LAPLACE'S METHOD, FOURIER ANALYSIS, AND RANDOM WALKS ON $\mathbb{Z}^d$

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1. LAPLACE'S METHOD OF ASYMPTOTIC EXPANSION

1.1. **Stirling's formula.** Laplace's approach to Stirling's formula is noteworthy first, because it makes a direct connection with the Gaussian (normal) distribution (whereas in other approaches the Gaussian distribution enters indirectly, or not at all), and second, because it provides a general strategy for the asymptotic approximation of a large class of integrals with a large parameter. Stirling's formula states that as $n \to \infty$,

$$n! \sim \sqrt{2\pi n} n^n e^{-n}. \tag{1}$$

The symbol $\sim$ means that the ratio of the two sides approaches 1 as $n \to \infty$, equivalently, that the relative error in the approximation goes to 0. Laplace's starting point is the gamma function representation

$$n! = \int_0^\infty x^n e^{-x}\, dx, \tag{2}$$

which can be verified by induction, using an integration by parts to reduce the power x^n to x^{n-1}. The integrand achieves its max at $x = n$ (as you should check), and the value there is $n^n e^{-n}$. This already accounts for the largest factors in the Stirling approximation. Factoring this out gives

$$n! = n^n e^{-n} \int_0^\infty \left(\frac{x}{n}\right)^n e^{-(x-n)}\, dx.$$

Now the substitution $y = x/n$ will have the effect of moving the location of the maximum of the integrand from $x = n$ to $y = 1$, and leaves us with the formula

$$n! = n^n e^{-n} n \int_0^\infty y^n e^{-n(y-1)}\, dy. \tag{3}$$

Thus, to complete the proof of Stirling's formula it is enough to prove that the last integral is well-approximated by $\sqrt{2\pi}/\sqrt{n}$ for large n.

1.2. **Spikes and the bell curve.** At this point you are well-advised to fire up your favorite graphing calculator and plot the integrand in (3) for several values of n, perhaps $n = 4, 400, 40000$ and see what happens. You will see that as n gets larger the integrand *spikes* around the point $y = 1$. Now change the scaling of the $y-$axis, so that the spike is opened up, and you will see what is unmistakably the bell curve. This is the whole point of Laplace's method: for large n nearly all of the integral is accounted for by the spike near 1, and this spike looks more and more (after rescaling the argument) like the normal density, which we know how to integrate.

This is a phenomenon that holds much more generally. Let (a, b) be an open interval and let $g : (a, b) \to \mathbb{R}$ be any twice continuously differentiable function that satisfies the following conditions: for some $x_* \in (a, b)$,

$$\begin{aligned} g(x_*) &= 0; \\ -g''(x_*) &= 1/\sigma^2 > 0; \quad \text{and} \\ g(x) &< 0 \quad \text{for all } \; x \neq x_*. \end{aligned} \tag{4}$$

We will be interested in the large-n behavior of the integral

$$J(n) := \int_a^b e^{ng(x)}\,dx \tag{5}$$

(Note that the integral in (3) has this form, with $g(x) = \log x - x + 1$.) For any $n \geq 1$ the integrand $g(x)$ attains its maximum *uniquely* at $x = x_*$, and at this point the value is 1. Raising the integrand to the nth power has the effect of sending everything rapidly to 0 *except* in the close vicinity of $x = x_*$; that is, the integrand once again *spikes*. The second assumption in (4) guarantees that the spike is approximately a normal curve, because the other two assumptions imply that the first two terms in the Taylor series for g around $x = x_*$ vanish, leaving the quadratic second term, which by the middle assumption is nondegenerate. Thus, in principle, Laplace's method should apply to any integral of the form (5) provided the function g satisfies the restrictions (4).

Unfortunately there are a few things that might still go wrong. Assumptions (4) ensure that $e^{ng(x)}$ will approach 0 everywhere except at $x = x_*$, but they don't guaranteee that it will do so *uniformly*, nor will that the total area in the tails of the spikes are finite. The following proposition gives relatively simple *sufficient* conditions that keep bad things from happening.

