1. Weak Convergence

**Definition 1.** Let \((\mathcal{X}, d)\) be a complete, separable metric space (also known as a Polish space). The Borel \(\sigma\)-algebra on \(\mathcal{X}\) is the minimal \(\sigma\)-algebra containing the open (and hence also closed) subsets of \(\mathcal{X}\). If \(\mu_n\) and \(\mu\) are finite Borel measures on \(\mathcal{X}\), then \(\mu_n\) converges weakly (or in distribution), written \(\mu_n \Rightarrow \mu\), if for every bounded, continuous function \(f : \mathcal{X} \to \mathbb{R}\),

\[
\lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu.
\]

**Proposition 1.** If \(\mathcal{X}\) is a compact metric space then the space \(C(\mathcal{X})\) of continuous, real-valued functions on \(\mathcal{X}\) is separable in the topology of uniform convergence, that is, there is a countable subset \(G \subset C(\mathcal{X})\) such that for every function \(f \in C(\mathcal{X})\) and every \(\varepsilon > 0\) there exists \(g \in G\) such that

\[
\|f - g\|_{\infty} < \varepsilon.
\]

**Proof.** See your favorite analysis textbook. In the special case where \(\mathcal{X}\) is a compact subset of \(\mathbb{R}^d\) for some \(d < \infty\) (which is probably the only case that will occur in this course) the proposition follows easily from Weierstrass' Polynomial Approximation Theorem. 

**Proposition 2.** (Weierstrass) Let \(\mathcal{X}\) be a compact subset of \(\mathbb{R}^d\). Then the polynomial functions in \(d\) variables on \(\mathcal{X}\) are uniformly dense in \(C(\mathcal{X})\), that is, for every function \(f \in C(\mathcal{X})\) and every \(\varepsilon > 0\) there is a polynomial \(p(x) = p(x_1, x_2, \ldots, x_d)\) such that

\[
\|f - p\|_{\infty} < \varepsilon.
\]

**Proposition 3.** (Riesz Representation Theorem) (A) Let \(\mathcal{X}\) be a compact metric space. Then every bounded, positive, linear functional \(\lambda : C(\mathcal{X}) \to \mathbb{R}\) is gotten by integration against a finite, positive, Borel measure \(\mu\): that is,

\[
\lambda(g) = \int g \, d\mu \quad \forall g \in C(\mathcal{X}).
\]

(B) Let \(\mathcal{X}\) be a Polish space. Then every bounded, linear functional \(\lambda : C_c(\mathcal{X}) \to \mathbb{R}\) on the space \(C_c(\mathcal{X})\) of continuous functions with compact support is gotten by integration against a positive, Borel measure \(\mu\) that attaches finite mass to every compact subset of \(\mathcal{X}\). Thus, (2) holds for all \(g \in C_c(\mathcal{X})\).

**Note:** A linear functional \(\lambda : C(\mathcal{X}) \to \mathbb{R}\) is said to be bounded if there is a constant \(A < \infty\) such that \(|\lambda(g)| \leq A\|g\|_{\infty}\) for every \(g \in C(\mathcal{X})\), and positive if for every nonnegative continuous function \(g : \mathcal{X} \to \mathbb{R}\) the value \(\lambda(g) \geq 0\).

**Proof.** See Rudin, *Real and Complex Analysis*, ch. 2 for the standard proof.

**Proposition 4.** (Helly's Selection Principle) Let \(\mathcal{X}\) be a compact metric space. Then every sequence \(\mu_n\) of bounded positive measures for which \(\sup_n \mu_n(\mathcal{X}) < \infty\) has a weakly convergent subsequence.
Proof. Let \( G \subset C(\mathcal{X}) \) be a countable, uniformly dense set of continuous functions on \( \mathcal{X} \). For every \( g \in G \) the sequence \( \int g \, d\mu_n \) is bounded in \( \mathbb{R} \), since \( \sup_n \mu_n(\mathcal{X}) := A < \infty \), and so by Cantor’s diagonal argument there is a subsequence \( \mu_k \) of \( \mu_n \) such that

\[
\lambda(g) := \lim_{k \to \infty} \int g \, d\mu_k
\]

exists and is finite for all \( g \in G \). Since \( G \) is uniformly dense in \( C(\mathcal{X}) \), it follows by a routine argument that the convergence (3) holds for all \( g \in C(\mathcal{X}) \). Furthermore, it is easy to check that the limit \( \lambda(g) \) is a linear functional on the space \( C(\mathcal{X}) \), and is positive and bounded (because \( |\lambda(g)| \leq A\|g\|_\infty \)). Therefore, by the Riesz Representation Theorem, there is a finite, positive, Borel measure \( \mu \) on \( \mathcal{X} \) such that

\[
\lambda(g) = \int g \, d\mu \quad \forall \, g \in C(\mathcal{X}).
\]

This, together with the convergence (3), implies that \( \mu_k \Rightarrow \mu \). \( \square \)

Definition 2. Let \( \mathcal{X} \) be a Polish space. A family \( \{\mu_i\}_{i \in I} \) of finite, positive Borel measures on \( \mathcal{X} \) is said to be tight if for every \( \varepsilon > 0 \) there is a compact subset \( K \subset \mathcal{X} \) such that

\[
\mu_i(K^c) < \varepsilon \quad \forall \, i \in I.
\]

Proposition 5. Let \( \mathcal{X} \) be a Polish space. Then every tight sequence \( \mu_n \) of finite, positive Borel measures on \( \mathcal{X} \) such that \( A := \sup_n \mu_n(\mathcal{X}) < \infty \) has a weakly convergent subsequence.

