Harmonic Functions and Brownian motion

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1 Dynkin’s Formula

Denote by $W_t = (W_1^t, W_2^t, \ldots, W_d^t)$ a standard $d$–dimensional Wiener process on $(\Omega, \mathcal{F}, P)$, and let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be an admissible filtration. Assume that the measurable space $(\Omega, \mathcal{F})$ supports probability measures $P^x$, one for each $x \in \mathbb{R}^d$, such that under $P^x$ the process $W_t$ is a Wiener process with initial state $W_0 = x$. Denote by $\Delta$ the Laplace operator

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

**Theorem 1.** Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a bounded, $C^2$ function whose partial derivatives (up to order 2) are all bounded. Then for any $t \geq 0$ and any $x \in \mathbb{R}^d$,

$$E^x f(W_t) = f(x) + \frac{1}{2} E^x \int_0^t \Delta f(W_s) \, ds. \tag{1}$$

Consequently, the process

$$Y_t^f := f(W_t) - \frac{1}{2} E^x \int_0^t \Delta f(W_s) \, ds \tag{2}$$

is a martingale under $P^x$ relative to $\mathbb{F}$, and so for any stopping time $T$ such that $E^x T < \infty$,

$$E^x f(W_T) = f(x) + \frac{1}{2} E^x \int_0^T \Delta f(W_s) \, ds. \tag{3}$$

The integral formula (3) is a special case of Dynkin’s formula, which in the case of Brownian motion is an easy consequence of Itô’s formula. More on this later. Before proving it, though, let’s look at the formula (1) in another way. Define the heat semigroup to be the one-parameter family $\{T_t\}_{t \geq 0}$ of operators on the space of bounded, continuous functions $g$ by

$$T_t g(x) = E^x g(X_t). \tag{4}$$

It is easy to see (and you should check this) that each $T_t$ maps bounded, continuous functions to bounded, continuous functions, that it does not increase sup norm, and that it is
linear in $g$. The identity (1) can be re-written as

$$T_t g = g + (1/2) \int_0^t T_s \Delta g \, ds \quad \text{or, more simply,}$$

$$T_t = I + (1/2) \int_0^t T_s \Delta \, ds.$$ 

This can be interpreted, at least formally, as an ordinary differential equation

$$\frac{dT_t}{dt} = (1/2) T_t \Delta.$$ 

If you think of $T_t$ and $\Delta$ as matrices (of course they aren’t matrices, but they are linear operators) then this is just the ordinary differential equation for the matrix exponential function. This suggests (at least formally) that we can “solve” for $T_t$:

$$T_t = \exp\{\Delta t/2\}.$$ 

(5)

All of this is, of course, purely formal, but it is nevertheless highly suggestive, and as it turns out can be made into rigorous mathematics. Furthermore, the correspondence between the Laplace operator and Brownian motion extends to similar correspondences between other elliptic, second-order differential operators and other Markov processes known as diffusions. We will return to this later in the course when we have the Itô formula to work with.

Proof of Theorem 1. Theorem 1 as stated here is not especially difficult to prove, and there are a number of different strategies one could use. The shortest and simplest is based on the fact that the transition probabilities of Brownian motion,

$$p_t(x, y) := \exp\{-\|x - y\|^2/2t\}/(2\pi t)^d/2,$$ 

(6)

satisfy the (forward) heat equation

$$\frac{\partial p_t(x, y)}{\partial t} = \frac{1}{2} \Delta_y p_t(x, y)$$ 

(7)

for all $t > 0$ and $x, y \in \mathbb{R}^d$. Here $\Delta_y$ denotes the Laplace operator with respect to the $x$ variables$^1$ At $t = 0$ and $x = y$ the partial derivatives blow up; however, for any $\varepsilon > 0$ all of the partial derivatives (both $\partial/\partial t$ and $\Delta_x$) are uniformly bounded and uniformly continuous on the region $t \in [\varepsilon, \infty)$ and $x, y \in \mathbb{R}^d$. Hence, by the fundamental theorem of calculus and Fubini’s theorem, for $t > \varepsilon > 0$,

$$E^\varepsilon(f(W_t) - f(W_\varepsilon)) = \int_{s=\varepsilon}^t \int_{y \in \mathbb{R}^d} \frac{\partial}{\partial s} p_s(x, y) f(y) \, ds \, dy$$

$$= \frac{1}{2} \int_{s=\varepsilon}^t \int_{y \in \mathbb{R}^d} (\Delta_y p_s(x, y)) f(y) \, dy$$

$^1$Of course the same equation but with $\Delta_y$ replaced by $\Delta_x$ also holds, by symmetry.
Assume now that \( f \) is not only bounded and \( C^2 \), but has compact support. Then the last integral above can be evaluated using integration by parts. (Put a big box around the support of \( f \) and integrate by parts twice in each variable, and use the fact that the boundary terms will all vanish because the boundary lies outside the support of \( f \).) This gives

\[
\frac{1}{2} \int_{s=\varepsilon}^t \int_{y \in \mathbb{R}^d} (\Delta_y p_s(x, y)) f(y) \, dy = \frac{1}{2} \int_{s=\varepsilon}^t \int_{y \in \mathbb{R}^d} p_s(x, y) (\Delta_y f(y)) \, dy = \frac{1}{2} \int_{s=\varepsilon}^t E^x \Delta f(W_s) \, ds.
\]

Thus, for each \( \varepsilon > 0 \),

\[
E^x (f(W_t) - f(W_{\varepsilon})) = \frac{1}{2} \int_{s=\varepsilon}^t E^x \Delta f(W_s) \, ds.
\]

Since both \( f \) and \( \Delta f \) are bounded and continuous, the path-continuity of Brownian motion and the bounded convergence theorem imply that this equality holds also at \( \varepsilon = 0 \). This proves the theorem for functions \( f \) with compact support.

**EXERCISE:** Finish the proof, using the fact that any \( C^2 \) function \( f \) can be approximated on large balls of \( \mathbb{R}^d \) by functions with compact support. (See Lemma 5 of the Appendix.)

