KINGMAN'S SUBADDITIVE ERGODIC THEOREM

Theorem 1. *(Kingman's theorem, ergodic case) Let (Ω, μ, T) be an ergodic measure-preserving system, and let $(g_n)_{n\geq 1}$ be a sequence of L^1 random variables that satisfy the subadditivity relation*

$$g_{m+n} \leq g_n + g_m \circ T^n. \tag{1}$$

Then

$$\lim_{n\to\infty} \frac{g_n}{n} = \gamma := \inf_{n\geq 1} \frac{Eg_n}{n} \quad a.s. \tag{2}$$

Remark 1. This is an extension of Birkhoff's theorem, because for any $f \in L^1$ the sequence $g_n := S_n f$ satisfies the condition (1) (in fact with $\leq$ replaced by $=$). The importance of Kingman's theorem lies in the fact that interesting *subadditive* sequences are more common than interesting *additive* sequences.

Proof. This consists of establishing the two inequalities

$$\limsup g_n/n \leq \gamma; \quad \text{and} \tag{3}$$

$$\liminf g_n/n \geq \gamma. \tag{4}$$

(Here and in the remainder of the proof we drop the a.s. qualification.) In doing this, we may assume without loss of generality that the random variables g_n are *non-positive*. To see this, observe that for any subadditive sequence g_n (that is, a sequence satisfying the subadditivity relations (1)), the sequence

$$\tilde{g}_n := g_n - S_n g_1$$

is also subadditive, and consists of non-positive random variables. Moreover, if $\lim \tilde{g}_n/n$ exists a.s. and is constant, then $\lim g_n/n$ also exists and is constant, because the difference between the two sequences is the sequence $S_n g_1/n$, which by Birkhoff's theorem converges to Eg_1.

Upper Estimate: The first inequality is an easy consequence of Birkhoff's theorem, but wtih one minor complication. The difficulty is that even though T is ergodic, its $m-$fold iterate T^m need not be ergodic. (Exercise: Provide an example.) Iterating the subadditivity condition (1) gives

$$g_{nm+k} \leq \sum_{j=1}^{n} g_m \circ T^{jm} + \sum_{j=1}^{k} g_1 \circ T^{nm+j}.$$

The obvious strategy would be to fix m, let $1 \leq k \leq m$, divide both sides by nm, let $n \to \infty$, and use Birkhoff on the right side to conclude that

$$\limsup g_n/n \leq Eg_m/m. \tag{5}$$

However, since T^m need not be ergodic, the use of Birkhoff isn't justified. Hence, we resort to the following indirect approach: iterate (1) in different ways to obtain

$$g_{nm+k} \leq \sum_{j=0}^{n-1} g_m \circ T^{jm} + \sum_{j=1}^{k+m} g_1 \circ T^{nm+j};$$

$$g_{nm+k} \leq g_1 + \sum_{j=0}^{n-1} g_m \circ T^{jm+1} + \sum_{j=2}^{k+m} g_1 \circ T^{nm+j};$$

$$g_{nm+k} \leq g_1 + g_1 \circ T + \sum_{j=0}^{n-1} g_m \circ T^{jm+2} + \sum_{j=3}^{k+m} g_1 \circ T^{nm+j};$$

$$\cdots$$

Note that in each of these inequalities there are at most $k + m \leq 2m$ terms of the form $g_1 \circ T^i$, but nm terms of the form $g_m \circ T^i$. Thus, upon taking the average of the first m inequalities, one obtains

$$g_{nm+k} \leq \frac{1}{m} \sum_{j=0}^{nm-1} g_m \circ T^j + \text{remainder},$$

where the remainder is an average of at most $mk \leq m^2$ terms of the form $g_1 \circ T^i$. Now divide both sides by nm and let $n \to \infty$: in the limit, the remainder terms divided by nm will converge to zero, and Birkhoff's theorem will apply to the rest, yielding the inequality (5). Since this holds for all values of m, this proves the upper bound (3).

Lower Estimate: For this estimate we use the assumption that the random variables g_n are non-positive. This implies that $\gamma \leq 0$, and together with the subadditivity relation (1) implies that for any finite increasing sequence of nonnegative integers

$$1 \leq j_1 < j_1 + k_1 - 1 < j_2 < j_2 + k_{2-1} < \cdots < j_r < j_r + k_r - 1 < N < \infty$$

we have

(6) $$g_N \leq g_{k_1} \circ T^{j_1} + g_{k_2} \circ T^{j^2} + \cdots + g_{k_r} \circ T^{j_r}.$$

Define, for any $\omega \in \Omega$,

$$G(\omega) = \liminf_{n \to \infty} g_n(\omega)/n \in [-\infty, 0].$$

The subadditivity hypothesis (1) implies that $g_{n+1} \leq g_1 + g_n \circ T$, and hence, since $g_1/n \to 0$, it follows that

$$G \leq G \circ T.$$

But because T is measure-preserving, the random variables G and $G \circ T$ have the same distribution, so it must be that $G = G \circ T$ a.s., and so the random variable G is *invariant*. This implies that G is *constant*, because we have assumed that the system (Ω, μ, T) is ergodic. Thus, our task is to show that this constant G is not smaller than γ.

Suppose to the contrary that $G < \gamma - \varepsilon$ for some $\varepsilon > 0$. Then $\inf g_n/n < \gamma - \varepsilon$ a.s. Hence, by the monotone convergence theorem, for any $\delta > 0$ there exists m so large that

$$\mu(B) \geq 1 - \delta \quad \text{where} \quad B = B_m = \{\omega \in \Omega : \min_{1 \leq n \leq m} g_n/n < \gamma - \varepsilon\}.$$

Fix n large and $\omega \in \Omega$, and label integers j in the range $[1, nm]$ *bad* if $T^j(\omega) \in B$. By construction, for each bad j, there exists $k \leq m$ so that

$$g_k \circ T^j(\omega) < k(\gamma - \varepsilon).$$

For each such pair j, k color the integers in the interval $[j, j+k-1]$ *blue*. Then it is possible to select a subset of the blue intervals so that their union contains all of the bad integers j. (Keep the leftmost blue interval $I_1 = [j_1, j_1 + k_1 - 1]$, then delete from the collection all blue intervals whose left endpoints lie in I_1, then keep the next leftmost blue interval I_2 among those remaining, and so on.) Denote the blue intervals in this subcollection by

$$I_i = [j_i, j_i + k_i - 1] \quad \text{where } \ i = 1, 2, \ldots, r,$$

and let L be the sum of their lengths. Now apply (6): by construction, for each interval $[j_i, j_i + k_i - 1]$ we have

$$g_{k_i} \circ T^{j_i}(\omega) < k_i(\gamma - \varepsilon),$$

so (6) implies that

$$g_{nm+m}(\omega) < L(\gamma - \varepsilon) < (\gamma - \varepsilon) \sum_{i=1}^{nm} \mathbf{1}_B(T^i(\omega)).$$

Finally, divide both sides by nm and take expecations:

$$E \frac{g_{nm+m}}{nm} \leq (\gamma - \varepsilon)\mu(B) < (\gamma - \varepsilon)(1 - \delta).$$

Letting $n \to \infty$ and using the definition of γ we deduce that

$$\gamma < (\gamma - \varepsilon)(1 - \delta).$$

But $\delta > 0$ was arbitrary, so this is a contradiction. □