Spatial Epidemics: Critical Behavior

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with the collaboration of

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Reed-Frost (SIR) Model

- Population Size $N < \infty$
- Individuals susceptible (S), infected (I), or recovered (R).
- Recovered individuals immune from further infection.
- Infecteds recover in time 1.
- Infecteds infect susceptibles with probability p.

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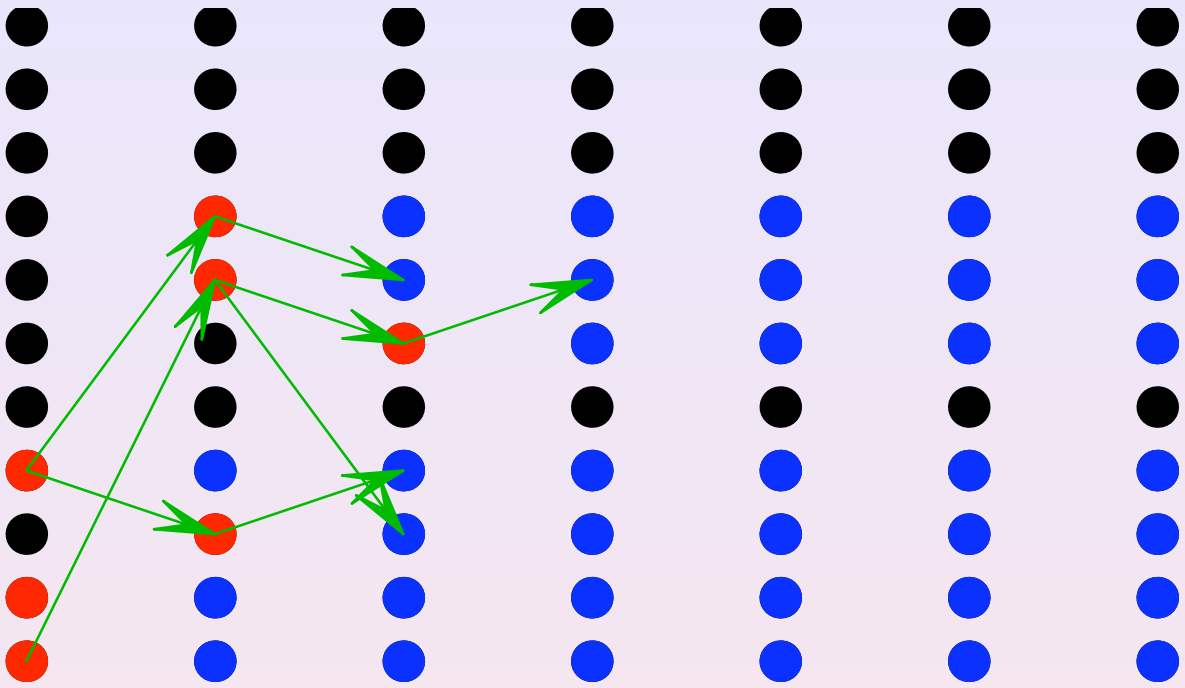
Critical Case: $p = 1/N$

SIR Model:Example

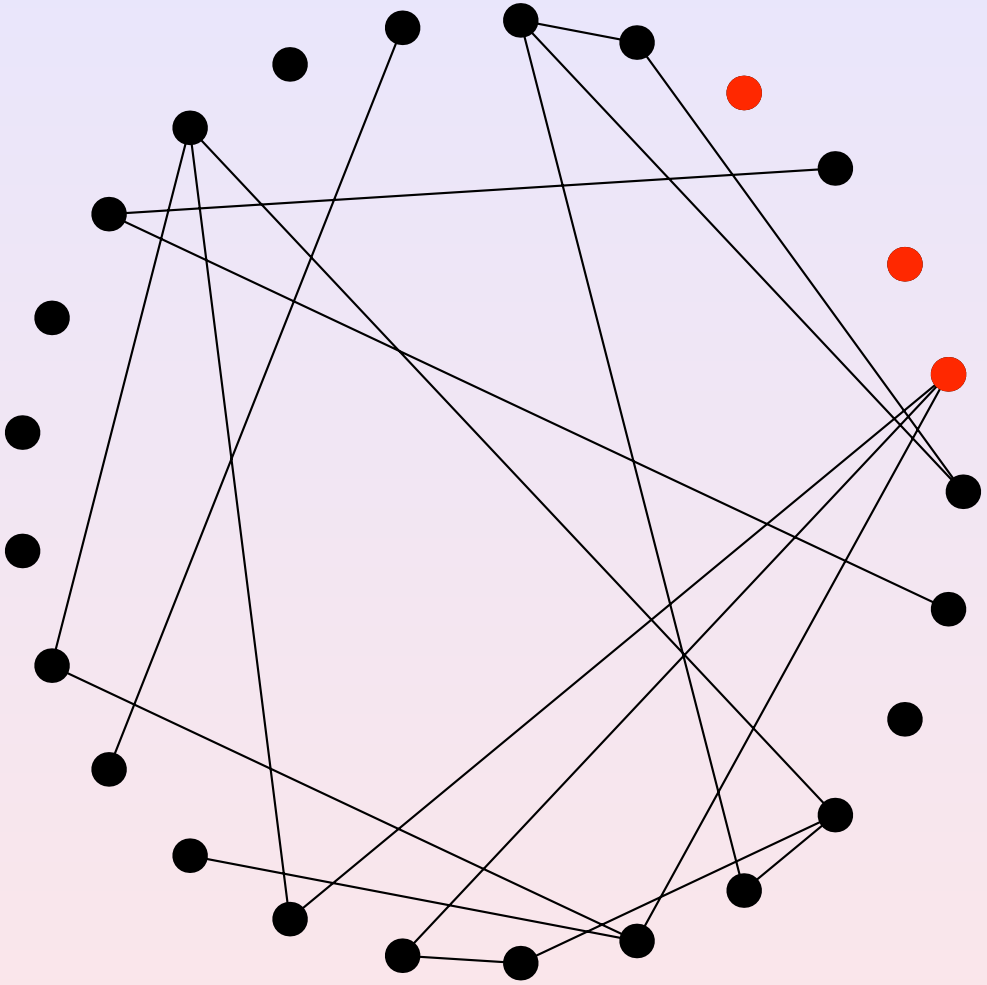


Reed-Frost and Erdös-Renyi Random Graphs

Reed-Frost model is equivalent to the Erdös-Renyi random graph model:

$$\begin{aligned} \text{Individuals} &\longleftrightarrow \text{Vertices} \\ \text{Infections} &\longleftrightarrow \text{Edges} \\ \text{Epidemic} &\longleftrightarrow \text{Connected Components} \end{aligned}$$

