Harmonic Functions and Brownian Motion in Several Dimensions

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1 $d-$-Dimensional Brownian Motion

Definition 1. A *standard $d-$dimensional Brownian motion* is an $\mathbb{R}^d-$valued continuous-time stochastic process $\{W_t\}_{t\geq 0}$ (i.e., a family of $d-$dimensional random vectors W_t indexed by the set of nonnegative real numbers t) with the following properties.
(A)′ $W_0 = 0$.
(B)′ With probability 1, the function $t \to W_t$ is continuous in t.
(C)′ The process $\{W_t\}_{t\geq 0}$ has *stationary, independent increments*.
(D)′ The increment $W_{t+s} - W_s$ has the $d-$dimensional normal distribution with mean vector 0 and covariance matrix tI.

The $d-$dimensional normal distribution with mean vector 0 and (positive definite) covariance matrix Σ is the Borel probability measure on $\mathbb{R}^d$ with density

$$\varphi_\Sigma(x) = ((2\pi)^d \det(\Sigma))^{-1/2} \exp\{-x^T\Sigma^{-1}x/2\}; \tag{1}$$

if $\Sigma = tI$ then this is just the product of d one-dimensional Gaussian distributions with mean 0 and variance t. Thus, the existence of $d-$dimensional Brownian motion follows directly from the existence of $1-$dimensional Brownian motion: if $\{W^{(i)}\}_{t\geq 0}$ are independent $1-$dimensional Brownian motions then

$$W_t = \begin{pmatrix} W_t^{(1)} \\ W_t^{(2)} \\ \cdots \\ W_t^{(d)} \end{pmatrix}$$

is a d-dimensional Brownian motion. Observe that when the covariance matrix $\Sigma = I$ is the identity, the transition density $\varphi_{tI}(x) = \varphi_{tI}(|x|)$ depends only on $|x|$, and hence is invariant under orthogonal transformations of $\mathbb{R}^d$. It follows that if $\{W_t\}_{t\geq 0}$ is a $d-$dimensional Brownian motion then for any orthogonal transformation U of $\mathbb{R}^d$ the process $\{UW_t\}_{t\geq 0}$ is also a $d-$dimensional Brownian motion.

Change of Initial Point. An $\mathbb{R}^d-$valued continuous-time stochastic process $\{W_t\}_{t\geq 0}$ is said to be a *$d-$dimensional Brownian motion started at x* if the process $\{W_t - x\}_{t\geq 0}$ is a standard $d-$dimensional Brownian motion. Clearly, on any probability space that supports a standard Brownian motion there is a Brownian motion started at x: just add x to each W_t.

Given the existence of a *standard* $d-$dimensional Brownian motion, it is not difficult to construct a single measurable space $(\Omega, \mathcal{F})$ equipped with measurable $\mathbb{R}^d-$valued random vectors W_t and a family of probability measures P^x on $(\Omega, \mathcal{F})$ such that under P^x the process $(W_t)_{t\geq 0}$ is a $d-$dimensional Brownian motion started at x. In the following sections, I will use the superscript x on probabilities and expectations to denote the measure P^x.

CONSTRUCTION: Let $(\Omega', \mathcal{F}', P')$ be a probability space on which is defined a standard $d-$dimensional Brownian motion $(W'_t)_{t\geq 0}$. Let $(\Omega, \mathcal{F})$ be the measurable space $\Omega = \Omega' \times \mathbb{R}^d$ with $\sigma-$algebra $\mathcal{F} = \mathcal{F}' \times \mathcal{B}_d$, where $\mathcal{B}_d$ the usual Borel field of $\mathbb{R}_d$. Define probability measures $P^x = P' \times \delta_x$ on $(\Omega, \mathcal{F})$, and random vectors $W_t : \Omega \to \mathbb{R}^d$ by

$$W_t(\omega, x) = W'_t(\omega) + x;$$

these are measurable relative to the product $\sigma-$algebra $\mathcal{F} = \mathcal{F}' \times \mathbb{B}_d$, and under P^x the process $(W_t)_{t\geq 0}$ is a $d-$dimensional Brownian motion started at x.

