MathFinance 346/Stat391
Homework 1
Due Wednesday January 21, 2004

The first two problems concern foreign exchange. In problem 1, the same assumptions as were made in the Lecture are assumed to be in force; in problem 2, the coefficients are assumed to be time-dependent.

Problem 1: Compute the arbitrage price at time $t = 0$ of the call option discussed in the notes for Lecture 9 by calculating the discounted risk-neutral expectation under the risk-neutral measure Q_B for pound-sterling investors. Verify that the price agrees with that computed using the risk-neutral measure Q_A for dollar investors.

Problem 2: Assume that the shart rates r_t^A and r_t^B for the US and UK Money Markets are time-dependent, but nonrandom and continuous. Assume also that the exchange rate Y_t between \$ and £ (that is, $Y_t =$ number of Pounds Sterling that one dollar will buy at time t) obeys the Stochastic Differential Equation

$$dY_t = \mu_t Y_t \, dt + \sigma_t Y_t \, dW_t$$

where σ_t is a positive, nonrandom, continuous function of t. Here W_t is a standard Wiener process under Q^B, the risk-neutral probability measure for £ investors.

What relation holds among the functions r_t^A, r_t^B, μ_t, and σ_t?

The remaining problems concern an important SDE , the "mean-reversion" equation that governs the *Ornstein-Uhlenbeck* process. For these you will need the Itô formula, the Girsanov theorem, and the exponential martingales discussedc in the Lecture. Assume that under the probability measure P, the process W_t is a standard Wiener process.

Problem 3: Let $f(t)$ be a continuous, nonrandom function of t. What is the distribution of the random variable

$$\int_0^t f(s)\, dW_s \ ?$$

Hint: Evaluate the moment generating function $E \exp\{\lambda \int_0^t f(s)\, dW_s\}$. Using what you saw in the Lecture, you sould be able to do this with virtually no calculations.

Problem 4: The *Ornstein-Uhlenbeck* process X_t with spring constant $\alpha > 0$ is the solution to the Stochastic Differential Equation

(1) $$dX_t = -\alpha X_t \, dt + dW_t.$$

The initial point X_0 is arbitrary. (A) Find an explicit solution to this stochastic differential equation of the form

(2) $$X_t = X_0 + g(t) \int_0^t f(s)\, dW_s$$

where $g(t)$ and $f(t)$ are continuous, nonrandom functions of t. Hint: Itô. (B) What is the distribution of the random variable X_t? (C) What is the covariance of the random variables $X(t), X(s)$?

Problem 5: Suppose that under P the process W_t is a standard Wiener process, and that Q is the probability measure on the class $\mathcal{F}_T$ of events observable up to time T that is defined by

(3) $$\left(\frac{dQ}{dP}\right)_{\mathcal{F}_T} = \exp\{-\int_0^T W_s\, dW_s - \int_0^T W_s^2\, ds/2\}.$$

(A) What is the nature of the process W_t under the measure Q?
(B) Show that the Radon-Nikodym derivative may be written as

(4) $$\left(\frac{dQ}{dP}\right)_{\mathcal{F}_T} = \exp\{-\frac{1}{2}(W_T^2 - T) - \int_0^T W_s^2\, ds/2\}.$$

In this form, the Radon-Nikodym derivative is useful both for simulations and for theoretical investigations, the latter using the *Feynman-Kac* formula, about which you will undoubtedly hear more later.