Proposition 1. *Assume in addition to conditions* (4) *that* $J_1 < \infty$*, and that for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $g(x) < -\varepsilon$ *for all* x *such that* $|x - x_*| > \delta$*. Then as* $n \to \infty$*,*

$$J_n \sim \sqrt{\frac{2\pi}{n\sigma}}. \tag{6}$$

Proof. It suffices to prove that for any $\varepsilon > 0$ the value J_n will eventually be larger than $(1 - \varepsilon)\times$ the right side and smaller than $(1 + \varepsilon)\times$ the right side. Choose $\delta > 0$ such that $g(x) < -\varepsilon$ for all $|x - x_*| > \delta$ and such that

$$-(1+\varepsilon)^2 \frac{(x - x_*)^2}{2\sigma^2} < g(x) < -(1-\varepsilon)^2 \frac{(x - x_*)^2}{2\sigma^2} \tag{7}$$

for all $|x - x_*| \leq \delta$. (That this is possible follows from Taylor's theorem and the assumptions (4).) Break the integral J_n into two pieces: the interval $[x_* - \delta, x_* + \delta]$ containing the spike, and the complementary interval(s). Consider the complementary intervals first:

$$\int_{(a,b)-[x_*-\delta,x_*+\delta]} e^{ng(x)}\,dx \leq \int_{(a,b)} e^{g(x)} e^{-(n-1)\varepsilon}\,dx \leq J_1 e^{-(n-1)\varepsilon}.$$

By assumption $J_1 < \infty$, so this piece of the integral is bounded by a constant times $e^{-n\varepsilon}$. Exponentials go to 0 much more rapidly than any polynomial, so it will suffice to show that the integral over the spike interval is of polynomial size. (Exercise: Fill in the details.)

Now let's consider the spike. In the interval $|x - x_*| \leq \delta$ the inequalities (7) hold, so in this region the integrand $e^{ng(x)}$ is trapped between two bell curves, one with variance $\sigma^2/n(1-\varepsilon)$, the other with variance $\sigma^2/n(1+\varepsilon)$. Doing the change of variable $t = \sqrt{n}(x - x_*)$, we find that the integral

$$\int_{|x-x_*|\leq\delta} e^{ng(x)}\,dx$$

is trapped between

$$\int_{-\sqrt{n}\delta}^{\sqrt{n}\delta} e^{-t^2(1+\varepsilon)^2/2\sigma^2}\,dt/\sqrt{n} \sim \frac{\sqrt{2\pi}}{\sqrt{n}\sigma}(1+\varepsilon)$$

and

$$\int_{-\sqrt{n}\delta}^{\sqrt{n}\delta} e^{-t^2(1-\varepsilon)^2/2\sigma^2}\,dt/\sqrt{n} \sim \frac{\sqrt{2\pi}}{\sqrt{n}\sigma}(1-\varepsilon).$$

□

2. Fourier series

2.1. **Absolutely convergent Fourier series.** Let $(a_m)_{m\in\mathbb{Z}}$ be an absolutely summable sequence of complex numbers. (Absolutely summable means that $\sum|a_m| < \infty$.) The *Fourier series* of the sequence $(a_m)_{m\in\mathbb{Z}}$ is the series

$$A(\theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta}, \quad \text{where } \theta \in \mathbb{R}. \tag{8}$$

This series converges uniformly for $\theta \in \mathbb{R}$, and the limit $A(\theta)$ is a uniformly continuous, $2\pi-$periodic function of θ. It is natural in many contexts to view $A(\theta)$ as a (continuous) function on the unit circle

$$\mathbb{T} = \mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\}, \tag{9}$$

by making the identification $\theta \leftrightarrow e^{i\theta}$. This point of view is useful for various reasons, most important of which are that (a) $\mathbb{T}$ is an abelian *group* under multiplication, and (b) functions on $\mathbb{T}$ extend, in certain circumstances, to analytic functions on domains containing the unit circle.