□

2 Dynkin's Formula

2.1 Dynkin's formula for Brownian motion

Assume that the measurable space $(\Omega, \mathcal{F})$ supports probability measures P^x, one for each $x \in \mathbb{R}^d$, such that under P^x the process W_t is a Wiener process with initial state $W_0 = x$. Denote by Δ the Laplace operator

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

Theorem 1. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded, C^2 function whose partial derivatives (up to order 2) are all bounded. Then for any $t \geq 0$ and any $x \in \mathbb{R}^d$,*

$$E^x f(W_t) = f(x) + \frac{1}{2} E^x \int_0^t \Delta f(W_s)\, ds. \tag{2}$$

Consequently, the process

$$Y_t^f := f(W_t) - \frac{1}{2} \int_0^t \Delta f(W_s)\, ds \tag{3}$$

is a martingale under P^x relative to $\mathbb{F}$, and so for any stopping time T such that $E^x T < \infty$,

$$E^x f(W_T) = f(x) + \frac{1}{2} E^x \int_0^T \Delta f(W_s)\, ds. \tag{4}$$

Proof. The assertion that the process Y_t^f is a martingale follows directly from formula (2) and the independent increments property of Brownian motion (as you should check). Given this, the conclusion (4) follows from Doob's optional sampling formula and the dominated convergence theorem, since Δf is bounded.

To prove equation (2), we will use the fact that the transition probabilities of Brownian motion,

$$p_t(x,y) := \exp\{-\|x-y\|^2/2t\}/(2\pi t)^{d/2}, \tag{5}$$

satisfy the (forward) heat equation

$$\frac{\partial p_t(x,y)}{\partial t} = \frac{1}{2}\Delta_y p_t(x,y) \tag{6}$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$. Here Δ_y denotes the Laplace operator with respect to the y variables.[1] At $t = 0$ and $x = y$ the partial derivatives blow up; however, for any $\varepsilon > 0$ all of the partial derivatives (both $(\partial/\partial t)$ and Δ_x) are uniformly bounded and uniformly continuous on the region $t \in [\varepsilon, \infty)$ and $x, y \in \mathbb{R}^d$. Hence, by the fundamental theorem of calculus and Fubini's theorem, for $t > \varepsilon > 0$,

$$\begin{aligned} E^x(f(W_t) - f(W_\varepsilon)) &= \int_{s=\varepsilon}^t \int_{y\in\mathbb{R}^d} \frac{\partial}{\partial s} p_s(x,y) f(y)\, ds \\ &= \frac{1}{2}\int_{s=\varepsilon}^t \int_{y\in\mathbb{R}^d} \left(\Delta_y p_s(x,y)\right) f(y)\, dy \end{aligned}$$

Assume now that f is not only bounded and C^2, but has compact support. Then the last integral above can be evaluated using integration by parts. (Put a big box around the support of f and integrate by parts twice in each variable, and use the fact that the boundary terms will all vanish because the boundary lies outside the support of f.) This gives

$$\begin{aligned} \frac{1}{2}\int_{s=\varepsilon}^t \int_{y\in\mathbb{R}^d} \left(\Delta_y p_s(x,y)\right) f(y)\, dy &= \frac{1}{2}\int_{s=\varepsilon}^t \int_{y\in\mathbb{R}^d} p_s(x,y) \left(\Delta_y f(y)\right)\, dy \\ &= \frac{1}{2}\int_{s=\varepsilon}^t E^x \Delta f(W_s)\, ds. \end{aligned}$$

Thus, for each $\varepsilon > 0$,

$$E^x(f(W_t) - f(W_\varepsilon)) = \frac{1}{2}\int_{s=\varepsilon}^t E^x \Delta f(W_s)\, ds.$$

Since both f and Δf are bounded and continuous, the path-continuity of Brownian motion and the bounded convergence theorem imply that this equality holds also at $\varepsilon = 0$. This proves the theorem for functions f with compact support.