Reed-Frost and Random Graphs



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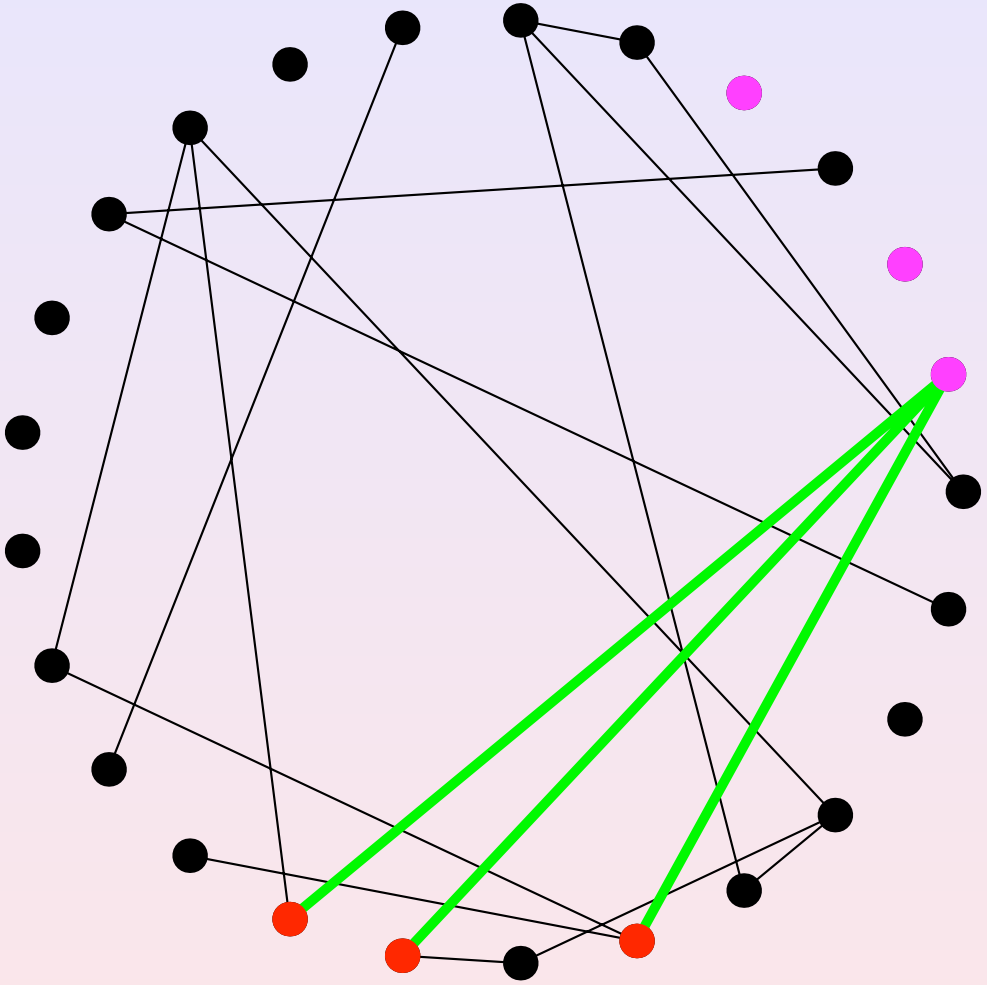

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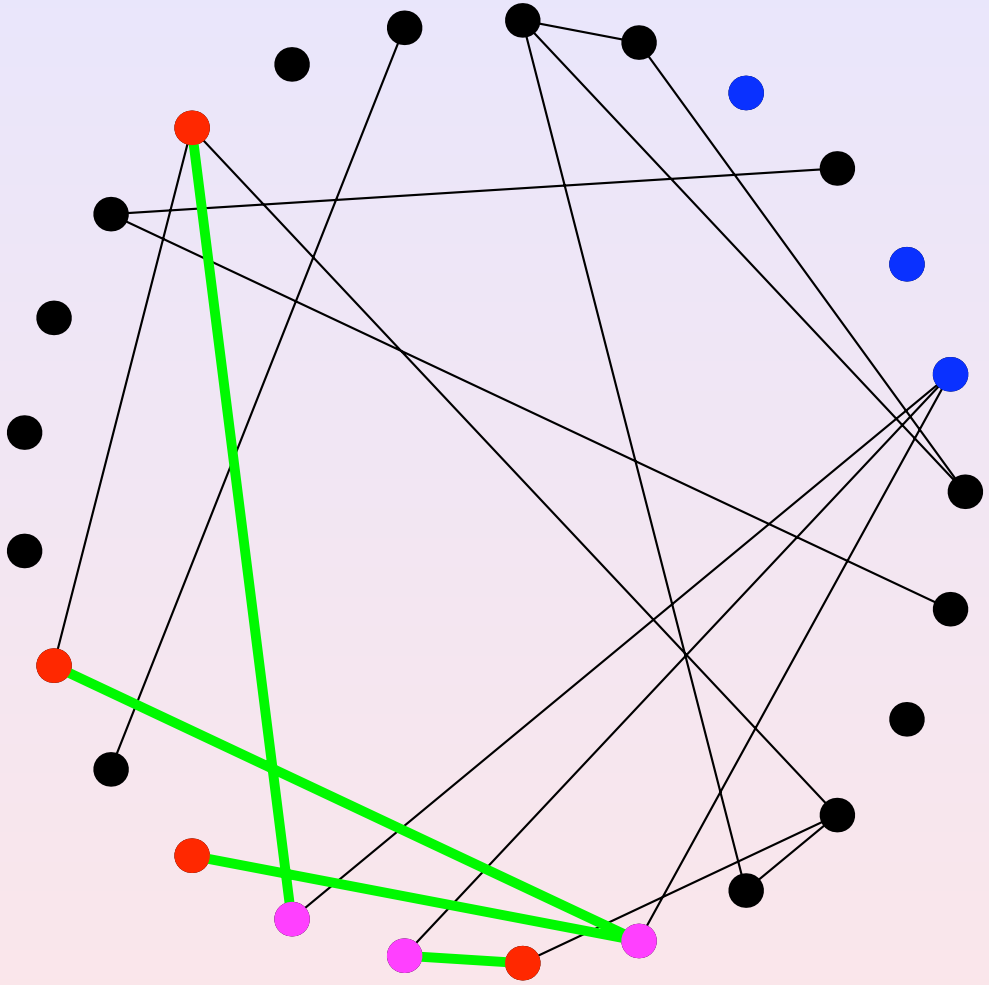