Proposition 2. *If $\{a_n\}_{n\in\mathbb{Z}}$ is absolutely summable then $A(\theta)$ is uniformly continuous and $2\pi-$periodic. Furthermore, if in addition $\sum|ka_k| < \infty$ then $A(\theta)$ is continuously differentiable, and*

$$A'(\theta) = \sum_{k=-\infty}^{\infty} ika_k e^{ik\theta}. \tag{10}$$

Remark 1. It follows from (10) by an easy induction argument that if $\sum|k|^r|a_k| < \infty$ for some integer $r \geq 1$ then $A(\theta)$ is r times continuously differentiable, with derivatives

$$A^s(\theta) = \sum_{k=-\infty}^{\infty} (ik)^s a_k e^{ik\theta} \quad \text{for } s \leq r. \tag{11}$$

Proof. The absolutely summability of the coefficients a_n implies that for $\theta \in [-\pi, \pi]$ the function $A(\theta)$ is the uniform limit of the finite trig sums

$$\sum_{n=-m}^{m} a_n e^{in\theta}.$$

Since finite trig sums are uniformly continuous on $[-\pi, \pi]$, it follows that the limit $A(\theta)$ is also uniformly continuous.

If $\sum |ka_k| < \infty$ then the right side of (10) is the Fourier series of an absolutely summable sequence, and therefore defines a uniformly continuous function on $[-\pi, \pi]$. Integration of this function can be done term by term (by Fubini or dominated convergence theorem). This implies the equality (10).

□

The usefulness of Fourier series stems from the fact that *the Fourier series of a convolution is the product of the Fourier series*. Here is a more precise statement. For any two absolutely summable sequences $(a_m)_{m\in\mathbb{Z}}$ and $(b_m)_{m\in\mathbb{Z}}$ define their convolution

$$c_m = (a*b)_m = \sum_{k=-\infty}^{\infty} a_k b_{m-k} = \sum_{k=-\infty}^{\infty} a_{m-k} b_k. \tag{12}$$

Observe (in other words, you supply the proof!) that the sequence $a * b$ is absolutely summable, and that

$$\sum_{m=-\infty}^{\infty} |c_m| \leq \sum_{m=-\infty}^{\infty} |a_m| \sum_{m=-\infty}^{\infty} |b_m| \tag{13}$$

Proposition 3. *If $C(\theta) = \sum c_m e^{im\theta}$ is the Fourier series of the convolution $c_m = (a*b)_m$ then*

$$C(\theta) = A(\theta)B(\theta). \tag{14}$$

Proof. Exercise. (Note that changing the order of summation is justified by the dominated convergence theorem and/or Fubini's theorem, because both $(a_m)_{m\in\mathbb{Z}}$ and $(b_m)_{m\in\mathbb{Z}}$ are assumed to be absolutely summable.) □

Fourier Inversion Formula: *If $A(\theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta}$ for some absolutely summable sequence $(a_m)_{m\in\mathbb{Z}}$ then the coefficients a_m can be recovered by*

$$a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\theta)e^{-im\theta}\, d\theta \tag{15}$$

Proof. Substitute the infinite series (8) for $A(\theta)$ in the integral (15) and interchange the integral with the infinite sum. (This is justified by the dominated convergence theorem – you figure out why.) The result is

$$\int_{-\pi}^{\pi} A(\theta)e^{-im\theta}\, d\theta = \sum_{n=-\infty}^{\infty} a_n \int_{-\pi}^{\pi} e^{in\theta - im\theta}\, d\theta.$$

The inversion formula follows by verifying that all of the integrals $\int \exp\{i(n-m)\theta\}\, d\theta$ are 0 except when $n = m$, in which case the integral is 2π. □