Remark 1. This is usually called Prohorov’s theorem. The converse is also true, but will not be needed. Note that the characterization of weak sequential compactness provided by this theorem is highly specific to the notion of weak convergence given in Definition 1: If (1) were only required to hold for continuous functions \( f : \mathcal{X} \to \mathbb{R} \) with compact support, then the tightness hypothesis would not be needed. See Billingsley, *Convergence of Probability Measures*, ch. 1 for further information.

Proof. The hypothesis that \( \mathcal{X} \) is a Polish space guarantees that there is a sequence of compact subsets \( K_n \) whose union is \( \mathcal{X} \), and such that each \( K_n \) is contained in the interior of \( K_{n+1} \). (For \( \mathcal{X} = \mathbb{R}^d \) this is obvious — just take \( K_n = [-n, n]^d \). Note that the requirement \( K_n \subset \text{int}(K_{n+1}) \) ensures that there is a continuous function \( f : \mathcal{X} \to [0, 1] \) that is 1 on \( K_n \) and 0 off \( K_{n+1} \), by the Urysohn Lemma.) By the Helly Selection Principle, for each \( K_n \) there is a subsequence of \( \mu_n \) that converges weakly on \( K_n \). Hence, by the Cantor diagonal argument, there is a subsequence \( \mu_k \) that converges weakly to a measure \( \nu_n \) on each \( K_n \). Observe that \( \nu_n(K_n) \leq A \) for each \( n \). The problem is to show that \( \mu_k \) converges weakly to a finite measure on \( \mathcal{X} \).

(1) Choose \( f \in C_b(\mathcal{X}) \). For any \( \varepsilon > 0 \) there exists \( n \) so large that \( \mu_k(K_n^c) < \varepsilon \) for every \( k \geq 1 \).

(Exercise: why?) Now

\[
\lim_{k \to \infty} \int_{K_n^c} f \, d\mu_k = \int_{K_n^c} f \, d\nu_n \quad \text{and} \quad \left| \int_{K_n^c} f \, d\mu_k \right| \leq \varepsilon \|f\|_\infty \forall \, k.
\]

Since \( \varepsilon > 0 \) is arbitrary, it follows routinely that

\[
\lim_{k \to \infty} \int f \, d\mu_k := \lambda(f)
\]

exists. Since \( \sup_k \mu_k(\mathcal{X}) \leq A \), it is clear that \( |\lambda(f)| \leq A\|f\|_\infty \), so the functional \( \lambda \) is bounded. It is easy to check that \( \lambda \) is linear and positive.

(2) It remains to show that the functional \( f \to \lambda(f) \) is given by integration against a finite Borel measure on \( \mathcal{X} \). For this, note that the restriction of \( \lambda \) to the space \( C_c(\mathcal{X}) \) of continuous
functions with compact support is bounded, linear, and positive, so the Riesz Representation Theorem implies that there is a positive measure $\nu$ such that

$$\lambda(f) = \int f \, d\nu \quad \forall f \in C_c(X).$$

To see that this equality extends to all $f \in C_b(X)$, observe that each such $f$ can be arbitrarily well-approximated by functions in $C_c(X)$. In particular, for each $n$ there is a continuous function $h_n : X \to [0,1]$ that is 1 on $K_n$ and 0 off $K_{n+1}$. For each $n$ the function $h_n f$ is continuous, has support $K_{n+1}$, and agrees with $f$ on $K_n$. Furthermore, since $f - f h_n$ is bounded in absolute value by $|f|$, $|\lambda(f) - \lambda(f h_n)| \leq 2\varepsilon \|f\|_{\infty}$ for all $n$ large enough that $\mu_k(K^c_n) < \varepsilon$. Therefore, by the Dominated Convergence Theorem,

$$\lambda(f) = \int f \, d\nu$$

Finally, observe that this identity, with $f \equiv 1$, implies that the total mass of $\nu$ is $\leq A$, because $\lambda(1) = \lim_{k \to \infty} \mu_k(X) \leq A$.

**Proposition 6.** Let $\mu, \nu$ be finite Borel measures both with compact support $K \subset \mathbb{R}^d$. If $\mu$ and $\nu$ have the same moments, that is, if for every monomial $x_1^{n_1} x_2^{n_2} \ldots x_d^{n_d}$,

$$\int x_1^{n_1} x_2^{n_2} \ldots x_d^{n_d} \, d\mu = \int x_1^{n_1} x_2^{n_2} \ldots x_d^{n_d} \, d\nu,$$

then they are equal as measures. More generally, if $\mu$ and $\nu$ are finite positive Borel measures on $\mathbb{R}^d$ with finite moment generating functions in a neighborhood of the origin, then equality of moments (4) implies that $\mu = \nu$.

**Remark 2.** In general, a probability measure is not uniquely determined by its moments, even when they are all finite. For further information on this subject, see, e.g., Rudin, *Real and Complex Analysis*, chapter on the Denjoy-Carleman theorem.

**Proof.** Consider first the case where both $\mu$ and $\nu$ are supported by a compact set $K$. Equality of moments (4) implies that $\int f \, d\mu = \int f \, d\nu$ for every polynomial $f$, and therefore, by the Weierstrass theorem, for all continuous functions $f$ on $K$. This in turn implies that $\mu(F) = \nu(F)$ for every rectangle $F$, by an easy approximation argument, and therefore for every Borel set $F$.