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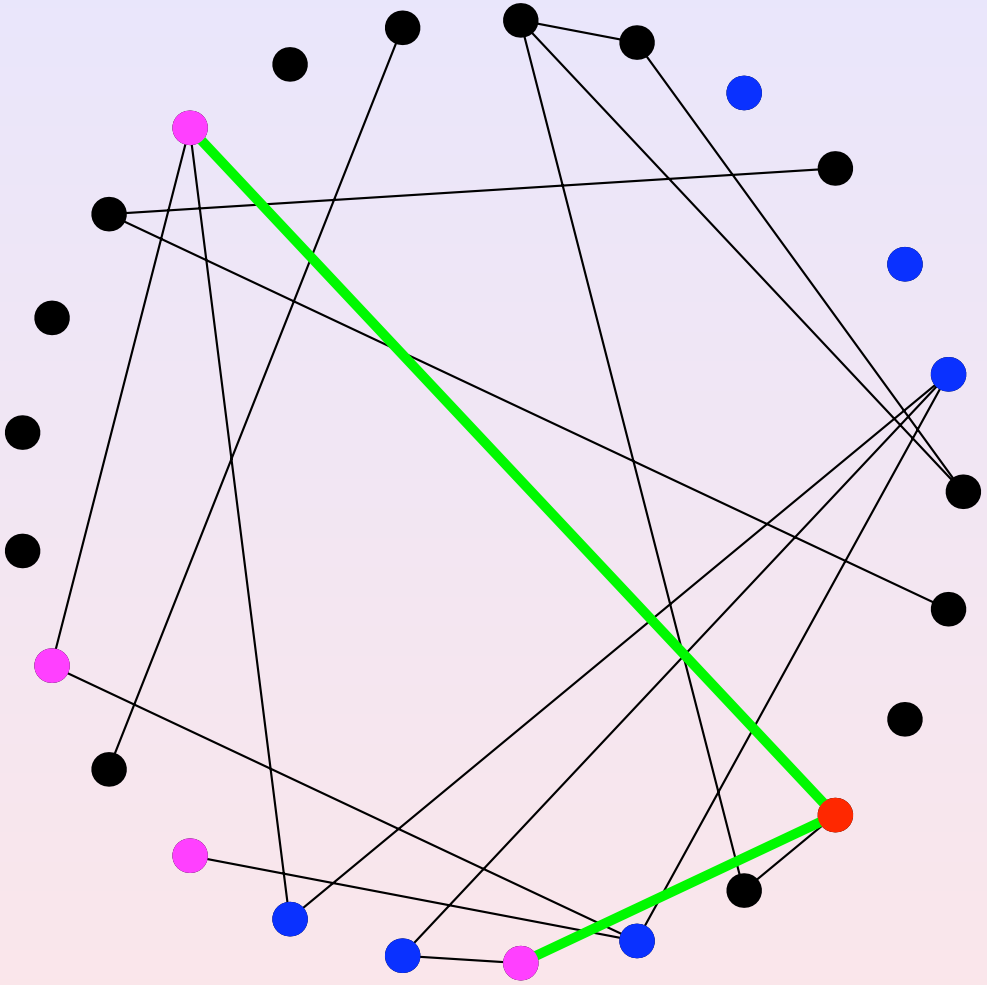

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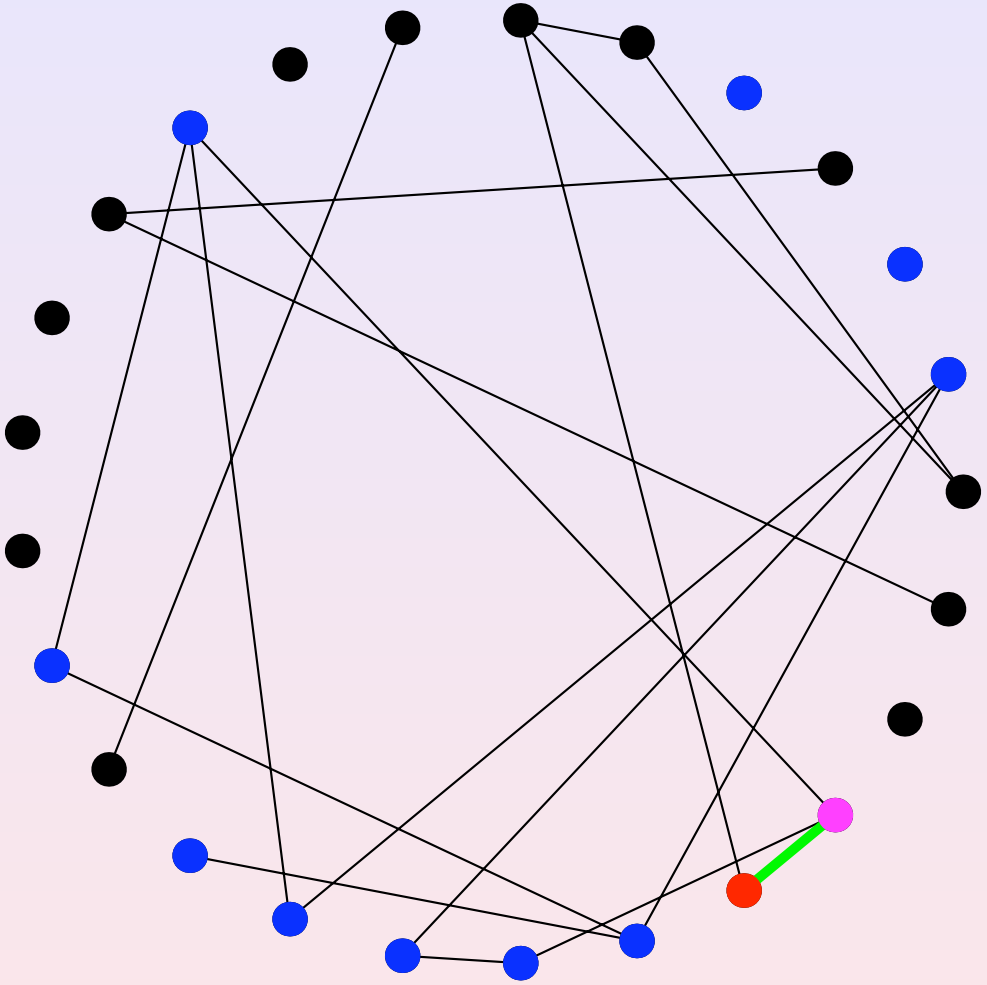