□

Exercise 1. (a) Finish the proof. (b) Better yet, show that the formula (2) holds for any function f such that f and all of its partial derivatives of order ≤ 2 have at most polynomial growth at ∞.

[1] Of course the same equation but with Δ_y replaced by Δ_x also holds, by symmetry.

Exercise 2. Let $u : \mathbb{R}_+ \times \mathbb{R}^d$ be a bounded C^2 function whose first and second partial derivatives are uniformly bounded (or, more generally, have at most polynomial growth as $|x| \to \infty$) on $[0, T] \times \mathbb{R}^d$, for any $0 \leq T < \infty$. Show that for any $t \geq 0$ and any $x \in \mathbb{R}^d$,

$$E^x u(t, W_t) = u(0, x) + E^x \int_0^y \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_x \right) u(s, W_s) \, ds,$$

and conclude that under P^x the process

$$u(t, W_t) - u(0, x) - \left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta_x \right) u(t, W_t)$$

is a martingale.

2.2 Local form of Dynkin's formula

Definition 2. A *domain* is a nonempty, open, connected subset of $\mathbb{R}^d$. A domain is said to be *transient* if for every $x \in D$, Brownian motion started at x exits D in finite time with probability one, that is,

$$P^x\{\tau_D < \infty\} = 1 \tag{7}$$

where

$$\begin{aligned} \tau_D &:= \inf\{t > 0 \, : \, W_t \not\in D\} \\ &= \infty \quad \text{if there is no such } \; t > 0. \end{aligned} \tag{8}$$

Theorem 2. *Let D be a transient domain, and assume that $E^x \tau_D < \infty$ for every $x \in D$. Let $f : \bar{D} \to \mathbb{R}$ be a bounded, continuous function defined on the closure of D. If f is C^2 in D with bounded first and second partial derivatives then for every $x \in D$,*

$$E^x f(W_{\tau_D}) = f(x) + \frac{1}{2} E^x \int_0^{\tau_D} \Delta f(W_s) \, ds. \tag{9}$$

Proof. If f extends to a bounded, C^2 function on all of $\mathbb{R}^2$ with bounded partials of order ≤ 2, then (9) follows directly from Dynkin's formula (4). Unfortunately, not all functions $f : D \to \mathbb{R}$ that satisfy the hypotheses of the theorem extend to C^∞ functions on $\mathbb{R}^d$. Thus, we must resort to some indirection: we will modify the function f near the boundary of D so as to guarantee that it *does* extend to a bounded, C^2 on all of $\mathbb{R}^d$. Fix $\varepsilon > 0$, and let $\varphi(x)$ be a C^∞ probability density on $\mathbb{R}^d$ with support contained in the ball

$$B_\varepsilon(0) = \{x \in \mathbb{R}^d \, : \, |x| < \varepsilon\};$$

see Lemma in the Appendix below for a proof that there is such a thing. Let D_ε be the set of all points $x \in D$ such that distance(x, D^c) is at least ε and such that $|x| \leq 1/\varepsilon$. Note that D_ε is compact, and that $\cup_{\varepsilon > 0} D_\varepsilon = D$. Define

$$g_\varepsilon = \varphi * \mathbf{1}_{D_{2\varepsilon}}.$$

Then g_ε is C^∞ (because any convolution with a C^∞ function is C^∞), it is identically 1 on $D_{3\varepsilon}$ and identically 0 on D_ε^c, it satisfies $0 \leq g \leq 1$ everywhere, and its partial derivatives of all orders are bounded, because D_ε is compact. Define

$$\begin{aligned} h(x) &= f(x)g_\varepsilon(x) \quad \text{for all } \ x \in D, \\ &= 0 \quad \text{for all } \ x \in D^c. \end{aligned}$$

Then h is C^2, bounded, with bounded partials up to order 2, and $h = f$ in $D_{3\varepsilon}$. Thus, Dynkin's formula applies to the function h with stopping time $\tau_{3\varepsilon} = \tau_{D_{3\varepsilon}}$. Since $f = h$ in $D_{3\varepsilon}$, it follows that

$$E^x f(W_{\tau_{3\varepsilon}}) = f(x) + E^x \int_0^{\tau_{3\varepsilon}} \Delta f(W_s)\, ds.$$

Now as $\varepsilon \to 0$, the stopping times $\tau_{3\varepsilon}$ converge monotonically to τ_D, and so the path-continuity of Brownian motion and the hypothesis that f and its partial derivatives are bounded imply that (9) holds, by the bounded convergence theorem for integrals.