Now consider the case where both measures have finite moment generating functions in a neighborhood of 0, that is,

$$\int e^{\theta^T x} \, d\mu(x) + \int e^{\theta^T x} \, d\nu(x) < \infty$$

for all $\theta$ in a neighborhood of the origin in $\mathbb{R}^d$. In this case the moment generating functions extend to complex arguments $z = (z_1, z_2, \ldots, z_d)$, and define holomorphic (see Remark 3 below) functions of $d$ variables:

$$\varphi_\mu(z) := \int e^{z^T x} \, d\mu(x) \quad \text{and} \quad \varphi_\nu(z) := \int e^{z^T x} \, d\nu(x).$$

Equality of moments implies that these two holomorphic functions have the same power series coefficients, and therefore are equal in a neighborhood of zero. It now follows that the two measures are the same (e.g., by the Fourier inversion theorem).
Remark 3. That the functions $\varphi_\mu(z)$ and $\varphi_\nu$ defined by (5) are holomorphic in their arguments $z_1, z_2, \ldots, z_d$ is a consequence of the Cauchy, Morrera, and Fubini theorems, by a standard argument, using the fact that $e^{z^T x}$ is holomorphic in $z$ for each $x$. The Morrera theorem states that a function $f(\zeta)$ is holomorphic in a domain $\Omega \subset \mathbb{C}$ if it integrates to 0 on every closed curve $\gamma$ that is contractible in $\Omega$. The Cauchy integral theorem asserts that if $f(\zeta)$ is holomorphic in a domain $\Omega$ then it integrates to zero on every closed curve $\gamma$ that is contractible in $\Omega$. Thus, if $f(\zeta, x)$ is holomorphic in $\zeta$ for each $x \in X$, and $\mu$ is a Borel measure on $X$, then $\int_X f(\zeta, x) \, d\mu(x)$ is holomorphic in $\zeta$ provided the conditions of the Fubini theorem are satisfied.

**Proposition 7.** (Method of Moments) Let $\mu$ be a probability measure on $\mathbb{R}^d$ with all moments finite, and such that it is the only finite measure on $\mathbb{R}^d$ with these moments. Let $\mu_n$ be a sequence of probability measures on $\mathbb{R}^d$ whose moments converge as $n \to \infty$ to the moments of $\mu$. Then

$$\mu_n \Rightarrow \mu.$$ 

**Proof.** Convergence of the second moments $\int x_j^2 \, d\mu(x)$ is enough to ensure that the sequence $\mu_n$ is tight (by Chebyshev’s inequality). Thus, every subsequence of $\mu_n$ has a subsequence that converges weakly, by Prohorov’s theorem. But any possible limit must have moments that agree with those of $\mu$; since $\mu$ is uniquely determined by its moments, it is the only possible accumulation point of the sequence $\mu_n$. \qed

The method of moments is the most useful tool for proving convergence of empirical spectral distributions in the theory of random matrices. There is one other useful tool, the Stieltjes transform:

**Definition 3.** Let $\mu$ be a finite, positive Borel measure on $\mathbb{R}$. The Stieltjes transform $F_\mu(z)$ is the holomorphic (see Remark 3) function of $z \in \mathbb{C} \setminus \mathbb{R}$ defined by

$$F_\mu(z) = \int (x - z)^{-1} \, d\mu(x).$$

**Remark 4.** Since the probability measure $\mu$ is supported by $\mathbb{R}$, the Stieltjes transform satisfies

$$F_\mu(z) = F_\mu(\overline{z}).$$

**Proposition 8.** Let $\mu_n$ be a tight sequence of probability measures on $\mathbb{R}$. If the Stieltjes transforms $F_n(z)$ of the measures $\mu_n$ converge for all $z$ in a set $A \subset \mathbb{C} \setminus \mathbb{R}$ with an accumulation point in $\mathbb{C} \setminus \mathbb{R}$ to a limit function $F(z)$ defined on $A$, then the sequence $\mu_n$ converges weakly to a probability measure whose Stieltjes transform agrees with $F(z)$ on $A$.

**Proof.** Since the sequence $\mu_n$ is tight, every subsequence has a weakly convergent subsequence. For every convergent subsequence $\mu_{k_n}$, the Stieltjes transforms $F_{k_n}(z)$ converge to the Stieltjes transform of the limit measure for all $z \in \mathbb{C} \setminus \mathbb{R}$, because $x \mapsto (x - z)^{-1}$ is a bounded, continuous function of $x \in \mathbb{R}$. (Note: This function is complex-valued, but its real and imaginary parts will be bounded, continuous, real-valued functions.) Thus, the limit measure must have a Stieltjes transform that agrees with $F(z)$ on the set $A$. Since the set $A$ has an accumulation point, analyticity guarantees that the Stieltjes transform of the limit measure is uniquely determined by its values $F(z)$ on $A$. \qed
2. MATRIX THEORY: TRACE AND DETERMINANT

2.1. Trace. The trace of a square matrix $M = (m_{i,j})$, denoted by $\text{tr}(M)$, is the sum $\sum_i m_{i,i}$ of its diagonal entries. The following is elementary but important:

**Proposition 9.** Let $A = (a_{i,j})_{m \times n}$ and $B = (b_{i,j})_{n \times m}$ be real or complex square matrices. Then

\[ \text{tr}(AB) = \text{tr}(BA) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} b_{j,i}. \]

Consequently, if $A$ and $B$ are similar square matrices, that is, if there is an invertible matrix $U$ such that $B = U^{-1}AU$, then

\[ \text{tr}(A) = \text{tr}(B). \]

A square matrix $A$ is diagonalizable if there exist an invertible matrix $U$ and a diagonal matrix $D$ such that $A = U^{-1}DU$. You should recall that $A$ is diagonalizable if and only if the underlying vector space has a basis consisting of eigenvectors of $A$; in this case, the diagonalization $A = U^{-1}DU$ is obtained by letting $U$ be the matrix whose columns are the eigenvectors, and $D$ the diagonal matrix whose diagonal entries are the corresponding eigenvalues. Thus, for diagonalizable $A$, the trace $\text{tr}(A)$ is just the sum of the eigenvalues. More generally:

**Corollary 10.** Assume that $A$ is diagonalizable. Then for any integer $k \geq 0$,

\[ \text{tr}(A^k) = \sum_{i} \lambda_i^k. \]

Furthermore, for any analytic function $f(z)$ defined by a power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with radius of convergence $R > 0$, if $A$ has spectral radius $< R$ (that is, if all eigenvalues of $A$ have absolute value $< R$) then

\[ \text{tr}(f(A)) = \sum_{i} f(\lambda_i). \]

Both assertions are easy consequences of Proposition 9. Relation (8) is especially useful in conjunction with the method of moments, as it gives an effective way of computing the moments of the empirical spectral distribution (defined below). Similarly, relation (9) gives a handle on various transforms of the empirical spectral distribution, in particular, the Stieltjes transform.

**Definition 4.** If $A$ is a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_i$, the empirical spectral distribution $F^A$ of $A$ is defined to be the uniform distribution on the eigenvalues (counted according to multiplicity), that is

\[ F^A := n^{-1} \sum_{i=1}^{n} \delta_{\lambda_i}. \]

Relation (8) implies

\[ \int \lambda^k F^A(d\lambda) = n^{-k} \text{tr}(A^k). \]
2.2. **Determinant.** The determinant of an $n \times n$ matrix $A = (a_{i,j})$ is defined by the combinatorial formula

\[
\det(A) := \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)}.
\]

The sum is over the set $S_n$ of permutations of the index set $[n]$, and for each permutation $\sigma \in S_n$, the sign (or parity) $(-1)^{\sigma}$ is defined to be $(-1)^{N(\sigma)}$, where $N(\sigma)$ is the number of cycles in the cycle representation of $\sigma$. Note that if $A$ is diagonal, then its determinant is just the product of the diagonal entries (the only permutation that gives a nonzero contribution to the sum (12) is the identity permutation). Also, the combinatorial formula (12) implies

\[
\det(A^T) = \det(A),
\]

where $A^T$ is the transpose of $A$. (Proof: exercise. You will need the fact – which you should also prove – that the parity $(-1)^{\sigma}$ of any permutation $\sigma$ is the same as that of its inverse.) The following recursive algorithm for computing the determinant is another routine consequence of the definition:

\[
\det(A) = \sum_{j=1}^{n} (-1)^{j+k-1} a_{k,j} \det(A^{k,j})
\]

where $A^{k,j}$ is the $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting the $k$th row and the $j$th column.

**Proposition 11.** The determinant $\det(A)$ is multilinear and antisymmetric in the columns of $A$, that is, for column vectors $a_i$ and $a'_i$, scalars $b, b'$, and permutations $\sigma$

\[
\det(ba_1 + b'a'_1, a_2, a_3, \ldots, a_n) = b \det(a_1, a_2, \ldots, a_n) + b' \det(a'_1, a_2, \ldots, a_n) \quad \text{and}
\]

\[
\det(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}) = (-1)^{\sigma} \det(a_1, a_2, \ldots, a_n).
\]

These are both easy consequences of the definition (12). (For the second equation (16), use the fact that the parity functional is multiplicative, that is, $(-1)^{\sigma T} = (-1)^{\sigma} (-1)^{T}$. Proposition 11 states that the determinant is unaffected by column operations (recall that a column operation consists of adding a scalar multiple of a column to another column), and that column transpositions change the sign of the determinant. In view of the transpose rule (13), this also implies that the determinant is unaffected by row operations, and that row transpositions change the sign of the determinant.

**Corollary 12.** The determinant $\det(A)$ is zero if $A$ is not full rank, that is, if the columns are linearly dependent.

**Proof.** If a column of $A$ is 0, then the determinant is zero, because each product in the sum (12) has a factor from this column. Also, if two columns of $A$ are identical then $\det(A) = 0$, by antisymmetry. (Apply (16) with $\sigma = \text{the transposition that switches the two identical columns}$. Since the parity $(-1)^{\sigma}$ of a transposition is always $-1$, (16) implies that $\det(A) = -\det(A)$. Hence, if two columns of $A$ are proportional, then $\det(A) = 0$.

Now suppose that the columns of $A$ are linearly dependent. Then there is a nontrivial relation among the columns; without loss of generality (by applying a suitable permutation to the columns and using (16)), the relation is of the form

\[
a_1 + \beta_2 a_2 + \cdots + \beta_n a_n = 0
\]
where \(a_i\) is the \(i\)th column of \(A\) and \(\beta_i\) are scalars. But this implies that a sequence of column operations (successively replace the first column by its sum with \(\beta_j \times \) the \(j\)th column) will lead to a matrix whose first column is 0. Thus, the determinant of \(A\) is 0. □

**Proposition 13.** If \(A\) and \(B\) are both \(n \times n\) matrices, then

\[
(17) \quad \det(AB) = \det(A) \det(B) = \det(B) \det(A) = \det(BA).
\]

Consequently, if \(A\) is invertible then \(\det(A^{-1}) = \frac{1}{\det(A)} \neq 0\), and if the matrices \(A\) and \(B\) are similar, then they have the same determinant. In particular, if \(A\) is diagonalizable with eigenvalues \(\lambda_i\), then

\[
(18) \quad \det(A) = \prod_{i=1}^{n} \lambda_i.
\]

