RENEWAL THEORY

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1. Renewal processes

A *renewal process* is the increasing sequence of random nonnegative numbers $S_0, S_1, S_2, \dots$ gotten by adding i.i.d. *positive* random variables $X_0, X_1, \dots$, that is,

$$S_n = S_0 + \sum_{i=1}^{n} X_i \tag{1}$$

When $S_0 = 0$ the renewal process is an *ordinary* renewal process; when S_0 is a nonnegative random variable the renewal process is a *delayed* renewal process. In either case, the individual terms S_n of this sequence are called *renewals*, or sometimes *occurrences*. With each renewal process is associated a *renewal counting process* $N(t)$ that tracks the total number of renewals (not including the initial occurrence) to date: the random variable $N(t)$ is defined by

$$N(t) = \max\{n : S_n \leq t\} = \tau(t) - 1 \quad \text{where} \tag{2}$$

$$\tau(a) = \min\{n \geq 1 : S_n > a\}. \tag{3}$$

Two cases arise in applications, the *arithmetic* case, in which the inter-occurrence times X_i are integer-valued, and the *non-arithmetic* case, in which the distribution of X_i is not supported by any arithmetic progression $h\mathbb{Z}$. The arithmetic case is of particular importance in the theory of discrete-time Markov chains, because the sequence of times at which the Markov chain returns to a particular state x is an arithmetic renewal process, as we will show. Since the theories in the arithmetic and non-arithmetic cases follow mostly parallel tracks, we shall limit our discussion to the arithmetic case.

2. The Feller-Erdös-Pollard Renewal Theorem

Assume that $\{S_n\}_{n\geq 0}$ is an ordinary, arithmetic renewal process with inter-occurrence times $X_i = S_i - S_{i-1}$ and inter-occurrence time distribution $f(k) = f_k = P\{X_i = k\}$. Define the *renewal measure*

$$u(k) = u_k = P\{S_n = k \text{ for some } n \geq 0\} = \sum_{n=0}^{\infty} P\{S_n = k\}. \tag{4}$$

Proposition 1. *The renewal measure u satisfies the* renewal equation

$$u_m = \delta_{0,m} + \sum_{k=1}^{m} f_k u_{m-k} \tag{5}$$

where $\delta_{0,m}$ is the Kronecker delta function (1 if $m = 0$ and 0 otherwise).

Proof. Exercise. (Condition on the first step X_1 of the random walk.) □

The cornerstone of renewal theory is the *Feller-Erdös-Pollard* theorem, which describes the asymptotic behavior of hitting probabilities in a renewal process.

Theorem 1. *(Feller-Erdös-Pollard) Let u be the renewal measure of an ordinary, arithmetic renewal process whose inter-occurrence time distribution $f_k = P\{X_i = k\}$ has finite mean $0 < \mu < \infty$ and is not supported by any proper additive subgroup of the integers (i.e., there is no $m \geq 2$ such that $P\{X_i \in m\mathbb{Z}\} = 1$). Then*

$$\lim_{k\to\infty} u(k) = 1/\mu. \tag{6}$$

Corollary 1. *If $\{S_n\}_{n\geq 0}$ is a* delayed *renewal process whose inter-occurrence time distribution $f_k = P\{X_1 = k\}$ satisfies the hypotheses of the Feller-Erdös-Pollard theorem, then*

$$\lim_{k\to\infty} P\{S_n = k \text{ for some } n \geq 0\} = 1/\mu. \tag{7}$$

Proof. Condition on the initial delay:

$$P\{S_n = k \text{ for some } n \geq 0\} = \sum_{m=0}^{\infty} P\{S_0 = m\} P\{S'_n = k - m \text{ for some } n \geq 0\}$$

where $S'_n = S_n - S_0 = \sum_{i=1}^{n} X_i$. The Feller-Erdös-Pollard theorem implies that for each m the hitting probability $P\{S'_n = k - m \text{ for some } n \geq 0\}$ converges to $1/\mu$ as $k \to \infty$, and so the dominated convergence theorem (applied to the infinite sum above) implies (7). □

The proof of the Feller-Erdös-Pollard theorem will rely on a generally useful technique known as *coupling*. The basic strategy is to construct on the same probability space two versions $\{S_n\}_{n\geq 0}$ and $\{\tilde{S}_n\}_{n\geq 0}$ of the random walk with different initial states in such a way that for some $m \geq 0$,

$$S_{n+m} = \tilde{S}_n \text{ eventually.} \tag{8}$$

It will then follow that for all sufficiently large $k \in \mathbb{Z}$ either both sequences visit k or neither will; consequently,

$$\lim_{k\to\infty} P\{S_n = k \text{ for some } n \geq 0\} - P\{\tilde{S}_n = k \text{ for some } n \geq 0\} = 0. \tag{9}$$