$\square$

3 Recurrence and Transience of Brownian Motion

3.1 Brownian motion in $\mathbb{R}^2$

Denote points of $\mathbb{R}^2$ by $z = (x, y)$, and let $|z| = \sqrt{x^2 + y^2}$. The function $\log|z|$ is well-defined and harmonic for $z \in \mathbb{R}^2 - \{(0,0)\}$, as is easily checked by calculation of the relevant derivatives. Furthermore, it is bounded, along with all partials, on any annulus $A(r_1; r_2) = \{z : 0 < r_1 < |z| < r_2\}$. Let τ be the exit time for the annulus $A(\frac{1}{2}; 2)$, and let $R_t = |W_t|$. Then $\tau < \infty$ (why?) and $\log R_\tau = \pm \log 2$ according to which of the two boundary circles contains the exit point W_τ. Consequently, Theorem 2 implies that for any initial point z on the circle $|z| = 1$,

$$0 = \log|z| = E^z \log|W_\tau| = \log 2 (P^z\{|W_\tau| = 2\} - P^z\{|W_\tau| = \frac{1}{2}\}).$$

Hence,

$$P^z\{|W_\tau| = 2\} = P^z\{|W_\tau| = \frac{1}{2}\} = \frac{1}{2}.$$

Brownian scaling now implies that for any initial point z such that $|z| = 2^k$, where $k \in \mathbb{Z}$, Brownian motion started at z will exit the annulus $A(2^{k-1}; 2^{k+1})$ on either boundary circle $|z| = 2^{k-1}$ or $|z| = 2^{k+1}$ with equal probabilities $1/2$.

Define stopping times $0 = T_0 < T_1 < T_2 < \cdots$ by

$$T_{n+1} = \min\{t > T_n : |W_t|/|W_{T_n}| = 2^{\pm 1}\}.$$

Then by the strong Markov property and the preceding calculation, the sequence $Y_n = \log|W_{T_n}|$ is a simple random walk on the integers $\mathbb{Z}$ under P^z, for any initial point $|z| = 1$. Simple random walk on $\mathbb{Z}$ is recurrent, so it follows that under P^z the Brownian motion W_t must visit every circle $|z'| = 2^k$; in particular, it must enter every neighborhood of the origin $(0,0)$. On the other hand, W_t cannot visit the point $(0,0)$ itself, by the following reasoning. For it to do so would require that the random walk Y_n visit every negative integer, and it cannot do this without first returning to 0 infinitely often. This would force the Brownian path to zigzag back and forth between the circles $|z'| = 1/2$ and $|z'| = 1$ infinitely often before hitting the origin, and this would entail a violation of path-continuity if the origin were hit in finite time.

This shows that if Brownian motion starts at a point on the unit circle then with probability one it will never hit the origin, but it will visit every neighborhood of the origin. By the Brownian scaling law it follows that the same is true for any initial point $W_0 = z$ except $z = (0,0)$. Since Brownian motion started at z is just a translate of Brownian motion started at the origin, this proves the following.