**Proof.** (Sketch) The only nontrivial assertion is the product rule \(\det(AB) = \det(A) \det(B)\). It is easy to check that this identity holds for upper triangular matrices \(A\) and \(B\). The general case can be deduced from this using the fact that any square matrix can be put into upper triangular form by either row or column operations and transpositions. □

### 3. Hermitian, Unitary, and Orthogonal Matrices

**3.1. Spectral Theorems.** A **Hermitian matrix** is a square complex matrix that equals its conjugate transpose. A matrix with real entries is therefore Hermitian if and only if it is symmetric. A **unitary matrix** is a square complex matrix whose conjugate transpose is its inverse. In spectral problems, it is often advantageous to take a coordinate-free approach, using an **inner product** to define Hermitian and unitary operators. Thus, let \(V\) be a complex vector space, and let \(\langle \cdot, \cdot \rangle\) be a complex inner product on \(V\). Recall the definition (see Rudin, *Real and Complex Analysis*, ch. 4):

**Definition 5.** An inner product on a complex vector space \(V\) is a mapping \(V \times V \to \mathbb{C}\) that satisfies

- (a) \(\langle u, v \rangle = \overline{\langle v, u \rangle}\).
- (b) \(\langle \alpha u + \alpha' u', v \rangle = \alpha \langle u, v \rangle + \alpha' \langle u', v \rangle\).
- (c) \(\langle u, u \rangle > 0\) for all \(u \neq 0\).
- (d) \(\langle 0, 0 \rangle = 0\).

The difference between the real and complex cases is rule (a); this together with (b) implies that \(\langle u, \alpha v \rangle = \bar{\alpha} \langle u, v \rangle\). The natural inner product on \(\mathbb{C}^n\) is \(\langle u, v \rangle = \sum_i u_i \overline{v_i}\). More generally, the natural inner product on the space \(L^2(\mu)\) of square-integrable complex-valued functions on a measure space \((\Omega, \mathcal{F}, \mu)\) is

\[
\langle f, g \rangle = \int f \overline{g} \, d\mu;
\]

when \(\mu\) is a probability measure this is a complex analogue of the covariance. An **inner product space** is a vector space equipped with an inner product. Two vectors \(u, v\) in an inner product space are said to be **orthogonal** if \(\langle u, v \rangle = 0\). An **orthonormal set** is a set of unit vectors such that any two are orthogonal. You should recall that the **Gram-Schmidt algorithm** produces orthonormal bases.

**Definition 6.** Let \(V\) be a finite-dimensional inner product space. For any operator (i.e., linear transformation) \(T : V \to V\) the **adjoint** \(T^*\) is the unique linear transformation such that

\[
(19) \quad \langle Tu, v \rangle = \langle u, T v \rangle \quad \forall \ u, v \in V.
\]
The linear transformation \( T \) is called Hermitian if \( T = T^* \), and unitary if \( T^{-1} = T^* \), equivalently,
\begin{align*}
\langle Tu, v \rangle &= \langle u, Tv \rangle & \text{(Hermitian)} \quad (20) \\
\langle Tu, Tv \rangle &= \langle u, v \rangle & \text{(Unitary)} \quad (21)
\end{align*}

**Theorem 14.** (Spectral Theorem for Hermitian Operators) Let \( T \) be a Hermitian operator on a finite-dimensional inner product space \( V \). Then all eigenvalues \( \lambda_i \) of \( T \) are real, and there is an orthonormal basis \( \{u_i\} \) consisting of eigenvectors of \( T \). Thus,
\begin{equation}
Tv = \sum_i \lambda_i \langle v, u_i \rangle u_i \quad \forall \ v \in V. 
\end{equation}

**Theorem 15.** (Spectral Theorem for Unitary Operators) Let \( T \) be a unitary operator on a finite-dimensional inner product space \( V \). Then all eigenvalues \( \lambda_i \) of \( T \) have absolute value 1, and there is an orthonormal basis \( \{u_i\} \) consisting of eigenvectors of \( T \). Thus,
\begin{equation}
Tv = \sum_i \lambda_i \langle v, u_i \rangle u_i \quad \forall \ v \in V. 
\end{equation}

Some of the important elements of the proofs are laid out below. Consider first the case where \( T \) is Hermitian. Suppose that \( T v = \lambda v \); then
\begin{align*}
\lambda \langle v, v \rangle &= \langle \lambda v, v \rangle \\
&= \langle Tv, v \rangle \\
&= \langle v,Tv \rangle \\
&= \langle v, \lambda v \rangle \\
&= \bar{\lambda} \langle v, v \rangle.
\end{align*}

Thus, \( \lambda = \bar{\lambda} \), so all eigenvalues of \( T \) are real. A similar argument shows that eigenvalues of unitary operators must be complex numbers of absolute value 1.

Next, there is the notion of an invariant subspace: a linear subspace \( W \) of \( V \) is invariant for \( T \) if \( TW \subset W \). If \( T \) is Hermitian (respectively, unitary) and \( W \) is an invariant subspace, then the restriction \( T|_W \) of \( T \) to \( W \) is also Hermitian (respectively, unitary). Also, if \( T \) is invertible, as is the case if \( T \) is unitary, then \( W \) is an invariant subspace if and only if \( TW = W \). The following is an easy exercise:

**Proposition 16.** Let \( T \) be either Hermitian or unitary. If \( T \) has an invariant subspace \( W \), then the orthogonal complement\(^1\) \( W^\perp \) of \( W \) is also an invariant subspace for \( T \).