Proposition 2. *Under the hypotheses of the Feller-Erdös-Pollard theorem,*

$$\begin{aligned} &\lim_{k\to\infty} u(k) - u(k-1) = 0 \quad \textit{and so} \\ &\lim_{k\to\infty} u(k) - u(k-j) = 0 \quad \textit{for every } j \geq 1. \end{aligned} \tag{10}$$

Proof. Assume first that the distribution $\{f_k\}_{k\geq 1}$ of the inter-occurrence times X_i is not supported by any *coset* of a proper subgroup of the integers (i.e., there do not exist integers k and $m \geq 2$ such that $P\{X_i \in k + m\mathbb{Z}\} = 1$). Let $\{X_i\}_{i\geq 1}$ and $\{X'_i\}_{i\geq 1}$ be two independent sequences of identically distributed random variables, all with distribution $\{f_k\}_{k\geq 1}$. Since the inter-occurrence time distribution has finite mean, the differences $Y_i := X_i - X'_i$ have mean zero. Furthermore, since $\{f_k\}_{k\geq 1}$ of the inter-occurrence times X_i is not supported by any coset of a proper subgroup of the integers, the distribution of the differences Y_i is not supported by any proper subgroup of the integers, and so by the recurrence theorem for one-dimensional random walks the sequence

$$S_n^Y = \sum_{i=1}^n Y_i$$

will visit every integer infinitely often, with probability one. Let T be the smallest n such that $S_n^Y = 1$; then T is a stopping time for the sequence $\{(X_n, X_n')\}_{n\geq 1}$ (that is, for the minimal filtration generated by this sequence). Define

$$S_n = \sum_{i=1}^n X_i \quad \text{and}$$
$$S_n' = 1 + \sum_{i=1}^{n\wedge T} X_i' + \sum_{i=1+n\wedge T}^{n} X_i.$$

By construction, $S_n = S_n'$ for all $n \geq T$, Moreover, since T is a stopping time, the process $\{S_n' - 1\}_{n\geq 0}$ has the same joint distribution as does $\{S_n\}_{n\geq 0}$. Therefore, the limit relation (10) follows by the coupling principle (9).

If the distribution $\{f_k\}_{k\geq 1}$ is supported by $\ell + m\mathbb{Z}$ for some $m \geq 2$ and $1 \leq k < m$ then the differences Y_i take their values in the subgroup $m\mathbb{Z}$, so the recurrence theorem no longer implies that the random walk S_n^Y will visit every integer. However, S_n^Y will visit every point of $m\mathbb{Z}$, so the coupling argument above (with the obvious modifications) shows that

$$\lim_{k\to\infty} u(k) - u(k-m) = 0 \quad \Longrightarrow$$
$$\lim_{k\to\infty} u(k) - u(k-jm) = 0 \quad \forall\, j \in \mathbb{N}. \tag{11}$$

Recall that u satisfies the renewal equation $u(k) = Eu(k - K_1)$ for all integers $k \geq 1$ (compare equation (5)), so if the distribution $\{f_k\}_{k\geq 1}$ is supported by $\ell + m\mathbb{Z}$ for some $m \leq 2$ and $1 \leq \ell \leq m-1$ then (11) implies that

$$\lim_{k\to\infty} u(k) - u(k - jm - \ell) = 0 \quad \forall\, j \in \mathbb{N}, \quad \text{which further implies}$$
$$\lim_{k\to\infty} u(k) - u(k - jm - 2\ell) = 0 \quad \forall\, j \in \mathbb{N},$$
$$\lim_{k\to\infty} u(k) - u(k - jm - 3\ell) = 0 \quad \forall\, j \in \mathbb{N},$$
$$\text{etc.}$$

The sequence $\ell, 2\ell, 3\ell, \ldots$ must exhaust the integers mod m, because otherwise the distribution $\{f_k\}_{k\geq 1}$ would be supported by a proper subgroup of $\mathbb{Z}$, contrary to our standing assumptions. The proposition now follows. □

Proof of the Feller-Erdös-Pollard theorem. Proposition 2 implies that for large k the renewal measure $u(k)$ differs by only a negligible amount from $u(k-j)$ for any j. To deduce that $u(k)$ converges to $1/\mu$ we use the renewal equations $u_m = \delta_0(m) + \sum_{k=1}^m f_k u_{m-k}$. Summing over all m from 0 to n gives

$$\sum_{k=0}^n u_{n-k} = 1 + \sum_{k=0}^n u_{n-k} \sum_{j=1}^k f_j,$$

which can be re-written as

$$\sum_{k=0}^{n} u_{n-k}(1-F_k) = 1 \tag{12}$$

where $F_k = \sum_{j=1}^{k} f_j$.