**Definition 1.** A **domain** is a nonempty, open, connected subset of \( \mathbb{R}^d \). A domain is said to be **transient** if for every \( x \in D \), Brownian motion started at \( x \) exits \( D \) in finite time with probability one, that is,

\[
P^x \{ \tau_D < \infty \} = 1 \tag{8}
\]

where

\[
\tau_D := \inf\{ t > 0 : W_t \notin D \} \tag{9}
\]

\[
= \infty \quad \text{if there is no such } t > 0.
\]

**Theorem 2.** Let \( D \) be a transient domain, and assume that \( E^x \tau_D < \infty \) for every \( x \in D \). Let \( f : D \to \mathbb{R} \) be a bounded, continuous function defined on the closure of \( D \). If \( f \) is \( C^2 \) in \( D \) with bounded partials of order \( \leq 2 \) then for every \( x \in D \),

\[
E^x f(W_{\tau_D}) = f(x) + \frac{1}{2} E^x \int_0^{\tau_D} \Delta f(W_s) \, ds. \tag{10}
\]

**Proof.** If \( f \) extends to a bounded, \( C^2 \) function on all of \( \mathbb{R}^2 \) with bounded partials of order \( \leq 2 \), then (10) follows directly from Dynkin’s formula (3). Unfortunately, not all functions \( f : D \to \mathbb{R} \) that satisfy the hypotheses of the theorem extend to \( C^\infty \) functions on \( \mathbb{R}^d \). Thus, we must resort to some indirection: we will modify the function \( f \) near the boundary of \( D \) so as to guarantee that it does extend to a bounded, \( C^2 \) on all of \( \mathbb{R}^d \). Fix \( \varepsilon > 0 \), and let \( \varphi(x) \) be a \( C^\infty \) probability density on \( \mathbb{R}^d \) with support contained in the ball

\[
B_\varepsilon(0) = \{ x \in \mathbb{R}^d : |x| < \varepsilon \}.
\]
see Lemma in the Appendix below for a proof that there is such a thing. Let $D_{\varepsilon}$ be the set of all points $x \in D$ such that distance$(x, D^c)$ is at least $\varepsilon$, and define

$$g_{\varepsilon} = \varphi \ast 1_{D_{2\varepsilon}}.$$ 

Then $g_{\varepsilon}$ is $C^\infty$ (because any convolution with a $C^\infty$ function is $C^\infty$), it is identically 1 on $D_{3\varepsilon}$ and identically 0 on $D_{\varepsilon}^c$, it satisfies $0 \leq g \leq 1$ everywhere, and its partial derivatives of all orders are bounded. Define

$$h(x) = f(x)g_{\varepsilon}(x) \quad \text{for all} \quad x \in D,$$

$$= 0 \quad \text{for all} \quad x \in D^c.$$

Then $h$ is $C^2$, bounded, with bounded partials up to order 2, and $h = f$ in $D_{3\varepsilon}$. Thus, Dynkin’s formula applies to the function $h$ with stopping time $\tau_{3\varepsilon} = \tau_{D_{3\varepsilon}}$. Since $f = h$ in $D_{3\varepsilon}$, it follows that

$$E^x f(W_{\tau_{3\varepsilon}}) = f(x) + E^x \int_0^{\tau_{3\varepsilon}} \Delta f(W_s) \, ds.$$

Now as $\varepsilon \to 0$, the stopping times $\tau_{3\varepsilon}$ converge monotonically to $\tau_D$, and so the path-continuity of Brownian motion and the hypothesis that $f$ and its partial derivatives are bounded imply that (10) holds, by the bounded convergence theorem for integrals.

\[\square\]

2 Harmonic Functions

2.1 Representation by Brownian expectations

A function $f : D \to \mathbb{R}$ defined on a domain $D$ of $\mathbb{R}^d$ is said to be harmonic in $D$ if it is $C^2$ and satisfies the Laplace equation

$$\Delta f = 0 \quad \text{in} \quad D.$$

Theorem 3. Assume that $f : \bar{D} \to \mathbb{R}$ is continuous and bounded on the closure of a domain $D$ and harmonic in $D$, with bounded partial derivatives up to order 2. Let $\tau = \tau_D$ be the first exit time from $D$, that is, $\tau = \inf \{ t : W_t \notin D \}$ or $\tau = \infty$ if there is no such $t$. If $P^x \{ \tau < \infty \} = 1$ for every $x \in D$ then

$$f(x) = E^x f(W_{\tau}) \quad (11)$$

Note 1. We do not assume here that $E^x \tau_D < \infty$, as in Theorem 2, but only that the domain $D$ is transient, i.e., that $P^x \{ \tau_D < \infty \} = 1$.

Theorem 3 is a probabilistic form of the Poisson integral formula for harmonic functions. The distribution of $W_{\tau}$ under $P^x$ (which we would usually call the exit distribution) is known in analysis as the harmonic measure

$$\omega(x; dy) := P^x \{ W_{\tau} \in dy \}.$$

In order that this be well-defined, the exit time $\tau$ must be finite with $P^x-$probability 1 for all $x \in \bar{D}$. Call a domain $D$ that has this property a transient domain. Every bounded domain
is transient (why?). In \( \mathbb{R}^2 \) any domain whose complement contains a ball is transient, because two-dimensional Brownian motion visits every ball w.p.1 (see section 3 below), but this isn’t true in higher dimensions \( d \geq 3 \).