Sup

2.2. **Fourier Series in Dimensions $d \geq 2$.** There is a completely analogous theory for "sequences" defined on higher-dimensional integer lattices. Denote by $\mathbb{Z}^d$ the $d-$dimensional lattice, that is, the set of all $d-$tuples $m = (m_1, m_2, \dots, m_d)$ with integer entries, and by $\mathbb{T}^d$ the $d-$fold Cartesian power of the circle group $\mathbb{T}$. Both $\mathbb{Z}^d$ and $\mathbb{T}^d$ are abelian groups; the group $\mathbb{T}^d$ is the $d-$dimensional torus, and for all practical purposes can be viewed as the $d-$cube $[-\pi, \pi]^d$ with opposite faces glued together. The $d-$dimensional analogue of a sequence is an assignment $(a_m)_{m\in\mathbb{Z}^d}$ of complex numbers to the points of the lattice, i.e., a function $a : \mathbb{Z}^d \to \mathbb{C}$. The spaces $\ell^p(\mathbb{Z}^d)$ consist of the sequences show absolute pth powers are summable. For a sequence $(a_m)_{m\in\mathbb{Z}^d}$ in $\ell^1(\mathbb{Z}^d)$, the Fourier series $A(\theta)$ can be defined as

$$A(\theta) = \sum_{m\in\mathbb{Z}^d} a_m e^{i\langle m,\theta\rangle} = \sum_{m\in\mathbb{Z}^d} a_m e_m(\theta) \tag{16}$$

where $\langle m,\theta\rangle = \sum_{j=1}^d m_j\theta_j$ is just the usual dot product.

Theorem 4. *If $(a_m)_{m\in\mathbb{Z}^d}$ is absolutely summable with Fourier series $A(\theta)$ then $A(\theta)$ is continuous on $\mathbb{T}^d$ and the Fourier coefficients can be recovered by the Fourier inversion formula*

$$a_m = (2\pi)^{-d} \int_{[-\pi,\pi]^d} A(\theta)e^{-i\langle m,\theta\rangle}\, d\theta. \tag{17}$$

The proof is nearly identical to that used to prove the analogous statement in the one-dimensional case. Take note, however, of one important difference: the Fourier integral in equation (17) is a $d-$fold multiple integral with respect to (normalized) $d-$dimensional Lebesgue measure

$$d\theta = d\theta_1 d\theta_2 \cdots d\theta_d.$$

3. Random walk

3.1. **Recurrence of random walks in dimension $d = 1$.** Suppose now that $X_1, X_2, \dots$ are independent, identically distributed random variables taking values in the integers. Thus, there is a probability distribution $\{q_k\}_{k\in\mathbb{Z}}$ such that for each $n \geq 1$ and all choices $k_1, k_2, \dots, k_n \in \mathbb{Z}$,

$$P(X_i = k_i \text{ for all } i = 1, 2, \dots, n) = \prod_{i=1}^n q_{k_i}. \tag{18}$$

Proposition 5. *Let $S_n = \sum_{i=1}^n X_i$; then*

$$\sum_{k\in\mathbb{Z}} P\{S_n = k\}e^{ik\theta} = \varphi(\theta)^n \quad \textit{where} \quad \varphi(\theta) = \sum_{k\in\mathbb{Z}} q_k e^{ik\theta}. \tag{19}$$

Proof. This could be deduced from the convolution formula proved earlier, but it is perhaps easier to reason from first principles. For each k, decompose the probability $P(S_n = k)$ by summing the probabilities (18) over all $k_1, k_2, \dots, k_n$ whose sum is k. This yields

$$\sum_{k\in\mathbb{Z}} P\{S_n = k\}e^{ik\theta} = \sum_{k_1}\sum_{k_2}\cdots\sum_{k_n}\prod_{i=1}^n q_{k_i} e^{i\theta\sum_{j=1}^n k_j}.$$

This is the nth power of $\varphi(\theta)$. □

Corollary 6. *Under the hypotheses of Proposition 5,*

$$P\{S_n = k\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\theta)^n e^{-ik\theta}\, d\theta. \tag{20}$$

The integral (20) is similar in form to the integral (5) studied in section 1 above. There are two important differences: first the integrands considered in section 1 were assumed to be real-valued; and second, the complex exponential factor $e^{-ik\theta}$ is not an nth power. Thus, Proposition 1 does not apply directly, at least not in all cases. Nevertheless, the *Laplace method* used to obtain Proposition 1 does apply, with some minor modifications.