The Feller-Erdös-Pollard theorem follows easily from equation (12) and Proposition 2. Since $\sum_{k=1}^{\infty}(1-F_k) = \mu$, the sequence $\{(1-F_k)/\mu\}_{k\geq 1}$ is a probability distribution on the positive integers. For any $\varepsilon > 0$ there exists $k(\varepsilon) < \infty$ such that this probability distribution puts at least $1-\varepsilon$ of its mass on the interval $[1, k(\varepsilon)]$. By Proposition 2, for sufficiently large m, say $m \geq m(\varepsilon)$, the function $u(m-j)$ will not differ by more than ε from $u(m)$, and in any case $u(j) \leq 1$ for all j. Consequently, for $m \geq m(\varepsilon)$,

$$|u_m \mu - 1| \leq \varepsilon + \varepsilon\mu.$$

Since $\varepsilon > 0$ is arbitrary, the theorem follows. □

3. The Renewal Equation and the Key Renewal Theorem

3.1. **The Renewal Equation.** The usefulness of the Feller-Erdös-Pollard theorem derives partly from its connection with another basic theorem called the Key Renewal Theorem (see below) which describes the asymptotic behavior of solutions to the *Renewal Equation*. The Renewal Equation is a convolution equation relating bounded sequences $\{z(m)\}_{m\geq 0}$ and $\{b(m)\}_{m\geq 0}$ of real numbers:

Renewal Equation, First Form:

$$z(m) = b(m) + \sum_{k=1}^{m} f(k) z(m-k). \tag{13}$$

Here $f(k) = f_k$ is the interoccurrence time distribution for the renewal process. There is an equivalent way of writing the Renewal Equation that is more suggestive of how it actually arises in practice. Set $z(m) = b(m) = 0$ for $m < 0$; then the upper limit $k = m-1$ in the sum in the Renewal Equation may be changed to $m = \infty$ without affecting its value. The Renewal Equation may now be written as follows, with X_1 representing the first interoccurrence time:

Renewal Equation, Second Form:

$$z(m) = b(m) + Ez(m - X_1). \tag{14}$$

Renewal equations crop up all over the place. In many circumstances, the sequence $z(m)$ is some scalar function of time whose behavior is of some interest; the renewal equation is gotten by conditioning on the value of the first interoccurrence time. In carrying out this conditioning, it is crucial to realize that the sequence $S_1^*, S_2^*, \dots$ defined by

$$S_n^* = S_n - X_1 = \sum_{j=2}^{n} X_j$$

is itself a renewal process, independent of X_1, and with the same interoccurrence time distribution $f(x)$.

Example 1. *(Age and Residual Lifetime)* Let $A_m = m - S_{\tau(m)-1}$ and $R_m = S_{\tau(m)} - m$; these random variables are the *age* and *residual lifetime* at time m. Fix $r \geq 1$, and set $z(m) = P\{A_m = r\}$. Then z satisfies the Renewal Equation (14) with

$$b(m) = P\{X_1 > m\} \tag{15}$$

EXERCISE: Derive this, and derive a similar renewal equation for the distribution of the residual lifetime.

3.2. **Solution of the Renewal Equation.** Consider the Renewal Equation in its second form $z(m) = b(m) + Ez(m - X_1)$ where by convention $z(m) = 0$ for all negative values of m. Since the function $z(\cdot)$ appears on the right side as well as on the left, it is possible to resubstitute on the right. This leads to a sequence of equivalent equations:

$$\begin{aligned} z(m) &= b(m) + Ez(m - X_1) \\ &= b(m) + Eb(m - X_1) + Ez(m - X_1 - X_2) \\ &= b(m) + Eb(m - X_1) + Eb(m - X_1 - X_2) + Ez(m - X_1 - X_2 - X_3) \end{aligned}$$

and so on. After m iterations, there is no further change (because $S_{m+1} > m$ and $z(l) = 0$ for all negative integers l), and the right side no longer involves z. Thus, it is possible to solve for z in terms of the sequences b and p:

$$\begin{aligned} z(m) &= \sum_{n=0}^{\infty} Eb(m - S_n) \\ &= \sum_{n=0}^{\infty} \sum_{x=0}^{\infty} b(m - x) P\{S_n = x\} \\ &= \sum_{x=0}^{\infty} \sum_{n=0}^{\infty} b(m - x) P\{S_n = x\} \\ &= \sum_{x=0}^{\infty} b(m - x) u(x). \end{aligned}$$

Note that only finitely many terms in the series are nonzero, so the interchange of summations is justified. Thus, the solution to the Renewal Equation is the convolution of the sequence $b(m)$ with the renewal measure:

$$\boxed{z(m) = \sum_{x=0}^{\infty} b(m - x) u(x)} \tag{16}$$

3.3. **The Key Renewal Theorem.** The formula (16) and the Feller-Erdös-Pollard theorem now combine to give the asymptotic behavior (as $m \to \infty$) of the solution z.