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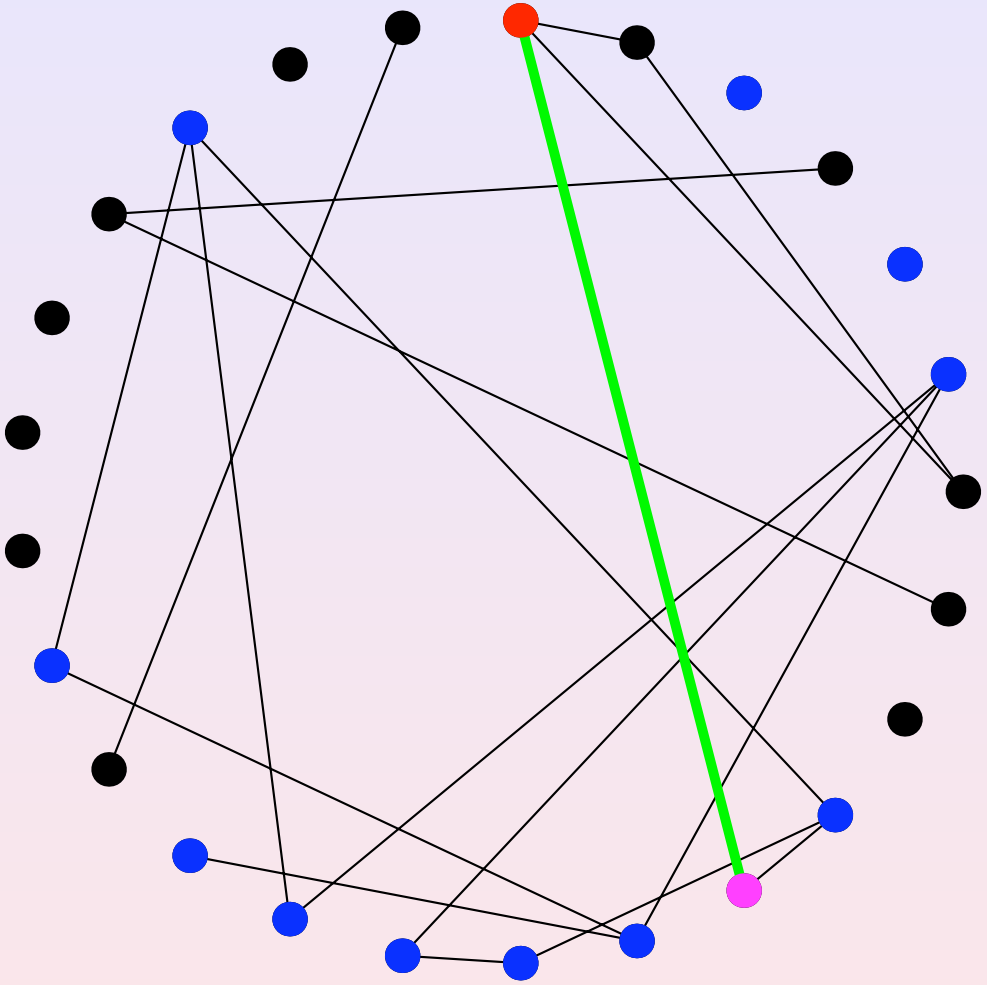

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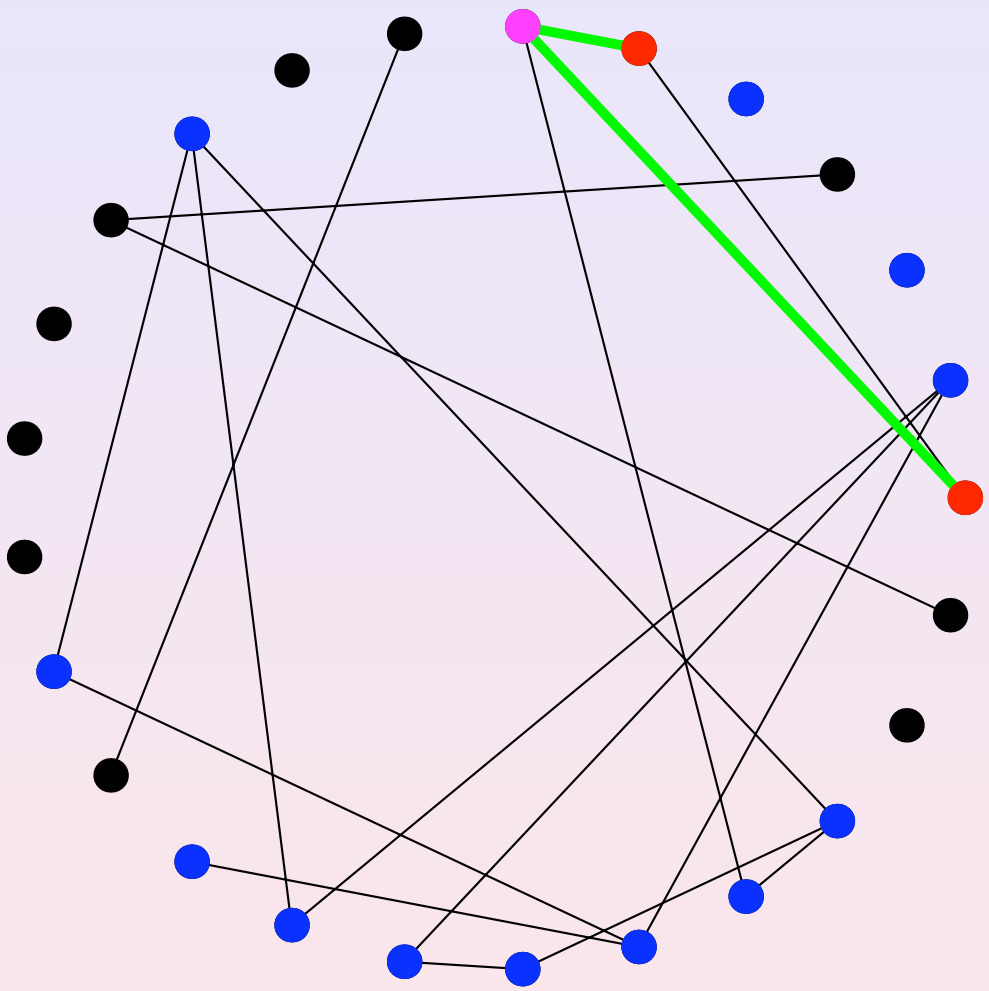