Before we turn to the general case let's see what we can deduce from Proposition 1 directly. Suppose that the distribution $(q_k)_{k\in\mathbb{Z}}$ is *symmetric* about 0, that is, $q_k = q_{-k}$, and that its first two moments are finite. Then the characteristic function $\varphi(\theta)$ is real and even (that is, $\varphi(\theta) = \varphi(-\theta)$) and the mean $\mu = 0$. Consequently, the characteristic function $\theta(\theta)$ has the Taylor expansion

$$\varphi(\theta) = 1 - \frac{\sigma^2}{2}\theta^2 + o(\theta)^2$$

in a neighborhood of the origin, and so the first two terms of the Taylor series of $\log \varphi$ around zero vanish. Thus, at least in a neighborhood of the origin, the Fourier integral (20) has the form (5) when $k = 0$. Unfortunately, $\varphi(\theta)$ might not attain its maximum *uniquely* at $\theta = 0$, and in fact, if the probability distribution $(q_k)_{k\in\mathbb{Z}}$ is supported by (say) the even integers $2\mathbb{Z}$, then $\varphi(\theta)$ will take the value 1 not only at 0 but at $\theta = \pm\pi$. The following lemma characterizes those situations in which the characteristic function will have multiple maxima in $[-\pi, \pi]$.

Lemma 7. *If $(q_k)_{k\in\mathbb{Z}}$ is* not *supported by any proper subgroup $m\mathbb{Z}$ of $\mathbb{Z}$ (some $m \geq 2$) then $\varphi(\theta) = 1$ only at $\theta = 0$ in the interval $[-\pi, \pi]$. If $(q_k)_{k\in\mathbb{Z}}$ is not supported by any* coset *of a proper subgroup (that is, by a proper arithmetic progression) then $|\varphi(\theta)| < 1$ for all $\theta \neq 0$ in $[-\pi, \pi]$.*

Proof. Exercise. □

Example 1. The Rademacher distribution (that is, the distribution that assigns probability 1/2 to ± 1) is not supported by a proper subgroup of $\mathbb{Z}$, but it is supported by the arithmetic progression $2\mathbb{Z} + 1$. It characteristic function

$$\varphi(\theta) = \cos\theta \tag{21}$$

assumes the value $+1$ only at $\theta = 0$, but it assumes the value -1 at $\theta = \pm\pi$. Note that in general if ξ_1 takes its values in the arithmetic progression $m\mathbb{Z} + k$, for some k relatively prime to m, then S_{nm} takes values in $m\mathbb{Z}$, and S_n can only return to 0 at integer multiples of m.

Proposition 8. *Assume that the distribution $(p_k)_{k\in\mathbb{Z}}$ of ξ_1 is symmetric about 0 and has variance $0 < \sigma^2 < \infty$. Then the random walk S_n is recurrent.*

Proof. The idea is to use Laplace's method and the Fourier inversion formula (20) to show that $P\{S_n = 0\} \geq Cn^{-1/2}$ for some $C > 0$, at least for even n. Polya's criterion for recurrence will then imply that the random walk returns to 0 infinitely often with probability 1.

Consider first the case where the distribution is not supported by any proper arithmetic progression. Then $|\varphi(\theta)| < 1$ for all $\theta \in [-\pi, \pi] - \{0\}$. Fix $\delta > 0$ and let $\varepsilon = \max |\varphi(\theta)|$ where the max is over all $\theta \in [-\pi, \pi]$ such that $|\theta| \geq \delta$. Then $\varepsilon < 1$, and so

$$\int_{[-\pi,\pi]-[-\delta,\delta]} |\varphi(\theta)|^n \, d\theta \leq \varepsilon^n.$$

This goes to 0 faster than any polynomial in n, so it will be enough to show that

$$\int_{[-\delta,\delta]} \varphi(\theta)^{2n} \, d\theta \geq C/\sqrt{2n}.$$

But if $\delta > 0$ is sufficiently small then $\varphi(\theta) > 0$ for all $\theta \in [-\delta, \delta]$, and so in this case the integral is of the form (5). Consequently, Proposition 1 implies that the integral is $\sim C/\sqrt{2n}$ with $C = 1/\sqrt{2\pi n}\sigma$ (I think!).