Theorem 2. *(Key Renewal Theorem) Let $z(m)$ be the solution to the Renewal Equation (14). If the sequence $b(m)$ is absolutely summable, then*

$$\lim_{m \to \infty} z(m) = \mu^{-1} \sum_{k=0}^{\infty} b(k). \tag{17}$$

Proof. The formula (16) may be rewritten as

$$z(m) = \sum_{k=0}^{\infty} b(k)u(m-k) \tag{18}$$

For each fixed k, the sequence $u(m-k) \to \mu^{-1}$ as $m \to \infty$, by the Feller-Erdös-Pollard theorem. Thus, as $m \to \infty$, the kth term of the series (18) converges to $b(k)/\mu$. Moreover, because $u(m-k) \le 1$, the kth term is bounded in absolute value by $|b(k)|$. By hypothesis, this sequence is summable, so the Dominated Convergence Theorem implies that the series converges as $m \to \infty$ to the right side of (17).

Example 2. *Residual Lifetime.* For each fixed $r \ge 1$, the sequence $z(m) = P\{R(m) = r\}$ satisfies the renewal equation

$$z(m) = P\{X_1 = m + r\} + \sum_{k=1}^{m} z(m-k)P\{X_1 = k\} = f_{m+r} + \sum_{k=1}^{m} z(m-k) f_k. \tag{19}$$

This reduces to (14) with $b(m) = f(m+r)$. The sequence $b(m)$ is summable, because $\mu = EX_1 < \infty$ (why?). Therefore, the Key Renewal Theorem implies that for each $r = 1, 2, 3, \dots,$

$$\lim_{m\to\infty} P\{R(m) = r\} = \mu^{-1} \sum_{k=0}^{\infty} f(k+r) = \mu^{-1} P\{X_1 \ge r\}. \tag{20}$$

This could also be deduced from the convergence theorem for Markov chains, using the fact that the sequence R_m is an irreducible, positive recurrent Markov chain with stationary distribution (**??**).

Example 3. *Total Lifetime.* Recall (Example 3 above) that the sequence $z(m) = P\{L(m) = r\}$ satisfies the Renewal Equation (14) with $b(m)$ defined by (**??**). Only finitely many terms of the sequence $b(m)$ are nonzero, and so the summability hypothesis of the Key Renewal Theorem is satisfied. Since $\sum_{k\ge 0} b(m) = r f(r)$, it follows from (17) that

Corollary 2.

$$\lim_{m\to\infty} P\{L(m) = r\} = r f(r)/\mu. \tag{21}$$

Example 4. *Fibonacci numbers.* Discrete convolution equations arise in many parts of probability and applied mathematics, but often with a "kernel" that isn't a proper probability distribution. It is important to realize that such equations can be converted to (standard) renewal equations by the device known as *exponential tilting*. Here is a simple example.

Consider the *Fibonacci sequence* 1,1,2,3,5,8,.... This is the sequence a_n defined by the recursion

$$a_{n+2} = a_{n+1} + a_n \tag{22}$$

and the initial conditions $a_0 = a_1 = 1$. To convert the recursion to a renewal equation, multiply a_n by a geometric sequence:

$$z_n = \theta^{-n} a_n$$

for some value $\theta > 0$. Also, set $z_n = 0$ for $n \le -1$. Then (22) is equivalent to the equation

$$z_n = z_{n-1}\theta^1 + z_{n-2}\theta^2 + \delta_{0,n} \tag{23}$$

where $\delta_{i,j}$ is the Kronecker delta function. (The delta function makes up for the fact that the original Fibonacci recursion (22) does not by itself specify a_0 and a_1.) Equation (23) is a renewal equation for any value of $\theta > 0$ such that $\theta^1 + \theta_2 = 1$, because then we can set $f_1 = \theta^1$, $f_2 = \theta^2$, and $f_m = 0$ for $m \geq 3$. Is there such a value of θ? Yes: it's the *golden ratio*

$$\theta = \frac{-1 + \sqrt{5}}{2}.$$

(The other root is negative, so we can't use it to produce a renewal equation from (22).) The Key Renewal Theorem implies that

$$\lim_{n \to \infty} z_n = \lim_{n \to \infty} \theta^{-n} a_n = 1/(\theta + 2\theta^2).$$

Thus, the Fibonacci sequence grows at an exponential rate, and the rate is the inverse of the golden ratio.