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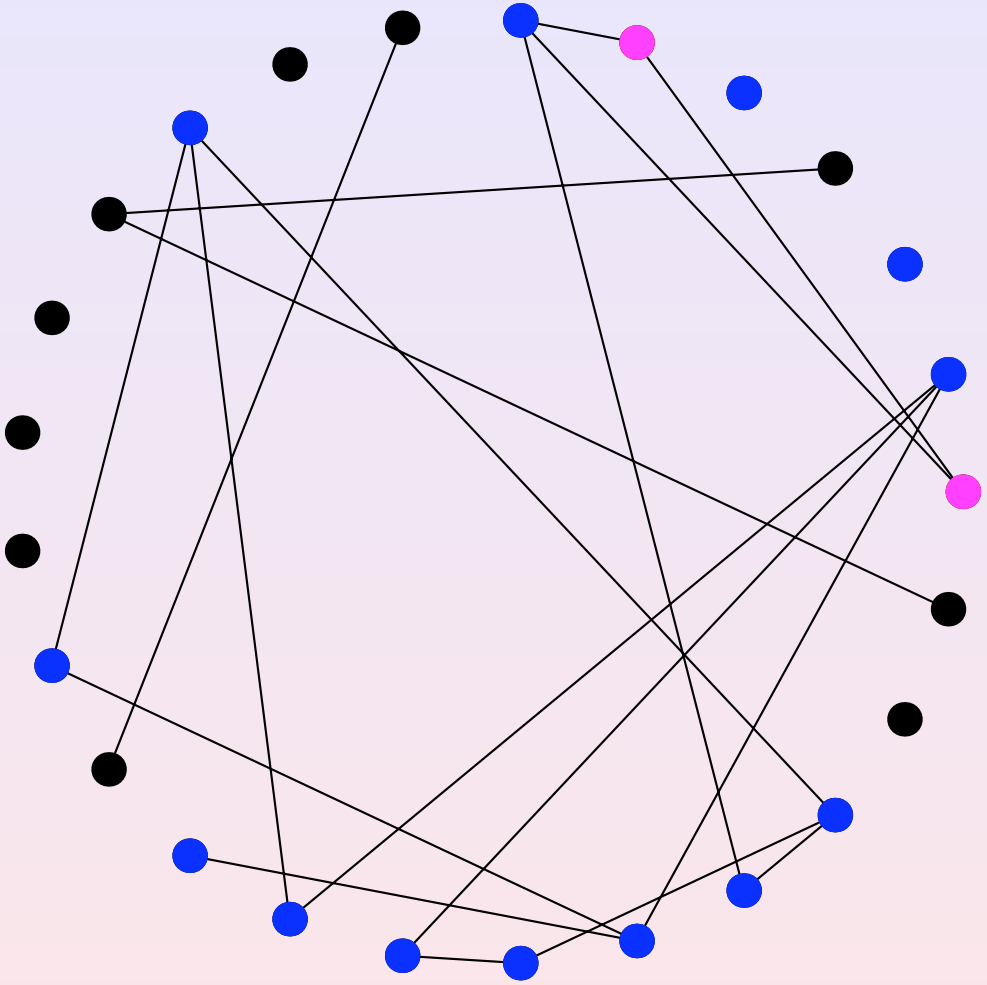
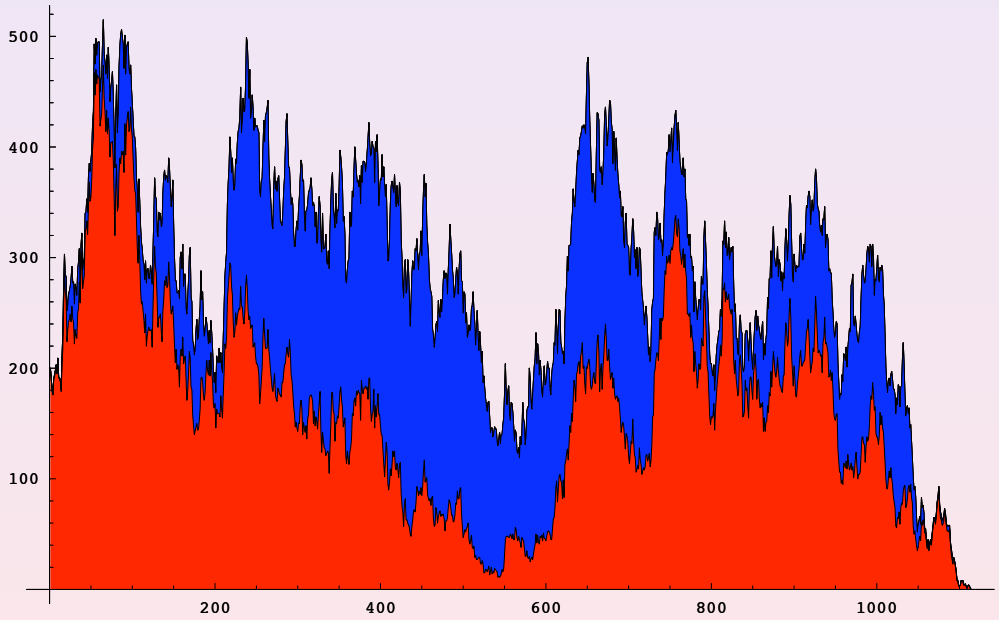


Branching Envelope of an Epidemic

- Each epidemic has a branching envelope (GW process)
- Offspring distribution: Binomial-$(N, p) \approx$ Poisson-1
- Epidemic is dominated by its branching envelope
- When $I_t \ll S_t$, infected set grows $\approx$ branching envelope