For domains \( D \) with smooth boundaries it can be shown that for each \( x \) the harmonic measure \( \omega(x; dy) \) is absolutely continuous with respect to surface area measure on the boundary \( \partial D \); the Radon-Nikodym derivative is known as the Poisson kernel. This kernel can be calculated explicitly for a number of important domains, including balls and half-spaces, and in two dimensions can be gotten for many more domains by conformal mapping. More on this later in the course. In any case the integral formula (11) can be rewritten as

\[
f(x) = \int_{\partial D} f(y) \omega(x; dy). \tag{12}
\]

**Proof of Theorem 3** Let \( D_n \) be an increasing sequence of domains, each satisfying the hypotheses of Theorem 2 whose union is \( D \). (For instance, let \( D_n \) be the intersection of \( D \) with the ball of radius \( n \) centered at the origin.) Denote by \( \tau_n \) the first exit time of the domain \( D_n \); then

\[
\tau_n \uparrow \tau_D,
\]

and so by path-continuity of Brownian motion, \( W(\tau_n) \to W_{\tau_D} \). For each \( n \geq 1 \), Theorem 2 implies that

\[
E^x f(W_{\tau_n}) = f(x) \quad \text{for all } x \in D_n.
\]

By the bounded convergence theorem, it follows that for every \( x \in D_n \) the formula (11) holds. Since the domains \( D_n \) exhaust \( D \), the formula must hold for all \( x \in D \). \( \square \)

**Example 1.** Although the hypothesis that \( f \) is bounded with bounded partials can be relaxed, it cannot be done away with altogether. Consider, for example, the upper half-plane \( D = \{(x, y) \in \mathbb{R}^2 : y > 0\} \) and the function

\[
f(y) = y \quad \text{for } (x, y) \in \bar{D}.
\]

This is obviously harmonic, but clearly the integral formula (11) fails at all points.

### 2.2 Maximum Principle

**Corollary 1.** (Weak Maximum Principle) Let \( D \) be a transient domain. If \( f : \bar{D} \to \mathbb{R} \) is continuous and bounded on the closure of a domain \( D \) and harmonic in \( D \), with bounded partial derivatives up to order 2, then it must attain its maximum value on the boundary \( \partial D \).

**Proof.** This is an obvious consequence of the integral representation (12). \( \square \)

**Example 2.** The hypothesis that \( D \) is transient is needed for this corollary. In section 3 below we will show that Brownian motion in dimensions \( d \geq 3 \) is transient, and in particular that if the initial point is outside the ball \( B_1(0) \) of radius 1 centered at the origin then there is positive probability that the Brownian motion will never hit \( \bar{B}_1(0) \). Thus, the domain
$B_1(0)^c$ is not a transient domain. It will follow from this that if $\tau$ is the first exit time of $B_1(0)^c$ then the function

$$u(x) := P^x \{ \tau = \infty \}$$

is harmonic and positive on $B_1(0)^c$ but identically 0 on the boundary. Thus, the Weak Maximum Principle fails for the region $B_1(0)^c$.

**Corollary 2. (Uniqueness Theorem)** Let $D$ be a transient domain. Suppose that $f : \bar{D} \to \mathbb{R}$ and $g : \bar{D} \to \mathbb{R}$ are both continuous and bounded on the closure of a domain $D$ and harmonic in $D$, with bounded partial derivatives up to order 2. If $f = g$ on $\partial D$ then $f = g$ in $D$.

**Proof.** Apply the maximum principle to the difference $f - g$. \hfill \Box

### 2.3 Harnack Principle

**Proposition 1.** Let $D$ be a transient domain. Then for any two points $x, x' \in D$ the exit distributions $\omega_x(dy) = \omega(x; dy)$ and $\omega_{x'}(dy) = \omega(x'; dy)$ are mutually absolutely continuous on $\partial D$, and the Radon-Nikodym derivative $d\omega_{x'} / d\omega_x$ is bounded away from 0 and $\infty$ on $\partial D$. Moreover, for each $x \in D$,

$$\lim_{x' \to x} \sup_{y \in \partial D} \left| \frac{d\omega_{x'}(y)}{d\omega_x(y)} - 1 \right| = 0.$$  \hspace{1cm} (13)

**Proof.** Fix $x \in D$ and let $r > 0$ be small enough that $\bar{B}_r(x) \subset D$. Let $\tau$ be the time of first exit from $\bar{D}$, and let $\tau(B_r(x))$ be the first exit time of the ball $B_r(x)$. If a Brownian motion starts at $x$ then it must exit the ball $B_r(x)$ before exiting $\bar{D}$, and so $\tau(B_r(x)) \leq \tau$. By the rotational symmetry of Brownian motion, the exit distribution of the ball is the uniform distribution on the boundary sphere. Consequently, by the strong Markov property, for every bounded, continuous function $f : \bar{D} \to \mathbb{R}$,

$$E^x f(W_\tau) = \int_{\partial D} f(y) \omega_x(dy)$$

$$= E^x E^x(f(W_\tau) \mid \mathcal{F}_{\tau(B_r(x))})$$

$$= \frac{1}{|\partial B_r(x)|} \int_{x' \in \partial B_r(x)} \int_{y \in \partial D} f(y) \omega_{x'}(dy).$$

This means that the exit distribution $\omega_x$ of $D$ must be the average, relative to the uniform distribution on $\partial B_r(x)$, of the exit distributions $\omega_{x'}$ of $D$, where $x' \in \partial B_r(x)$. Since this also holds for every ball centered at $x$ of radius $r' \leq r$, it follows that for any $0 \leq r' < r$,

$$\omega_x(dy) = \frac{1}{|A_x(r', r)|} \int_{x' \in A_x(r', r)} \omega_{x'}(dy)$$

where $A_x(r', r)$ denotes the annular region $B_r(x) - B_{r'}(x)$.

Now let $x_*$ be a point of $D$ very near $x$. The exit distribution $\omega_{x_*}$ is, by the same reasoning as above, the average of the exit distributions $\omega_{x'}$ where $x'$ lies in an annular

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2The terminology is a bit unfortunate. The Brownian motion is itself transient, but the domain $B_1(0)^c$ is not transient. In fact, it is the transience of the Brownian motion that causes $B_1(0)^c$ not to be transient.
region centered at $x_*$. For $x_*$ near $x$ the annular region for $x_*$ can be chosen so that it is contained in the annular region for $x$, and hence $\omega_{x_*} \ll \omega_x$. Similarly, $\omega_x \ll \omega_{x_*}$. Moreover, if $x$ and $x_*$ are very close then these annular regions can be chosen so as to have the same inner and outer radii and nearly 100% overlap. This implies the relation (13).

