M^{5/2}/N\)

So if \(M \approx N^{2/5}\) then # Misfires Per Generation \(\approx 1\).
Critical Scaling: Heuristics (SIR Epidemics, $d = 1$)

- Duration: $\approx M$ generations.
- # Infected Per Generation: $\approx M$
- # Infected Per Site: $\approx \sqrt{M}$
- # Recovered Per Site: $\approx M\sqrt{M}$
- # Misfires Per Site: $\approx M^2/N$
- # Misfires Per Generation: $\approx M^{5/2}/N$

So if $M \approx N^{2/5}$ then # Misfires Per Generation $\approx 1$.

But how do we know that the infected individuals in generation $n$ don’t “clump”? 
Critical Scaling: Heuristics (SIR Epidemics, $d = 3$)

- # Infected Per Generation $\approx$ Duration $\approx M$
- # Sites Reachable $\approx M^{3/2}$.
- # Infected Per Infected Site: $\approx O(1)$
- # Recovered Per Site: $\approx M^2 / M^{3/2} = \sqrt{M}$
- # Misfires Per Generation: $\approx M \times \sqrt{M} / N$

So if $M \approx N^{2/3}$ then # Misfires Per Generation $\approx 1$. 
Proof Strategy I

**Lemma:** Assume that $L_n$ and $L$ are likelihood ratios under $P_n$ and $P$, and define $Q_n$ and $Q$ by

\[ dQ_n = L_n dP_n, \]
\[ dQ = L dP. \]

Assume that $X_n$ and $X$ are random variables whose distributions under $P_n$ and $P$ satisfy

\[ (X_n, L_n) \Rightarrow (X, L). \]

Then the $Q_n$--distribution of $X_n$ converges weakly to the $Q$--distribution of $X$. 
Proof Strategy II

**Theorem:** (Dawson) The law $Q$ of the Dawson-Watanabe process with location-dependent killing rate $\theta(x, t)$ is mutually a.c. relative to the law $P$ of the Dawson-Watanabe process with no killing (superBM), and the likelihood ratio is

$$
\frac{dQ}{dP} = \exp \left\{ - \int \theta(t, x) \, dM(t, x) - \frac{1}{2} \int \langle X_t, \theta(t, \cdot)^2 \rangle \, dt \right\}
$$

where $M$ is the orthogonal martingale measure attached to the superBM $X_t$. 
Proof Strategy III

\( P^M = \text{Law of } M\text{th branching random walk.} \)

\( Q^{M,N} = \text{Law of corresponding spatial epidemic.} \)

\[
\frac{dQ^{M,N}}{dP^M} = \prod_{\text{times } t} \prod_{\text{sites } x} \left( 1 + R^{M,N}(t, x) \right)
\]

where \( R^{M,N}(t, x) \) is a function of the number of misfires at site \( x \) at time \( t \). So the problem is to show that under \( P^M \), as \( M \to \infty \),

\[
\sum_{t} \sum_{x} R^{M,N}(t, x)
\]

converges to the exponent in Dawson’s likelihood ratio.
Local Behavior for Branching Random Walk: \(d = 1\)

\[ Y^k_n(\cdot) = \text{branching random walk on } \mathbb{Z} \text{ with Poisson-1 offspring distribution and initial state } Y^k_0, \text{ with scaling as in Watanabe's theorem}. \]
Local Behavior for Branching Random Walk: $d = 1$

$Y_n^k(\cdot) =$ branching random walk on $\mathbb{Z}$ with Poisson-1 offspring distribution and initial state $Y_0^k$, with scaling as in Watanabe's theorem.

**Theorem:** If $Y_0^k(\lceil \sqrt{kx} \rceil) \rightarrow Y_0(x)$ where $Y_0(x)$ is continuous with compact support then

$$\frac{Y_{kt}^k(\lceil \sqrt{kx} \rceil)}{\sqrt{k}} \Rightarrow X(t, x)$$

where $X(t, x)$ is the Dawson-Watanabe density process.

Lalley 2008
Local Time for Branching Random Walk: $d = 2, 3$

$Y^k_n(\cdot) =$ branching random walk on $\mathbb{Z}^d$ with Poisson-1 offspring distribution and initial state $Y^k_0$ scaling as in Watanabe's theorem.

$U^k_n(\cdot) = \sum_{i=0}^{t} Y^k_i(\cdot)$

Lalley-Zheng 2010
Local Time for Branching Random Walk: \( d = 2, 3 \)

\[ Y_n^k(\cdot) = \text{branching random walk on } \mathbb{Z}^d \text{ with Poisson-1 offspring distribution and initial state } Y_0^k \text{ scaling as in Watanabe’s theorem.} \]

\[ U_n^k(\cdot) = \sum_{i=0}^{t} Y_i^k(\cdot) \]

**Theorem:** If \( Y_0^k([\sqrt{k}x]) \to Y_0(x) \) where \( Y_0 \) satisfies hypotheses of Sugitani then in \( d = 2, 3 \),

\[
\frac{U_{5t}^k([\sqrt{k}x])}{k^{2-d/2}} \implies L(t, x)
\]

where \( L(t, x) \) is Sugitani local time.