Critical GW Processes

Theorem:(Feller;Jirina) Let Y_n^N be a sequence of Galton-Watson processes, all with offspring distribution Poisson-1, such that $Y_0^N = N$. Then

$$Y_{Nt}^N / N \xrightarrow{\mathcal{D}} Y_t$$

where Y_t is the Feller diffusion started at $Y_0 = 1$:

$$dY_t = \sqrt{Y_t}\, dB_t.$$

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Corollary: (Kolmogorov) If τ_N is the lifetime of Y^N then

$$\tau_N / N \xrightarrow{\mathcal{D}} Exponential - 1$$

Critical Behavior: Reed-Frost Epidemics

- Population size: $N \to \infty$
- # Infected in Generation n:$= I^N(n)$
- # Recovered in Generation n: $= R^N(n) = \sum_{j=0}^{n-1} I^N(j)$
- Initial Condition: $I^N(0) \sim bN^\alpha$

Theorem: As population size $N \to \infty$,

$$\begin{pmatrix} I^N(N^\alpha t)/N^\alpha \\ R^N(N^\alpha t)/N^{2\alpha} \end{pmatrix} \xrightarrow{\mathcal{D}} \begin{pmatrix} I(t) \\ R(t) \end{pmatrix}$$

The limit process satisfies $I(0) = b$ and

$$\begin{aligned} dR(t) &= I(t)\,dt \\ dI(t) &= +\sqrt{I(t)}\,dB_t && \text{if } \alpha < 1/3 \\ dI(t) &= +\sqrt{I(t)}\,dB_t - I(t)R(t)\,dt && \text{if } \alpha = 1/3 \end{aligned}$$

Dolgoarshinnykh & Lalley 2005

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Corollary: (Martin-Lof) If $\alpha = 1/3$ then as population size $N \to \infty$,

$$R^N(\infty)/N^{2/3} \Longrightarrow \tau(b)$$

where $\tau(b) =$ first passage time of $B(t) + t^2/2$ to b.

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Note: $R^N(\infty)$ is the number of vertices in the connected components of $bN^{1/3}$ randomly chosen vertices of the Erdös-Renyi graph. D. Aldous proved an equivalent result for the size of the maximal connected component.

Critical Behavior: Heuristics

- Critical Epidemic with $I_0 = m$ should last $\approx m$ generations.
- Number R_t recovered should be $\approx m^2$.
- Offspring in branching envelope :: attempted infections.
- Misfires: Infections of immunes not allowed.
- Critical Threshold: # misfires/generations $\approx O(1)$

Critical SIR Epidemic:

$$E(\#\text{misfires in generation } t+1) \approx I_t R_t / N$$

so there will be observable deviation from branching envelope when

$$I_t \approx N^{1/3} \quad \text{and} \quad R_t \approx N^{2/3}$$

Proof Strategy I

Lemma: Assume that L_n and L are likelihood ratios under P_n and P, and define Q_n and Q by

$$dQ_n = L_n \, dP_n,$$
$$dQ = L \, dP.$$

Assume that X_n and X are random variables whose distributions under P_n and P satisfy

$$(X_n, L_n) \Longrightarrow (X, L).$$

Then the Q_n-distribution of X_n converges weakly to the $Q-$distribution of X.

Proof Strategy II

$$P_N = \text{law of } Y^N \quad \text{(Galton-Watson)}$$
$$Q_N = \text{law of } I^N \quad \text{(Reed-Frost)}$$

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$$Q_N = \text{law of } I^N \quad \text{(Reed-Frost)}$$

$$\begin{aligned}
\frac{dQ_N}{dP_N} &\approx \prod_{n=0}^{\infty} (1 - R_n/N)^{Y_{n+1}} \exp\{Y_n R_n/N\} \\
&\approx \exp\left\{ \sum_{n=0}^{\infty} (Y_{n+1} - Y_n) R_n/N - \frac{1}{2} \sum_{n=0}^{\infty} Y_{n+1} R_n^2/N^2 \right\} \\
&\approx \text{Girsanov LR}
\end{aligned}$$

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Dolgoarshinnykh & Lalley 2005

Spatial SIR Epidemic:

- Villages V_x at Sites $x \in \mathbb{Z}^d$
- Village Size:=N
- Nearest Neighbor Disease Propagation
- SIR Rules Locally:
  - Infected individuals infect susceptibles at same or neighboring site with probability p_N
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Problem: What are the temporal and spatial extents of the epidemic under various initial conditions?

Percolation Representation

Spatial SIR epidemic is equivalent to critical bond percolation on the graph $G_N := K_N \times \mathbb{Z}^d$ with nearest neighbor connections:

- Vertex set $[N] \times \mathbb{Z}^d$
- Edges connect vertices (i, x) and (j, y) if $\text{dist}(x, y) \leq 1$