The foregoing argument shows that for every $x \in D$ there is a neighborhood of $x$ in which the exit distributions are all mutually a.c., with Radon-Nikodym derivatives bounded above and below by $1/2$ and 2. Since $D$ is pathwise connected, it follows that all exit distributions are mutually a.c, and that the Radon-Nikodym derivatives are bounded away from 0 and $\infty$.

**Corollary 3.** *(Strong Maximum Principle)* Let $D$ be a transient domain, and let $f$ be a bounded, harmonic function on $D$ that extends continuously to the boundary $\partial D$. If $f$ attains its maximum value at an interior point of $D$, then $f$ must be constant on $\bar{D}$.

**Proof.** Suppose that $f(x) = \max_{y \in \bar{D}} f(y)$. Then by the integral representation formula (12) it must be the case that $f(y) = f(x)$ for $\omega_x$—almost every point $y \in \partial D$. But Proposition 1 implies that all exit distributions $\omega_{x'}$, where $x' \in D$, are mutually a.c. Consequently, for any $x' \in D$,

$$f(y) = f(x) \quad \text{for } \omega_{x'} - \text{almost every } y \in \partial D.$$  

Therefore, by the Poisson integral representation (12), $f'(x) = f(x)$ for every $x' \in D$. 

**Corollary 4.** *(Harnack Principle)* For any domain $D$ and compact subset $K \subset D$ there is a positive constant $C = C_{K,D}$ with the following property. For every harmonic function $u : D \to (0, \infty)$ and all $x, x' \in K$,

$$\frac{u(x')}{u(x)} \leq C \quad (14)$$

**Proof.** The domain $D$ is not assumed to be transient. However, since $K$ is compact there is an open set $U \subset D$ containing $K$ such that $U$ has compact closure; this domain $U$ must therefore be transient. Furthermore, any function $u$ that is harmonic in $D$ is continuous on $\bar{U}$, and so the Poisson integral formula applies in $U$. Now Proposition 1 implies that there is a finite constant $C \geq 1$ such that for all $x, x' \in K$ and $y \in D$,

$$\frac{\omega_{x'}(dy)}{\omega_x(dy)} \leq C.$$  

This, together with the Poisson integral formula (12), implies (14).

**Corollary 5.** Assume that $f : D \to \mathbb{R}$ is continuous and bounded on the closure of a domain $D$ and harmonic in $D$, with bounded partial derivatives up to order 2. If $f$ has a local maximum at an interior point $x \in D$ then $f$ is constant on $D$.

**Proof.** Suppose that $f$ has a local maximum at $x \in D$, and let $B_r(x)$ be a ball centered at $x$ whose closure is contained in $D$. If $r > 0$ is sufficiently small then $f(x) \geq f(y)$ for every $y \in B_r(x)$, because $f(x)$ is a local max. Applying Theorem 3 for $f$ in the ball $B_r(x)$ shows that on the boundary $\partial B_r(x)$ the function $f$ must be identically equal to $f(x)$, because the exit
distribution of Brownian motion started at the center of a ball is the uniform distribution on its boundary, by the rotational symmetry of Brownian motion. It follows that $f$ must be constant on the closed ball $B_r(x)$.

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3 Recurrence and Transience of Brownian Motion

3.1 Brownian motion in $\mathbb{R}^2$

Denote points of $\mathbb{R}^2$ by $z = (x, y)$, and let $|z| = \sqrt{x^2 + y^2}$. The function $\log |z|$ is well-defined and harmonic for $z \in \mathbb{R}^2 - \{(0, 0)\}$, as is easily checked by calculation of the relevant derivatives. Furthermore, it is bounded, along with all partials, on any annulus $A(r_1; r_2) = \{z : 0 < r_1 < |z| < r_2\}$. Let $\tau$ be the exit time for the annulus $A(1/2; 2)$, and let $R_t = |W_t|$. Then $\tau < \infty$ (why?) and $\log R_\tau = \pm \log 2$ according to which of the two boundary circles contains the exit point $W_\tau$. Consequently, Theorem 3 implies that for any initial point $z$ on the circle $|z| = 1$,

\[ 0 = \log |z| = E^z \log |W_\tau| = \log 2(P^z\{|W_\tau| = 2\} - P^z\{|W_\tau| = \frac{1}{2}\}). \]

Hence,

\[ P^z\{|W_\tau| = 2\} = P^z\{|W_\tau| = \frac{1}{2}\} = \frac{1}{2}. \]

Brownian scaling now implies that for any initial point $z$ such that $|z| = 2^k$, where $k \in \mathbb{Z}$, Brownian motion started at $z$ will exit the annulus $A(2^{k-1}; 2^{k+1})$ on either boundary circle $|z| = 2^{k-1}$ or $|z| = 2^{k+1}$ with equal probabilities $1/2$.

Define stopping times $0 = T_0 < T_1 < T_2 < \cdots$ by

\[ T_{n+1} = \min\{t > T_n : |W_t|/|W_{T_n}| = 2^{\pm 1}\}. \]

Then by the strong Markov property and the preceding calculation, the sequence $Y_n = \log |W_{T_n}|$ is a simple random walk on the integers $\mathbb{Z}$ under $P^z$, for any initial point $|z| = 1$. Simple random walk on $\mathbb{Z}$ is recurrent, so it follows that under $P^z$ the Brownian motion $W_t$ must visit every circle $|z'| = 2^k$; in particular, it must enter every neighborhood of the origin $(0, 0)$. On the other hand, $W_t$ cannot visit the point $(0, 0)$ itself, by the following reasoning. For it to do so would require that the random walk $Y_n$ visit every negative integer, and it cannot do this without first returning to 0 infinitely often. This would force the Brownian path to zigzag back and forth between the circles $|z'| = 1/2$ and $|z'| = 1$ infinitely often before hitting the origin, and this would entail a violation of path-continuity if the origin were hit in finite time.