**Proof of Theorems 14–15.** The proof is by induction on the dimension of \( V \). Dimension 1 is trivial. Now every linear operator \( T \) on a complex vector space \( V \) has at least one eigenvector \( v \). (Proof: The characteristic polynomial \( \det(\lambda I - T) \) has a zero, since \( \mathbb{C} \) is algebraically complete. For any such root \( \lambda \) the linear transformation \( \lambda I - T \) must be singular, by Proposition 13, and so the equation \( (\lambda I - T)v = 0 \) has a nonzero solution \( v \).) If \( v \) is an eigenvector of \( T \), then the one-dimensional subspace of \( V \) spanned by \( v \) is invariant, and so its orthogonal complement \( W \) is also invariant. But \( \dim(W) \) is less than \( \dim(V) \), so the induction hypothesis applies to \( T|_W \): in particular, \( W \) has an orthonormal basis consisting of eigenvectors of \( T \). When augmented by the vector \( v/\sqrt{\langle v, v \rangle} \), this gives an orthonormal basis of \( V \) made up entirely of eigenvectors of \( T \).

\(^1\)The orthogonal complement \( W^\perp \) is defined to be the set of all vectors \( u \) such that \( u \) is orthogonal to every \( w \in W \).
3.2. **Orthogonal Matrices: Spectral Theory.** An **orthogonal matrix** is a unitary matrix whose entries are real, equivalently, a real matrix whose transpose is its inverse. Because an orthogonal matrix is unitary, the Spectral Theorem for unitary operators implies that its eigenvalues are complex numbers of modulus 1, and that there is an orthonormal basis of eigenvectors. This doesn’t tell the whole story, however, because in many circumstances one is interested in the action of an orthonormal matrix on a real vector space. An orthogonal linear transformation of a real inner product space need not have real eigenvectors: for instance, the matrix

\[ R_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

acting on \( \mathbb{R}^2 \) has no nonzero eigenvectors unless \( \theta \) is an integer multiple of \( 2\pi \), because \( R_\theta \) rotates every nonzero vector through an angle of \( \theta \).

**Lemma 17.** Let \( T \) be an orthogonal \( n \times n \) matrix, and let \( v \) be a (possibly complex) eigenvector with eigenvalue \( \lambda \). Then the complex conjugate \( \bar{v} \) is also an eigenvector, with eigenvalue \( \bar{\lambda} \).

The proof is trivial, but the result is important because it implies the following structure theorem for orthogonal linear transformations of real inner product spaces.

**Corollary 18.** Let \( T \) be an orthogonal \( n \times n \) matrix acting on the real inner product space \( \mathbb{R}^n \). Then \( \mathbb{R}^n \) decomposes as an orthogonal direct sum of one- or two-dimensional invariant subspaces for \( T \), on each of which \( T \) acts as a rotation matrix \( R_\theta \). In other words, in a suitable orthonormal basis \( T \) is represented by a matrix in block form (where all but the last two blocks are of size \( 2 \times 2 \))

\[
\begin{pmatrix}
R_{\theta_1} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & R_{\theta_2} & 0 & \cdots & 0 & 0 & 0 \\
& & & \ddots & & & \\
0 & 0 & 0 & \cdots & R_{\theta_k} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -I & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & I \\
\end{pmatrix}
\]

**Proof.** The only possible real eigenvalues are \( \pm 1 \). On the space of eigenvectors with eigenvector +1 the matrix \( T \) acts as the identity, and on the space of eigenvectors with eigenvector +1 the matrix \( T \) acts as \(( -1) \times \) the identity. Consequently, each of these subspaces splits as a direct sum of one-dimensional subspaces.

Let \( v = u + i w \) be a complex eigenvector with real and imaginary parts \( u, w \) and eigenvalue \( \lambda = e^{i\theta} \). By Lemma 17, \( \bar{v} = u - i w \) is an eigenvector with eigenvalue \( e^{-i\theta} \). Adding and subtracting the eigenvector equations for these two eigenvectors shows that the two-dimensional real subspace of \( \mathbb{R}^n \) spanned by \( u, w \) is invariant for \( T \), and that the restriction of \( T \) to this subspace is just the rotation by \( \theta \). It is routine to check that the two-dimensional invariant subspaces obtained in this manner are mutually orthogonal, and that each is orthogonal to the one-dimensional invariant subspaces corresponding to eigenvalues \( \pm 1 \).

\[ \square \]

3.3. **Minimax Characterization of Eigenvalues.** Let \( T : V \to V \) be a Hermitian operator on a finite-dimensional vector space \( V \) of dimension \( n \). According to the Spectral Theorem 14, the eigenvalues \( \lambda_i \) of \( T \) are real, and there is an orthonormal basis consisting of eigenvectors \( u_i \). Because the eigenvectors are real, they are linearly ordered:

\[
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.
\]
**Proposition 19.**

\[
\lambda_n = \max_{u: |u|=1} \langle Tu, u \rangle \quad \text{and} \quad \lambda_1 = \min_{u: |u|=1} \langle Tu, u \rangle
\]

**Proof.** Any unit vector \(u\) has an expansion \(u = \sum_i \alpha_i u_i\) in the eigenvectors of \(T\), where the (complex) scalars \(\alpha_i\) satisfy \(\sum_i |\alpha_i|^2 = 1\). It follows that

\[
Tu = \sum_i \alpha_i \lambda_i u_i \implies \langle Tu, u \rangle = \sum_i \lambda_i |\alpha_i|^2
\]

Since \(u\) is a unit vector, the assignment \(i \mapsto |\alpha_i|^2\) is a probability distribution on the index set \([n]\). Clearly, the probability distribution that maximizes the expectation \(\sum_i \lambda_i |\alpha_i|^2\) puts all of its mass on the indices \(i\) for which \(\lambda_i\) is maximal. Thus, the maximal expectation is \(\lambda_n\). Similarly, the minimum expectation is \(\lambda_1\). \(\square\)