Lalley-Zheng 2010
Spatial Extent of Super-BM in $d = 1$

- $X_t =$ Dawson-Watanabe process,
- $\mathcal{R} := \bigcup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := -\log P(\mathcal{R} \subset D | X_0 = \delta_x)$
Spatial Extent of Super-BM in $d = 1$

- $X_t =$ Dawson-Watanabe process,
- $\mathcal{R} := \bigcup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := - \log P(\mathcal{R} \subset D \mid X_0 = \delta_x)$

**Theorem (Dynkin):** For any finite interval $D$, $u_D(x)$ is the maximal nonnegative solution in $D$ of the differential equation

$$u'' = u^2$$
Spatial Extent of Super-BM in $d = 1$

- $X_t$ = Dawson-Watanabe process,
- $\mathcal{R} := \bigcup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := - \log P(\mathcal{R} \subset D \mid X_0 = \delta_x)$

Solution: Weierstrass $\mathcal{P}$– Function

\[
u_D(x) = \mathcal{P}_L(x/\sqrt{6}) = \frac{1}{6x^2} + \sum_{\omega \in L^*} \left\{ \frac{1}{6(x - \omega)^2} - \frac{1}{6\omega^2} \right\}
\]

where the period lattice $L$ is generated by $Ce^{\pi i/3}$ for $C > 0$ depending on $D = [0, a]$ as follows:

\[C = \sqrt{6a}\]
Spatial Extent of Super-BM in $d = 1$

- $X_t =$ Dawson-Watanabe process,
- $\mathcal{R} := \bigcup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := - \log P(\mathcal{R} \subset D \mid X_0 = \delta_x)$

**Solution:** Weierstrass $\mathcal{P} -$ Function

$$u_D(x) = \mathcal{P}_L(x/\sqrt{6}) = \frac{1}{6x^2} + \sum_{\omega \in L^*} \left\{ \frac{1}{6(x - \omega)^2} - \frac{1}{6\omega^2} \right\}$$

where the period lattice $L$ is generated by $Ce^{\pi i/3}$ for $C > 0$ depending on $D = [0, a]$ as follows:

$C = \sqrt{6}a$

**Consequence:** For any finite Borel measure $\mu$ with support contained in the interior of $D$,

$$- \log P(\mathcal{R} \subset D \mid X_0 = \mu) = \int \mathcal{P}_L(x/\sqrt{6}) \mu(dx)$$
Measure-Valued Spatial Epidemics

The scaling limits of (near-)critical spatial epidemics at the critical threshold in dimensions $d = 1, 2, 3$ are Dawson-Watanabe processes with killing. These processes $X(t, x)$ are solutions of the SPDE

$$\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \theta X - LX + \sqrt{X} \, W'(t, x)$$

where

$W'(t, x) =$ \text{space-time white noise}
$L(t, x) =$ \text{local time density}
$\theta =$ \text{infection rate parameter.}$
Measure-Valued Spatial Epidemics

The scaling limits of (near-)critical spatial epidemics at the critical threshold in dimensions $d = 1, 2, 3$ are **Dawson-Watanabe processes with killing**. These processes $X(t, x)$ are solutions of the SPDE

$$
\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \theta X - LX + \sqrt{X} \ W'(t, x)
$$

where

$$
W'(t, x) = \text{space-time white noise}
$$

$$
L(t, x) = \text{local time density}
$$

$$
\theta = \text{infection rate parameter.}
$$

**Question 1**: Can the epidemic survive forever?

**Question 2**: Can the epidemic survive **locally**?
Survival Threshold

**Theorem 1:** In dimension $d = 1$ the epidemic dies out with probability 1 (that is, $X_t = 0$ eventually) for any value of $\theta$. In dimensions $d = 2, 3$ there exist critical points $0 < \theta_c(d) < \infty$ such that

- If $\theta < \theta_c(d)$ then $X$ dies out a.s.
- If $\theta > \theta_c(d)$ then $X$ survives w.p.p.

**Theorem 2:** For any infection rate $\theta$, in each dimension $d = 1, 2, 3$ the epidemic almost surely dies out **locally**, that is, for any compact set $K$

$$X_t(K) = 0 \text{ eventually.}$$

Lalley-Perkins-Zheng 2011
Critical Hybrid Percolation

- Vertex set $[N] \times \mathbb{Z}^d$
- Edges connect vertices $(i, x)$ and $(j, y)$ if $\text{dist}(x, y) \leq 1$
- Percolation: Remove edges with probability $(1 - p)$.
Critical Hybrid Percolation

- Vertex set $[N] \times \mathbb{Z}^d$
- Edges connect vertices $(i, x)$ and $(j, y)$ if $\text{dist}(x, y) \leq 1$
- Percolation: Remove edges with probability $(1 - p)$.

**Problem:** At critical point $p_d = 1/(2d + 1)N$,
- How does connectivity probability decay?
- How do sizes of largest connected clusters scale with $N$?
Critical Hybrid Percolation

- Vertex set $[N] \times \mathbb{Z}^d$
- Edges connect vertices $(i, x)$ and $(j, y)$ if $\text{dist}(x, y) \leq 1$
- Percolation: Remove edges with probability $(1 - p)$.

**Problem:** At critical point $p_d = 1/(2d + 1)N$, 
- How does connectivity probability decay?
- How do sizes of largest connected clusters scale with $N$?

**Conjecture (Theorem?):** (L.-Shao) Let $(C_1, C_2, \ldots)$ be the largest, 2nd largest, etc., cluster sizes and $(D_1, D_2, \ldots)$ be the corresponding cluster diameters. Then in $d = 1$, as $N \to \infty$,

\[
N^{-3/5}(C_1, C_2, \ldots) \xrightarrow{\mathcal{D}} \quad \text{and} \quad \quad N^{-1/5}(D_1, D_2, \ldots) \xrightarrow{\mathcal{D}}
\]