This shows that if Brownian motion starts at a point on the unit circle then with probability one it will never hit the origin, but it will visit every neighborhood of the origin. By the Brownian scaling law it follows that the same is true for any initial point $W_0 = z$ except $z = (0, 0)$. Since Brownian motion started at $z$ is just a translate of Brownian motion started at the origin, this proves the following.
Theorem 4. For any two points \( z \neq z' \),

\[
P^z \{ W_t \text{ visits } B_\varepsilon(z') \} = 1 \quad \text{but} \quad P^z \{ W_t \text{ visits } z' \} = 0.
\]

\( \square \)

Corollary 6. (Liouville’s Theorem.) There is no bounded, non-constant harmonic function \( h : \mathbb{R}^2 \to \mathbb{R} \).

Proof. If there were such a function \( h \), then by Theorem 1 the process \( h(W_t) \) would be a bounded martingale under any \( P^x \). Bounded martingales must converge, a.s. By the recurrence theorem above, \( W_t \) must visit and revisit every neighborhood of every rational point in \( \mathbb{R}^2 \) at indefinitely large times, and so \( h(W_t) \) would have to come arbitrarily close to every value \( h(z) \) of \( h \) at indefinitely large times. This cannot happen unless \( h \) is constant. \( \square \)

3.2 Brownian motion in dimensions \( d \geq 3 \)

Denote points of \( \mathbb{R}^d \) by \( x = (x_1, x_2, \ldots, x_d) \) and let \( |x| = \sqrt{\sum x_i^2} \). The function

\[
h(x) = \frac{1}{|x|^{d-2}}
\]

is well-defined and harmonic in \( \mathbb{R}^d - \{0\} \) (in \( d = 3 \) it is called the Newtonian potential). Let \( W_t \) be a \( d \)-dimensional Brownian motion started at some \( x \in A(r_1, r_2) \) where \( A(r; s) \) denotes the set of all points \( y \in \mathbb{R}^d \) with \( r < |y| < s \), and let \( \tau \) be the time of first exit from \( A(r_1, r_2) \). As in dimension 2, the stopping time \( \tau \) is finite a.s. under \( P^x \) for all \( x \in A(r_1, r_2) \), and the value of \( h(W_\tau) \) must be either \( r_1^{d-1} \) or \( r_2^{d-1} \), depending on whether \( W_\tau \) is on the inner boundary or the outer boundary of \( A(r_1; r_2) \). Thus, by Theorem 3,

\[
|x|^{2-d} = r_1^{2-d} P^x \{|W_\tau| = r_1\} + r_2^{2-d} P^x \{|W_\tau| = r_2\}.
\]

Consequently,

\[
P^x \{|W_\tau| = r_1\} = \frac{|x|^{2-d} - r_2^{2-d}}{r_1^{2-d} - r_2^{2-d}}.
\]

As in dimension 2, the Brownian scaling law shows that for any \( r_1 < |x| < r_2 \) the exit probabilities depend only on the ratios \( |x|/r_1 \) and \( |x|/r_2 \).

Fix an initial point \( x \) with \( |x| = 1 \), and as in \( d = 2 \) define stopping times \( 0 = T_0 < T_1 < T_2 < \cdots \) by

\[
T_{n+1} = \min \{ t > T_n : |W_t|/|W_{T_n}| = 2^{\pm 1} \}.
\]

Then by the strong Markov property, the sequence \( Y_n = \log |W_{T_n}| \) is a nearest neighbor random walk on the integers \( \mathbb{Z} \), with transition probabilities

\[
P^x \{ |W_{T_{n+1}}|/|W_{T_n}| = 2 \} = p = (1 - 2^{2-d})/(2^{d-2} - 2^{2-d}) \quad \text{and} \quad P^x \{ |W_{T_{n+1}}|/|W_{T_n}| = 2^{-1} \} = q = (1 - p).
\]
For any \( d \geq 3 \) the probability \( p > 1/2 \), so the random walk has a positive drift. In particular, by the strong law of large numbers,

\[
\lim_{n \to \infty} \frac{|W_{T_n}|}{n} = p - q > 0,
\]

and so \(|W_{T_n}| \to \infty\). This proves

**Theorem 5.** Brownian motion is transient in dimensions \( d \geq 3 \), in particular, \( \lim_{t \to \infty} |W_t| = \infty \) with \( P^x \)-probability one, for any \( x \in \mathbb{R}^d \).

**Exercise 1.** For any initial point \( x \) such that \(|x| = 1\) and any \( 0 < r < 1 \), calculate the probability that Brownian motion started at \( x \) will ever reach the ball of radius \( r \) centered at the origin.

### 4 The Dirichlet Problem

Given a transient domain \( D \) and a bounded, continuous function \( f : \partial D \to \mathbb{R} \) on the boundary, is it always the case that \( f \) has a continuous extension \( u : \bar{D} \to \mathbb{R} \) that is harmonic in \( D \)? (Henceforth we will call such a function a *harmonic extension* of \( f \).) Are such harmonic extensions unique?

**Example 3.** Let \( D \) be the open unit ball of \( \mathbb{R}^3 \) with the line segment \( L = \{(x,0,0) : 0 \leq x < 1\} \) removed. Then the boundary \( \partial D \) is the union of the unit sphere \( S^2 \) with the segment \( L \). Now a Brownian motion started at a point \( x \in D \) will never hit the line segment \( L \), because its two-dimensional projection doesn’t hit points. Consequently, for each \( x \in D \) the exit distribution \( \omega_x \) will be the same for \( D \) as for the unit ball. This implies that the Dirichlet problem for \( D \) doesn’t always have a solution. In particular, let \( f : \partial D \to [0,1] \) be a continuous function that is identically 0 on the sphere \( S^2 \) but positive on \( L \). If \( u : D \to \mathbb{R} \) were a harmonic extension, then the Poisson integral formula (12) would force \( u(x) = 0 \) at every \( x \in D \). But this doesn’t extend continuously, since \( f \) is positive on \( L \).

This example is somewhat artificial in that part of the boundary is contained in the interior of \( \bar{D} \). One might be tempted to conjecture that barring this sort of behavior would remove the obstruction, but Lebesgue showed that this isn’t the only difficulty.