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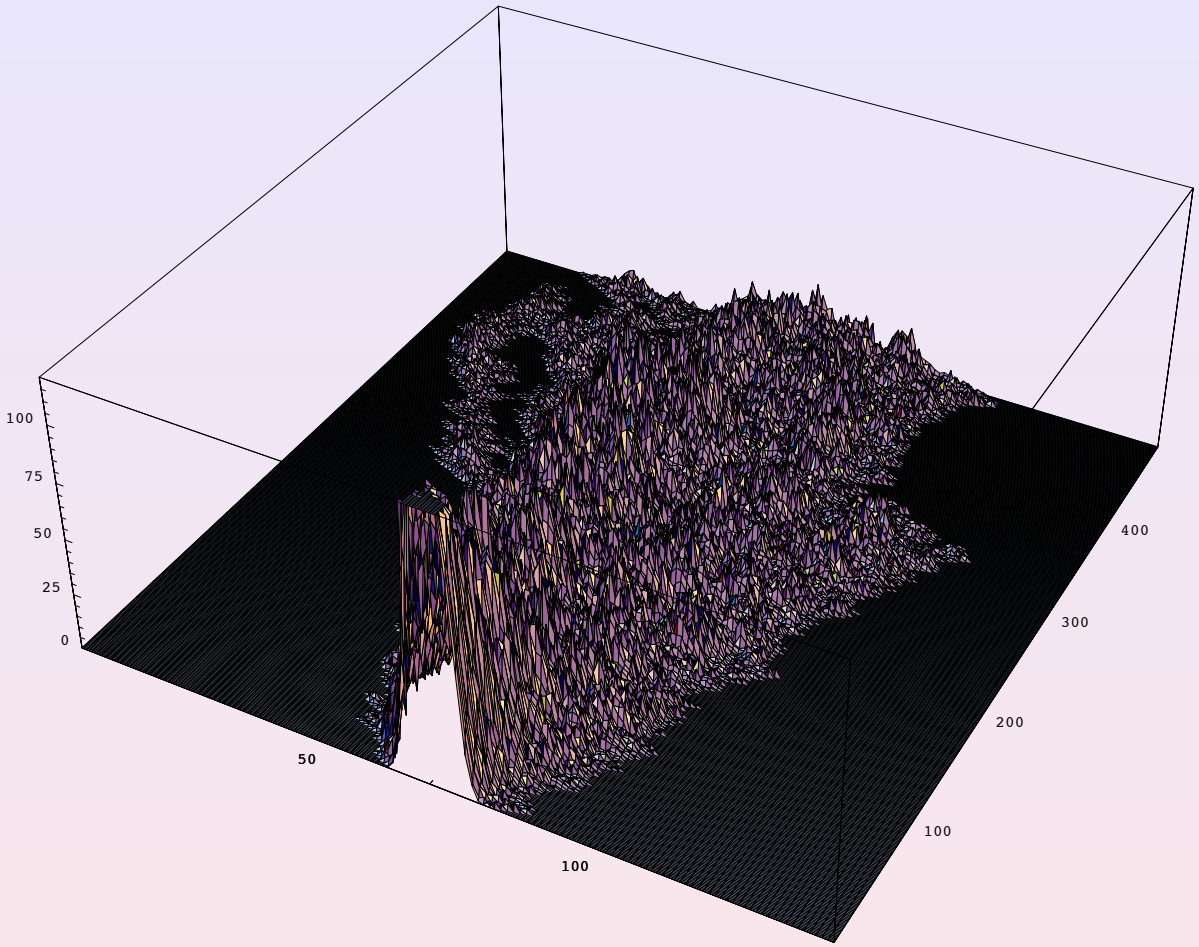
Problem: At critical point $p = 1/(2d + 1)N$,

- How does connectivity probability decay?
- How does size of largest connected cluster scale with N?
- Joint distribution of largest, 2nd largest, $\cdots$?

Critical Spatial SIS Epidemic: Simulation



Village Size: 20224
Initial State: 2048 infected at 0
Infection Probability: $p = 1/20224$

Branching Envelope

The branching envelope of a spatial SIR epidemic is a branching random walk: In each generation,

- A particle at x puts offspring at x or neighbors $x + e$.
- #Offspring are independent Binomial$-(N, p_N)$ or Poisson-Np_N
- Critical BRW : $p = p_N = 1/((2d + 1)N)$.

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Associated Measure-Valued Processes

$X_t^M =$ measure that puts mass $1/M$ at $x/\sqrt{M}$ for each particle at site x at time t.

Watanabe's Theorem

Let X_t^M be the measure-valued process associated to a critical nearest neighbor branching random walk. If

$$X_0^M \Longrightarrow X_0$$

then

$$X_{Mt}^M \Longrightarrow X_t$$

where X_t is the Dawson-Watanabe process (superBM). The DW process is a measure-valued diffusion.

Note 1: The total mass $\|X_t\|$ is a Feller diffusion.
Note 2: In $1D$, X_t has a continuous density $X(t, x)$.

Scaling Limits: SIR Spatial Epidemics in $d = 1$

Let $X_t^{M,N}$ be the measure that puts mass $1/M$ at $x/\sqrt{M}$ for each infected individual at site x at time t.

Theorem: Assume that the epidemic is critical and that $M = N^\alpha$. If $X_0^{M,N} \Rightarrow X_0$ then

$$X_{Mt}^{M,N} \Longrightarrow X_t$$

where

- If $\alpha < 2/5$ then X_t is the Dawson-Watanabe process.
- If $\alpha = 2/5$ then X_t is the Dawson-Watanabe process with killing at rate

$$\int_0^t X(s,x)\,ds$$

Lalley 2008

Scaling Limits: SIR Spatial Epidemics in $d = 2, 3$

Recall: $X_t^{M,N}$ is the measure that puts mass $1/M$ at $x/\sqrt{M}$ for each infected individual at site x at time t.

Theorem: Assume that the epidemic is critical and that $M = N^\alpha$. If $X_0^{M,N} \Rightarrow X_0$ and X_0 satisfies a smoothness condition then

$$X_{Mt}^{M,N} \Longrightarrow X_t$$

where

- If $\alpha < 2/(6-d)$ then X_t is the Dawson-Watanabe process.
- If $\alpha = 2/(6-d)$ then X_t is the Dawson-Watanabe process with killing rate $L(t,x) =$ Sugitani local time density.

Lalley-Zheng 2010

Sugitani's Local Time

Theorem: Assume that $d = 2$ or 3 and that the initial configuration $X_0 = \mu$ of the super-BM X_t satisfies

Smoothness Condition:

$$\int_0^t \int_{x \in \mathbb{R}^d} \phi_t(x - y)\, d\mu(y)$$

is jointly continuous in t, x, where $\phi_t(x)$ is the heat kernel (Gaussian density). Then for each $t \geq 0$ the occupation measure

$$L_t := \int_0^t X_s\, ds$$

is absolutely continuous with jointly continuous density $L(t, x)$.

Critical Scaling: Heuristics (SIR Epidemics, $d = 1$)

- Duration: $\approx M$ generations.
- # Infected Per Generation: $\approx M$
- # Infected Per Site: $\approx \sqrt{M}$
- # Recovered Per Site: $\approx M\sqrt{M}$
- # Misfires Per Site: $\approx M^2/N$
- # Misfires Per Generation: $\approx M^{5/2}/N$

So if $M \approx N^{2/5}$ then # Misfires Per Generation ≈ 1.

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So if $M \approx N^{2/5}$ then # Misfires Per Generation ≈ 1.

But how do we know that the infected individuals in generation n don't "clump"?

Critical Scaling: Heuristics (SIR Epidemics, $d = 3$)

- # Infected Per Generation $\approx$ Duration $\approx M$
- # Sites Reachable $\approx M^{3/2}$.
- # Infected Per Infected Site: $\approx O(1)$
- # Recovered Per Site: $\approx M^2/M^{3/2} = \sqrt{M}$
- # Misfires Per Generation: $\approx M \times \sqrt{M}/N$

So if $M \approx N^{2/3}$ then # Misfires Per Generation ≈ 1.