The minimax characterization is a generalization of Proposition 19 to the entire spectrum. This characterization is best described using the terminology of game theory. The game is as follows: First, I pick a linear subspace \(W \subset V\) of dimension \(k\); then you pick a unit vector \(u\) in \(W\). I pay you \(\langle Tu, u \rangle\). If we both behave rationally (not always a sure thing on my end, but for the sake of argument let's assume that in this case I do) then I should choose the subspace spanned by the eigenvectors \(u_1, u_2, \ldots, u_k\), and then you should choose \(u = u_k\), so that the payoff is \(\lambda_k\). That this is in fact the optimal strategy is the content of the minimax theorem:

**Theorem 20.** (Minimax Characterization of Eigenvalues)

\[
\lambda_k = \min_{W: \dim(W) = k} \max_{u: |u|=1} \langle Tu, u \rangle = \max_{W: \dim(W) = n-k} \min_{u: |u|=1} \langle Tu, u \rangle
\]

**Proof.** The second equality is obtained from the first by applying the result to the Hermitian operator \(-T\), so only the first equality need be proved. It is clear that the right side of (27) is no larger than \(\lambda_k\) (see the comments preceding the statement of the theorem), so what must be proved is that for any subspace \(W\) of dimension \(k\),

\[
\max_{u: |u|=1} \langle Tu, u \rangle \geq \lambda_k.
\]

Let \(u_i\) be an orthonormal basis of \(V\) such that \(Tu_i = \lambda_i u_i\), where the eigenvalues \(\lambda_i\) are arranged in increasing order as in (24). The subspace \(W\) has an orthonormal basis \(w_i\) of cardinality \(k\), and the vectors in this basis can be expanded as linear combinations of the eigenvectors \(u_i\):

\[
w_i = \sum_{j=1}^n \beta_{i,j} u_j.
\]

Because the vectors \(w_i\) form an orthonormal basis of \(W\), the rows of the matrix \(\beta_{i,j}\) are orthonormal, that is, the \(n\)–vectors \(\beta_i\) with entries \(\beta_{i,j}\) are orthonormal. Consider the \(k \times (k-1)\) matrix

\[
B := \begin{pmatrix}
\beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,k-1} \\
\beta_{2,1} & \beta_{2,2} & \cdots & \beta_{2,k-1} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{k,1} & \beta_{k,2} & \cdots & \beta_{k,k-1}
\end{pmatrix};
\]
the rank of $B$ is at most $k - 1$, so there is a nontrivial linear combination of the rows that sums to zero. Without loss of generality, the coefficients in this linear combination can be scaled so as to form a $k$–vector $\alpha$ of norm 1, that is,

$$\sum_{i=1}^{k} |\alpha_i|^2 = 1 \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i \beta_{i,j} = 0 \; \forall \; 1 \leq j \leq k - 1.$$ 

Thus,

$$w := \sum_{i=1}^{k} \alpha_i w_i = \sum_{i=1}^{k} \sum_{j=1}^{n} \alpha_i \beta_{i,j} u_j = \sum_{i=k}^{n} \sum_{j=k}^{n} \alpha_i \beta_{i,j} u_j := \sum_{j=k}^{n} \gamma_j u_j$$

is a unit vector in $W$ that is orthogonal to the first $k - 1$ eigenvectors $u_j$. Since $w$ is a unit vector, $\sum_j |\gamma_j|^2 = 1$, so it follows that

$$\langle Tw, w \rangle = \sum_{j=k}^{n} \lambda_j |\gamma_j|^2 \geq \lambda_k. \Box$$

**Corollary 21.** (Eigenvalue interlacing) Let $T : V \rightarrow V$ be Hermitian, and let $W \subset V$ be a linear subspace of dimension $n - 1$, where $n = \dim(V)$. Then the eigenvalues of the restriction $T|W$ are interlaced with those of $T$ on $V$: that is, if the eigenvalues of $T$ are $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and the eigenvalues of $T|W$ are $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$, then

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n. \quad (28)$$

**Proof.** It suffices to prove that $\mu_k \geq \lambda_k$, because the reverse inequalities then follow by considering $-T$. By Theorem 20,

$$\lambda_k = \min_{S \subseteq V : \dim(S) = k} \max_{u \in S : \|u\| = 1} \langle Tu, u \rangle \quad \text{and} \quad \mu_k = \min_{S \subseteq W : \dim(S) = k} \max_{u \in S : \|u\| = 1} \langle Tu, u \rangle,$$

where the minima are taken over linear subspaces $S$ of $V$ and $W$, respectively. Since $W \subset V$, every linear subspace of $W$ is a linear subspace of $V$, and so the first min is taken over a larger collection than the second. Thus, $\mu_k \geq \lambda_k. \Box$

3.4. **Empirical spectral distributions.** Recall that the empirical spectral distribution of a diagonalizable matrix is the uniform distribution on eigenvalues (counted according to multiplicity). The empirical spectral distribution is the object of primary interest in the study of random matrices. Thus, it is useful to know how changes in the entries of a matrix affect the empirical spectral distribution. In this section we give two useful bounds on the magnitude of the change in the empirical spectral distribution under certain types of matrix perturbations.