**Example 4.** ("Lebesgue’s Thorn") Let \( D \) be the unit ball of \( \mathbb{R}^3 \) with the region \( \Theta = \{(x,y,z) : x \geq 0 \text{ and } z^2 + y^2 \leq f(x)\} \) removed. Here \( f : [0,1] \to [0,1] \) should be a nondecreasing, continuous function such that \( f(0) = 0 \). If \( f(x) \to 0 \) rapidly as \( x \to 0 \) then the tip of the thorn won’t be “visible” to Brownian motion started at points very near the origin, that is, there will exist \( \delta > 0 \) such that

\[
P^x \{|W(\tau_D)| = 1\} \geq \delta
\]  

for all points \( x = (x_1,0,0) \) with \( x_1 < 0 \). If this is the case then there exist continuous functions on \( \partial D \) that do not extend continuously to harmonic functions on \( D \). In particular, let \( f = 1 \) on the unit sphere, \( 0 \leq f \leq 1 \) on \( \partial D \), and \( f = 0 \) on \( \partial D \cap \{x \in \mathbb{R}^d : |x| < 1/2\} \). Any harmonic extension of \( f \) would have to take values \( \geq \delta \) on the line segment \( \{(x_1,0,0) : x_1 < 0\} \), and therefore would be discontinuous at the origin.
Exercise 2. Fill in the gap by showing that if \( f \) is suitably chosen then (15) holds.

**Definition 2.** A point \( x \in \partial D \) in the boundary of a domain \( D \) is said to be a *regular point* if \( P^x \{ \tau_D = 0 \} = 1 \), where \( \tau_D = \inf \{ t > 0 : W_t \not\in \bar{D} \} \). A domain \( D \) with nonempty boundary is said to be regular if all its boundary points are regular.

**Theorem 6.** If \( D \) is a regular, transient domain then every bounded, continuous function \( f : \partial D \to \mathbb{R} \) has a unique bounded, harmonic extension.

Both hypotheses on the domain \( D \) (regularity and transience) are needed. Example 1 shows that a regular, transient domain may support several different harmonic extensions of a bounded, continuous function on the boundary, but Theorem 6 implies that only one of them can be bounded. Example 2 shows that the domain must in general be transient in order that the conclusion of Theorem 6 hold.

The Poisson integral formula (11) implies that if a continuous function \( f \) on \( \partial D \) has a bounded harmonic extension then it must be unique. So the real issue to be dealt with in proving Theorem 6 is the existence of a bounded harmonic extension. This will be deduced from the next two lemmas.

**Lemma 1.** Suppose that \( D \) is a regular, transient domain and that \( z \in \partial D \) is a regular boundary point. Then for every bounded, continuous function \( f : \partial D \to \mathbb{R} \) and every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that if \( x \in D \) is a point at distance less than \( \delta \) from \( z \) then

\[
|E^x f(W_{\tau_D}) - f(z)| < \varepsilon. \tag{16}
\]

**Proof.** By hypothesis, if a Brownian motion is started at \( z \) then almost surely it will exit \( \bar{D} \) immediately. Since the complement of \( \bar{D} \) is open, it follows that at arbitrarily small times the path will be at positive distance from \( \bar{D} \). Consequently, for every small \( \varrho > 0 \) there exists \( \delta > 0 \) such that

\[
P^z \{ \text{distance}(W_t, \bar{D}) > 2\delta \text{ for some } t \leq \varrho \} \geq 1 - \varrho. \tag{17}
\]

On the other hand, since Brownian motion has continuous paths, if \( \varrho > 0 \) is small then the path cannot move very far from its starting point, and so for any \( \delta' > 0 \), if \( \varrho > 0 \) is sufficiently small then

\[
P^z \{ \text{distance}(W_t, z) > \delta' \text{ for some } t \leq \varrho \} < \varrho. \tag{18}
\]

Now consider a Brownian motion started at a point \( x \) within distance \( \delta \) of \( z \). This may be obtained by translating a Brownian motion started at \( z \) be \( x - z \); the resulting path will always be within distance \( \delta > 0 \) of the original path. Consequently, (17) implies that

\[
P^x \{ \text{distance}(W_t, \bar{D}) > \delta \text{ for some } t \leq \varrho \} \geq 1 - \varrho,
\]

and so in particular the first exit time of \( \bar{D} \) will be less than \( \varrho \) with probability \( \geq 1 - \varrho \). Similarly, (18) implies that

\[
P^x \{ \text{distance}(W_t, z) > \delta + \delta' \text{ for some } t \leq \varrho \} < \varrho.
\]
Therefore, for any $\delta' > 0$, if $\varrho > 0$ and $\delta > 0$ are sufficiently small then for all $x$ within distance $\delta$ of $x$,

$$P^x \{ \text{distance}(W_{\tau_D}, z) > 2\delta' \} < \varrho.$$ 

Consequently, for any continuous function $f : \partial D \rightarrow \mathbb{R}$,

$$|E^x f(\cap_{\tau_D}) - f(z)| < (1 - \varrho) \sup_{y \in \partial D : d(y, z) < 2\delta'} |f(y) - f(z)| + \varrho(2\|f\|_\infty).$$

Since $\delta' > 0$ and $\varrho > 0$ can be made arbitrarily small, the conclusion (16) follows.

**Definition 3.** A bounded, measurable function $f : D \rightarrow \mathbb{R}$ on a domain $D$ is said to satisfy the mean value property if for every $x \in D$ and every ball $B_\varepsilon(x)$ centered at $x$ such that $B_\varepsilon(x) \subset D$,

$$f(x) = \int_{\partial B_\varepsilon(x)} f(y) \, d\sigma(y)$$

where $\sigma$ is normalized surface area measure on the sphere $\partial B_\varepsilon(x)$.

**Lemma 2.** If $f : D \rightarrow \mathbb{R}$ satisfies the mean value property in $D$ then it is $C^\infty$ and harmonic in $D$.