The general case can be handled the same way once the periodicities are accounted for. Suppose then that the distribution of ξ_1 is supported by $m\mathbb{Z}+k$ for some $m \geq 2$, and by no coarser arithmetic progression. Then S_m has distribution supported by $m\mathbb{Z}$, but by no proper arithmetic progression contained in $m\mathbb{Z}$; consequently S_m/m is integer-valued, and its distribution is contained in no arithmetic progression. Thus, the argument of the preceding paragraph implies that the random walk S_{mn}/m is recurrent. But this obviously implies that S_n is recurrent. □

The hypotheses of Proposition 8 are much more stringent than necessary. In fact the following is true: any one-dimensional random walk S_n on the integers whose increments ξ_j have finite absolute first moment and mean $E\xi_1 = 0$ is recurrent.

3.2. **Recurrence and transience in dimensions $d \geq 2$.** What about random walk on the higher-dimensional integer lattices $\mathbb{Z}^d$? To keep the discussion as elementary as possible let's consider only the *simple random walk* on $\mathbb{Z}^d$, which moves by choosing at each step one of the $2d$ nearest neighbors of the current state at random (each with probability $1/2d$) for its next state. If the random walk is started at the origin $S_0 = 0$ then the position at time n is $S_n = \sum_{j=1}^n \xi_j$ where the random vectors ξ_j are i.i.d. with characteristic function

$$\varphi(\theta) = Ee^{i\langle\theta,\xi_1\rangle} = d^{-1}\sum_{j=1}^{d}\cos\theta_j. \tag{22}$$

This is real-valued, but because the simple random walk has period 2 the characteristic function takes the value -1 at points θ whose entries θ_j are $\pm\pi$. Hence, it is easier to work with the random walk S_{2n}, whose step distribution has characteristic function $Ee^{i\langle S_2,\theta\rangle} = \varphi(\theta)^2$. This has the advantage that it is $\pi-$periodic, with absolute value strictly less than 1 for all nonzero $\theta \in [-\pi/2, \pi/2]^d$. Fourier inversion gives

$$P\{S_{2n} = 0\} = (\pi)^{-d}\int_{[-\pi/2,\pi/2]^d}\varphi(\theta/2)^{2n}\,d\theta. \tag{23}$$

(You should check to see if I got the factors of 2 correct.) This is of the form (5), but the integral is a $d-$fold multiple integral, so Proposition 1 doesn't apply except in $d = 1$. What is needed is a $d-$dimensional version of Laplace's method.

Suppose, then, that $g : [-A, A]^d \to \mathbb{R}$ is a C^2-function such that

$$g(0) = 0 \quad \text{and} \quad g(x) < 0 \;\text{ for all } x \neq 0,$$

and assume that the matrix H of negative second partials $-\partial^2 g/\partial x_j\partial x_k$ (at the origin) is strictly positive definite. Set

$$J(n) = \int_{[-A,A]^d} e^{ng(x)}\,dx.$$

Proposition 9. *Under the preceding hypotheses,*

$$J(n) \sim (2\pi)^{d/2} n^{-d/2} \det H \quad \textit{as} \;\; n \to \infty. \tag{24}$$

Proof. This is almost exactly the same as the proof of Proposition 1. The hypotheses on g imply that for large n, in a neighborhood of the origin the function $e^{ng(x)}$ is trapped between two Gaussian densities whose covariance matrices are $(1 \pm \varepsilon)H^{-1}/n$ for some $\varepsilon > 0$ which may be taken arbitrarily small by choosing a small neighborhood of the origin. Integrating these Gaussian densities and letting $\varepsilon \to 0$ yields (24). □

Theorem 10. *(Polya) Simple random walk is recurrent in dimensions $d = 1, 2$ and transient in dimensions $d \geq 3$.*

Proof. Proposition 9 and the Fourier inversion formula (23) imply that

$$P\{S_{2n} = 0\} \sim C_d n^{-d/2}$$

for certain positive constants C_d that I am too lazy to work out. The series $\sum n^{-d/2}$ is summable in dimensions $d \geq 3$, but diverges for $d = 1, 2$. $\square$