**Definition 7.** The Lévy distance between two probability distributions $\mu, \nu$ on $\mathbb{R}$ with cumulative distribution functions $F_\mu$ and $F_\nu$ is defined to be

$$L(\mu, \nu) := \inf \{ \varepsilon > 0 : F(x - \varepsilon) - \varepsilon \leq G(x) \leq F(x + \varepsilon) + \varepsilon \}. \quad (29)$$

Observe that if $\|F - G\|_\infty < \varepsilon$ then $L(F, G) < \varepsilon$; the converse, however, need not hold. Moreover, the Lévy distance characterizes convergence in distribution, that is, $\lim_{n \rightarrow \infty} L(F_n, F) = 0$ if and only if $F_n \Rightarrow F.$
Corollary 22. (Perturbation Inequality) Let $A$ and $B$ be Hermitian operators on $V = \mathbb{C}^n$ relative to the standard inner product, and let $F^A$ and $F^B$ be their empirical spectral distributions. Then
\begin{equation}
L(F^A, F^B) \leq n^{-1} \text{rank}(A - B).
\end{equation}

**Proof.** It suffices to prove this for Hermitian operators $A, B$ that differ by a rank-1 operator $\Delta = A - B$, because operators that differ by a rank-$k$ operator can be connected by a chain of $k + 1$ Hermitian operators whose successive differences are all rank-1. The operator $\Delta$ is Hermitian, so if it is rank-1 then it has a single nonzero eigenvalue $\delta$, with corresponding eigenvector $w$. Let $W$ be the $(n - 1)$-dimensional subspace orthogonal to $w$; then $\Delta|W = 0$ and so the restrictions $A|W$ and $B|W$ are identical. Let $\mu_1 \leq \cdots \leq \mu_{n-1}$ be the eigenvalues of $A|W = B|W$. By the Eigenvalue Interlacing Theorem (Corollary 21), the (ordered) eigenvalues $\lambda_i^B$ of $B$ are interlaced with the sequence $\mu_i$, and so are the eigenvalues $\lambda_i^A$ of $A$. Consequently, it is impossible for either $\lambda_{k+2}^A < \lambda_k^B$ or $\lambda_{k+2}^B < \lambda_k^A$ to occur for any $k$. □

Proposition 23. Let $A$ and $B$ be $n \times n$ Hermitian matrices with eigenvalues $\lambda_i$ and $\mu_i$, respectively, listed in decreasing order. Then
\begin{equation}
\sum_{i=1}^{n} (\lambda_i - \mu_i)^2 \leq \text{Tr} (A - B)^2.
\end{equation}

**Proof.** Expand the squares on both sides of the inequality to obtain the equivalent statement
\begin{equation}
\sum_{i} (\lambda_i^2 - 2\lambda_i \mu_i + \mu_i^2) \leq \text{Tr} A^2 + \text{Tr} B^2 - 2\text{Tr} AB
\end{equation}
(this also uses the fact that $\text{Tr} AB = \text{Tr} BA$). Since $\text{Tr} A^2 = \sum \lambda_i^2$ and $\text{Tr} B = \sum \mu_i^2$, proving inequality (32) is tantamount to proving
\begin{equation}
\text{Tr} AB \leq \sum \lambda_i \mu_i.
\end{equation}

Neither the trace nor the spectrum of a diagonizable matrix depends on which basis for the vector space is used. Consequently, we can work in the orthonormal basis of eigenvectors of $A$, that is to say, we may assume that $A$ is diagonal. Since $B$ is also Hermitian, there is a unitary matrix $U$ that diagonalizes $B$. Thus,

\begin{align*}
A &= \text{diag}(\lambda_1, \ldots, \lambda_n) \quad \text{and} \\
B &= U \text{diag}(\mu_1, \mu_2, \ldots, \mu_n) U^*,
\end{align*}

and so (33) is equivalent to
\begin{equation}
\sum \sum |U_{i,j}|^2 \lambda_i \mu_j \leq \sum \lambda_i \mu_i
\end{equation}

Now $U$ is unitary, so the matrix $(|U_{i,j}|^2)_{i,j}$ is doubly stochastic ($U$ unitary means that the rows and columns are orthonormal). Hence, inequality (34) will follow from the inequality
\begin{equation}
\sum \sum \lambda_i \mu_j p_{i,j} \leq \sum \lambda_i \mu_i
\end{equation}

where $p_{i,j}$ is any doubly stochastic matrix.

There are various ways to prove (35). Following is a short and painless proof that uses Birkhoff’s theorem on doubly stochastic matrices. Birkhoff’s theorem states that the convex set $\mathcal{F}$ of doubly
stochastic $n \times n$ matrices has as its extreme points the permutation matrices; that is, every doubly stochastic matrix is a convex combination of permutation matrices. To see how Birkhoff’s theorem applies to (35), consider the problem of maximizing the left side over all doubly stochastic matrices $p_{i,j}$. Since the left side is a linear form in the variables $p_{i,j}$, and since the set $\mathcal{S}$ of doubly stochastic $n \times n$ matrices is a convex set, the maximum must occur at one of the extreme points of $\mathcal{S}$. By Birkhoff’s theorem, the extreme points of $\mathcal{S}$ are the permutation matrices (see Exercise 1 below), so it suffices to check that

$$\max_{\sigma} \sum_i \lambda_i \mu_{\sigma(i)} = \sum_i \lambda_i \mu_i,$$

where the max is over the set of all permutations $\sigma$. For this, just check that if $\sigma$ is not the identity permutation, then the left side can be increased (or at least not decreased) by switching two indices $i, i'$ where $\lambda_i, \lambda_{i'}$ and $\mu_i, \mu_{i'}$ are in opposite relative order. \[\square\]

**Exercise 1.** **Prove Birkhoff theorem, that is, show that every doubly stochastic matrix can be written as a convex combination of permutation matrices.**

**Example:**

$$\begin{pmatrix} .3 & .7 \\ .7 & .3 \end{pmatrix} = .3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + .7 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$