**Proof.** First, let’s show that $f$ must be $C^\infty$ in $D$. Fix $\delta > 0$ small, and let $\varphi : \mathbb{R}^d \rightarrow [0, \infty)$ be a radial $C^\infty$ probability density with support contained in the ball $B_\varepsilon(0)$. Corollary guarantees that there is such a probability density. For any $x \in D$ whose distance from $\partial D$ is greater than $\delta$, define

$$f_\delta(x) = f * \varphi(x) = \int f(x - y) \varphi(y) \, dy.$$ 

This convolution is well-defined, because $\varphi$ has support contained in $B_\varepsilon(0)$ and $f$ is bounded (and measurable) in $D$, and since $\varphi$ is $C^\infty$, so is $f * \varphi$ (see Lemma 3 of the Appendix). Furthermore, since $\varphi$ is a radial function (that is, $\varphi(y) = \varphi(|y|)$ depends only on $|y|$, and since $f$ satisfies the mean value property in $D$,

$$f(x) = f_\delta(x) \quad \text{for all } x \in D \text{ such that } \text{distance}(x, \partial D) > \delta.$$ 

Hence, because $\delta > 0$ is arbitrary, $f$ is $C^\infty$ in $D$.

It remains to show that $f$ is harmonic in $D$. Now $D$ is not assumed to be a transient domain, so Theorem does not apply directly to $D$. However, it does apply to any ball $B$ whose closure is contained in $D$, and so for any such ball,

$$E^x f(W_{\tau_B}) = f(x) + \frac{1}{2} E^x \int_0^{\tau_B} \Delta f(W_s) \, ds.$$ 

Since $f$ satisfies the mean value property, this implies that

$$E^x \int_0^{\tau_B} \Delta f(W_s) \, ds = 0.$$ (19)
Suppose that at some point \( x \in D \) the Laplacian \( \Delta f(x) \neq 0 \). We have just shown that \( f \) is \( C^\infty \), so the Laplacian \( \Delta f \) is continuous in \( D \). Denote by \( \tau_\varepsilon \) the exit time of the ball \( B_\varepsilon(x) \). Then since \( \Delta f \) is continuous at \( x \),

\[
E^x \int_0^{\tau_\varepsilon} \Delta f(W_s) \, ds \Delta f(x) \sim E^x \tau_\varepsilon
\]
as \( \varepsilon \to 0 \). This contradicts (19).

Proof of Theorem \cite{ref1} Let \( f : \partial D \to \mathbb{R} \) be bounded and continuous. For each \( x \in D \), define \( u(x) = E^x f(W_{\tau_D}) \). Since \( D \) is a transient domain, this expectation is well-defined and finite. Proposition \cite{ref2} ensures that \( u(x) \) is continuous in \( x \), and hence measurable and locally integrable. Moreover, by the strong Markov property, \( u \) satisfies the mean value property in \( D \) (why?) and therefore is harmonic. Finally, Lemma \cite{ref3} implies that \( u \) is a continuous extension of \( f \).

5 Appendix: Smoothing and Localization

The usual way to approximate a not-so-smooth function by a smooth (that is, infinitely differentiable) one is by convolution with a smooth probability density. The next lemma shows that such a convolution will be smooth.

Lemma 3. (Smoothing) Let \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be of class \( C^k \) for some integer \( k \geq 0 \). Assume that \( \varphi \) has compact support. Let \( \mu \) be a finite measure on \( \mathbb{R}^d \) that is supported by a compact subset of \( \mathbb{R}^d \). Then the convolution

\[
\varphi * \mu(x) := \int \varphi(x-y) \, d\mu(y)
\]
is of class \( C^k \), has compact support, and its first partial derivatives satisfy

\[
\frac{\partial}{\partial x_i} \varphi * \mu(x) = \int \frac{\partial}{\partial x_i} \varphi(x-y) \, d\mu(y)
\]

Proof. (Sketch) The shortest proof is by induction on \( k \). For the case \( k = 0 \) it must be shown that if \( \varphi \) is continuous, then so is \( \varphi * \mu \). This follows by a routine argument from the dominated convergence theorem, using the hypothesis that the measure \( \mu \) has compact support (exercise). That \( \varphi * \mu \) has compact support follows easily from the hypothesis that \( \varphi \) and \( \mu \) both have compact support.

Assume now that the result is true for all integers \( k \leq K \), and let \( \varphi \) be a function of class \( C^{K+1} \), with compact support. All of the first partial derivatives \( \varphi_i := \partial \varphi / \partial x_i \) of \( \varphi \) are of class \( C^K \), and so by the induction hypothesis each convolution

\[
\varphi_i * \mu(x) = \int \varphi_i(x-y) \, d\mu(y)
\]

3The hypothesis that \( \varphi \) has compact support isn’t really necessary, but it makes the proof a bit easier.
is of class $C^K$ and has compact support. Thus, to prove that $\varphi \ast \mu$ has continuous partial derivatives of order $K+1$, it suffices to prove the identity (21). By the fundamental theorem of calculus and the fact that $\varphi_i$ and $\varphi$ have compact support, for any $x \in \mathbb{R}^d$

$$\varphi(x) = \int_{-\infty}^{0} \varphi_i(x + te_i) \, dt$$

where $e_i$ is the $i$th standard unit vector in $\mathbb{R}^d$. Convolving with $\mu$ and using Fubini’s theorem shows that

$$\varphi \ast \mu(x) = \int_{-\infty}^{0} \varphi_i(x - y + te_i) \, dt \, d\mu(y)$$

$$= \int_{-\infty}^{0} \int_{-\infty}^{0} \varphi_i(x - y + te_i) \, d\mu(y) \, dt$$

$$= \int_{-\infty}^{0} \varphi_i \ast \mu(x + te_i) \, dt$$

This identity and the fundamental theorem of calculus (this time used in the reverse direction) imply that the identity (21) holds for each $i = 1, 2, \ldots, d$. \hfill \Box

Is there a smooth probability density with compact support? Yes:

**Lemma 4.** There exists an even, $C^\infty$ probability density $\psi(x)$ on $\mathbb{R}$ with support $[-1, 1]$.

**Proof.** Let $U, U_1, U_2, \ldots$ be independent, identically distributed uniform-$[-1, 1]$ random variables and set

$$Y = \sum_{n=1}^{\infty} U_n / 2^n.$$ 

The random variable $Y$ is certainly between $-1$ and $1$, and its distribution is symmetric about 0, so if it has a density $\psi$ the density must be an even function with support $[-1, 1]$. That it does have a density follows because each of the summands $U_n / 2^n$ has a density: in general, if $V, W$ are independent random variables and if $V$ has a density then so does $V + W$, and the density of $V + W$ has at least as many continuous derivatives as does that of $V$. (Just use the convolution formula, and differentiate under the integral.)