Proof Strategy I

Lemma: Assume that L_n and L are likelihood ratios under P_n and P, and define Q_n and Q by

$$dQ_n = L_n \, dP_n,$$
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Assume that X_n and X are random variables whose distributions under P_n and P satisfy

$$(X_n, L_n) \Longrightarrow (X, L).$$

Then the Q_n-distribution of X_n converges weakly to the $Q-$distribution of X.

Proof Strategy II

Theorem: (Dawson) The law Q of the Dawson-Watanabe process with location-dependent killing rate $\theta(x,t)$ is mutually a.c. relative to the law P of the Dawson-Watanabe process with no killing (superBM), and the likelihood ratio is

$$dQ/dP = \exp\left\{-\int \theta(t,x)\, dM(t,x) - \frac{1}{2}\int \langle X_t, \theta(t,\cdot)^2\rangle\, dt\right\}$$

where M is the orthogonal martingale measure attached to the superBM X_t.

Proof Strategy III

P^M = Law of Mth branching random walk.
$Q^{M,N}$ = Law of corresponding spatial epidemic.

$$\frac{dQ^{M,N}}{dP^M} = \prod_{\text{times}\, t} \prod_{\text{sites}\, x} \left(1 + R^{M,N}(t,x)\right)$$

where $R^{M,N}(t,x)$ is a function of the number of misfires at site x at time t. So the problem is to show that under P^M, as $M \to \infty$,

$$\sum_t \sum_x R^{M,N}(t,x)$$

converges to the exponent in Dawson's likelihood ratio.

Local Behavior for Branching Random Walk: $d = 1$

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Theorem: If $Y_0^k([\sqrt{k}x]) \to Y_0(x)$ where $Y_0(x)$ is continuous with compact support then

$$\frac{Y_{kt}^k([\sqrt{k}x])}{\sqrt{k}} \Longrightarrow X(t,x)$$

where $X(t,x)$ is the Dawson-Watanabe density process.

Lalley 2008

Local Time for Branching Random Walk: $d = 2, 3$

$Y_n^k(\cdot) =$ branching random walk on $\mathbb{Z}^d$ with Poisson-1 offspring distribution and initial state Y_0^k scaling as in Watanabe's theorem.

$U_n^k(\cdot) = \sum_{i=0}^{t} Y_i^k(\cdot)$

Lalley-Zheng 2010

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Theorem: If $Y_0^k([\sqrt{k}x]) \to Y_0(x)$ where Y_0 satisfies hypotheses of Sugitani then in $d = 2, 3$,

$$\frac{U_{kt}^k([\sqrt{k}x])}{k^{2-d/2}} \Longrightarrow L(t, x)$$

where $L(t, x)$ is Sugitani local time.

Lalley-Zheng 2010

Spatial Extent of Super-BM in $d = 1$

- $X_t =$ Dawson-Watanabe process,
- $\mathcal{R} := \bigcup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := -\log P(\mathcal{R} \subset D \,|\, X_0 = \delta_x)$

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Theorem (Dynkin): For any finite interval D, $u_D(x)$ is the maximal nonnegative solution in D of the differential equation

$$u'' = u^2$$

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Solution: Weierstrass $\mathcal{P}-$ Function

$$u_D(x) = \mathcal{P}_L(x/\sqrt{6}) = \frac{1}{6x^2} + \sum_{\omega \in L^*} \left\{ \frac{1}{6(x-\omega)^2} - \frac{1}{6\omega^2} \right\}$$

where the period lattice L is generated by $Ce^{\pi i/3}$ for $C > 0$ depending on $D = [0, a]$ as follows:

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where the period lattice L is generated by $Ce^{\pi i/3}$ for $C > 0$ depending on $D = [0, a]$ as follows:

$$C = \sqrt{6}a$$

Consequence: For any finite Borel measure μ with support contained in the interior of D,

$$-\log P(\mathcal{R} \subset D \,|\, X_0 = \mu) = \int \mathcal{P}_L(x/\sqrt{6})\, \mu(dx)$$

Measure-Valued Spatial Epidemics

The scaling limits of (near-)critical spatial epidemics at the critical threshold in dimensions $d = 1, 2, 3$ are Dawson-Watanabe processes with killing. These processes $X(t, x)$ are solutions of the SPDE

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \theta X - LX + \sqrt{X}\, W'(t, x)$$

where

$$\begin{aligned} W'(t, x) &= \text{ space-time white noise} \\ L(t, x) &= \text{ local time density} \\ \theta &= \text{ infection rate parameter.} \end{aligned}$$

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Question 1: Can the epidemic survive forever?
Question 2: Can the epidemic survive locally?

Survival Threshold

Theorem 1: In dimension $d = 1$ the epidemic dies out with probability 1 (that is, $X_t = 0$ eventually) for any value of θ. In dimensions $d = 2, 3$ there exist critical points $0 < \theta_c(d) < \infty$ such that

- If $\theta < \theta_c(d)$ then X dies out a.s.
- If $\theta > \theta_c(d)$ then X survives w.p.p.

Theorem 2: For any infection rate θ, in each dimension $d = 1, 2, 3$ the epidemic almost surely dies out locally, that is, for any compact set K

$$X_t(K) = 0 \quad \text{eventually.}$$

Lalley-Perkins-Zheng 2011

Critical Hybrid Percolation

- Vertex set $[N] \times \mathbb{Z}^d$
- Edges connect vertices (i, x) and (j, y) if $\text{dist}(x, y) \leq 1$
- Percolation: Remove edges with probability $(1 - p)$.

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- Percolation: Remove edges with probability $(1 - p)$.

Problem: At critical point $p_d = 1/(2d + 1)N$,

- How does connectivity probability decay?
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Critical Hybrid Percolation

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Problem: At critical point $p_d = 1/(2d + 1)N$,

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Conjecture (Theorem?): (L.-Shao) Let $(C_1, C_2, \dots)$ be the largest, 2nd largest, etc., cluster sizes and $(D_1, D_2, \dots)$ be the corresponding cluster diameters. Then in $d = 1$, as $N \to \infty$,

$$N^{-3/5}(C_1, C_2, \dots) \xrightarrow{\mathcal{D}} \quad \text{and}$$

$$N^{-1/5}(D_1, D_2, \dots) \xrightarrow{\mathcal{D}}$$