Thus, what remains to be shown is the density of $Y$ is infinitely differentiable. For this we will use Fourier analysis. First, the Fourier transform of the uniform distribution on $[-1, 1]$ is

$$\mathcal{F} \{ e^{i\theta U} \} = \frac{\sin \theta}{\theta}, \quad \text{and this implies}$$

$$\mathcal{F} \{ e^{i\theta (U_n / 2^n)} \} = 2^n \frac{\sin(\theta / 2^n)}{\theta}$$

Consequently,

$$E \exp \{ i\theta \sum_{j=1}^{n} U_j / 2^j \} = 2^{(n)} \theta^{-n} \prod_{j=1}^{n} \sin(\theta / 2^j) := \psi_n(\theta)$$
This function decays like $\theta^{-n}$ as $|\theta| \to \infty$, and so for $n \geq 2$ it is integrable. Therefore, the Fourier inversion theorem implies that for any $n \geq 2$ the distribution of the random variable $\sum_{1}^{n} U_j/2^j$ has a density $f_n$. Furthermore, $|\theta|^{n-2} \psi_n(\theta)$ is integrable, so Fourier theory implies that the density $f_n$ has $n-2$ bounded, continuous derivatives. Finally, since the random variable $Y$ is gotten by adding an independent random variable to the finite sum $\sum_{1}^{n} U_j/2^j$, it follows that the distribution of $Y$ has a density, and that it has at least $n-2$ bounded, continuous derivatives.

Say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is a radial function if it is a function only of the distance to the origin, that is,

$$f(x) = f(|x|).$$

**Corollary 7.** For any $\varepsilon > 0$ and $d \geq 1$ there exists a $C^\infty$ radial probability density on $\mathbb{R}^d$ that is supported by the ball of radius $\varepsilon$ centered at the origin.

**Proof.** By Lemma 4 there is an even, $C^\infty$ probability density $\psi$ on the real line supported by the interval $[-1, 1]$. Fix $\delta > 0$, and define $g : \mathbb{R}^d \to \mathbb{R}_+$ by

$$g(x) = \psi(|x|^2/\delta^2).$$

This function is $C^\infty$ (why?), it is obviously nonnegative, and its support is contained in the ball of radius $\delta$ centered at the origin. Moreover, its integral over $\mathbb{R}^d$ is positive, because $\psi$ must be bounded away from 0 on some subinterval of $(0, 1)$. Thus, we can set

$$f(x) = g(x)/\int_{\mathbb{R}^d} f(y) \, dy.$$
Proof. By hypothesis, \( \bar{D} \subset G \) and \( \bar{G} \) is compact. Hence, there exists \( \varepsilon > 0 \) such that no two points \( x \in \partial D \) and \( y \in \partial G \) are at distance less than \( 4\varepsilon \). Let \( F \) be the set of points \( x \) such that \( \text{dist}(x, \partial D) \leq 2\varepsilon \), and let \( \varphi \) be a \( C^\infty \) probability density on \( \mathbb{R}^d \) with support contained in the ball of radius \( \varepsilon \) centered at the origin. Define

\[
f = \varphi \ast 1_F.
\]

By Lemma \([3]\) the function \( f \) is \( C^\infty \). By construction, \( f = 1 \) on \( D \) and \( f = 0 \) outside \( G \). (Exercise: Explain why.)

The existence of \( C^\infty \) probability densities with compact support has another consequence that is of importance in stochastic calculus: it implies that continuous functions can be arbitrarily well-approximated by \( C^\infty \) functions.

**Corollary 8.** Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a continuous function with compact support, and let \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be a \( C^\infty \) probability density that vanishes outside the ball of radius 1 centered at the origin. For each \( \varepsilon > 0 \), define

\[
\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d} \varphi(x/\varepsilon).
\]

Then \( \varphi_\varepsilon \ast f \) converges to \( f \) uniformly as \( \varepsilon \to 0 \):

\[
\lim_{\varepsilon \to 0} \|\varphi_\varepsilon \ast f - f\|_\infty = 0.
\]

**Proof.** If \( Y \) is a \( d \)-dimensional random vector with density \( \varphi \), then for any \( \varepsilon > 0 \) the random vector \( \varepsilon Y \) has density \( \varphi_\varepsilon \). Consequently, for every \( \varepsilon > 0 \) and every \( x \in \mathbb{R}^d \),

\[
\varphi_\varepsilon \ast f(x) = E f(x + \varepsilon Y).
\]

Since \( f \) has compact support, it is bounded and uniformly continuous. Clearly, \( \varepsilon Y \to 0 \) as \( \varepsilon \to 0 \), so the bounded convergence theorem implies that the expectation converges to \( f(x) \) for each \( x \). To see that the convergence is uniform in \( x \), observe that

\[
\varphi_\varepsilon \ast f(x) - f(x) = E \{ f(x + \varepsilon Y) - f(x) \}.
\]

Since \( f \) is uniformly continuous, and since \( |Y| \leq 1 \) with probability 1, for any \( \delta > 0 \) there exists \( \varepsilon_\delta > 0 \) such that for all \( \varepsilon < \varepsilon_\delta \),

\[
|f(x + \varepsilon Y) - f(x)| < \delta
\]

for all \( x \), with probability one. Uniform convergence in (23) now follows immediately. \( \Box \)