Theorem 3. *For any two points $z \neq z'$,*

$$\begin{aligned} P^z\{W_t \ \textit{visits every} \ \ B_\varepsilon(z')\} &= 1 \quad \textit{but} \\ P^z\{W_t \ \textit{visits} \ \ z'\} &= 0. \end{aligned}$$

□

Corollary 1. *(Liouville's Theorem.) There is no bounded, non-constant harmonic function $h : \mathbb{R}^2 \to \mathbb{R}$.*

Proof. If there were such a function h, then by Theorem 1 the process $h(W_t)$ would be a bounded martingale under any P^z. Bounded martingales must converge, a.s. But by the recurrence theorem above, W_t must visit and revisit every neighborhood of every rational point in $\mathbb{R}^2$ at indefinitely large times, and so $h(W_t)$ would have to come arbitrarily close to every value $h(z)$ of h at indefinitely large times. This cannot happen unless h is constant. □

3.2 Brownian motion in dimensions $d \geq 3$

Denote points of $\mathbb{R}^d$ by $x = (x_1, x_2, \ldots, x_d)$ and let $|x| = \sqrt{\sum_i x_i^2}$. The function

$$h(x) = \frac{1}{|x|^{d-2}}$$

is well-defined and harmonic in $\mathbb{R}^d - \{0\}$ (in $d = 3$ it is called the *Newtonian potential*). Let W_t be a $d-$dimensional Brownian motion started at some $x \in A(r_1, r_2)$ where $A(r;s)$ denotes the set of all points $y \in \mathbb{R}^d$ with $r < |y| < s$, and let τ be the time of first exit from $A(r_1, r_2)$. As in dimension 2, the stopping time τ is finite a.s. under P^x for all $x \in A(r_1;r_2)$, and the value of $h(W_\tau)$ must be either r_1^{d-1} or r_2^{d-1}, depending on whether W_τ is on the inner boundary or the outer boundary of $A(r_1;r_2)$. Thus, by Theorem 2,

$$|x|^{2-d} = |r_1|^{2-d} P^x\{|W_\tau| = r_1\} + r_2^{2-d} P^x\{|W_\tau| = r_2\}.$$

Consequently,

$$P^x\{|W_\tau| = r_1\} = \frac{|x|^{2-d} - r_2^{2-d}}{r_1^{2-d} - r_2^{2-d}}.$$

As in dimension 2, the Brownian scaling law shows that for any $r_1 < |x| < r_2$ the exit probabilities depend only on the ratios $|x|/r_1$ and $|x|/r_2$.

Fix an initial point x with $|x| = 1$, and as in $d = 2$ define stopping times $0 = T_0 < T_1 < T_2 < \cdots$ by

$$T_{n+1} = \min\{t > T_n : |W_t|/|W_{T_n}| = 2^{\pm 1}\}.$$

Then by the strong Markov property, the sequence $Y_n = \log |W_{T_n}|$ is a nearest neighbor random walk on the integers $\mathbb{Z}$, with transition probabilities

$$\begin{aligned} P^x\{\frac{|W_{T_{n+1}}|}{|W_{T_n}}| = 2\} &= p = (1 - 2^{2-d})/(2^{d-2} - 2^{2-d}) \quad \text{and} \\ P^x\{\frac{|W_{T_{n+1}}|}{|W_{T_n}}| = 2^{-1}\} &= q = (1 - p). \end{aligned}$$

For any $d \geq 3$ the probability $p > 1/2$, so the random walk has a positive drift. In particular, by the strong law of large numbers,

$$\lim_{n\to\infty} \frac{|W_{T_n}|}{n} = p - q > 0,$$

and so $|W_{T_n}| \to \infty$. This proves

Theorem 4. *Brownian motion is transient in dimensions $d \geq 3$, in particular, $\lim_{t\to\infty} |W_t| = \infty$ with P^x−probability one, for any $x \in \mathbb{R}^d$.*

Exercise 3. (A) For any initial point x such that $|x| = 1$ and any $0 < r < 1$, calculate the probability that Brownian motion started at x will ever reach the ball of radius r centered at the origin.

(B) Let S_n be the $p - q$ random walk started at 0 for some $p \in (0, \frac{1}{2})$. (A $p - q$ random walk is a nearest neighbor random walk on the integers where at each step the probability of moving one step to the right is p and the probability of moving one step to the left is $q = 1 - p$.) Let $M = \max_{n\geq 0} S_n$. Find the distribution of M. HINT: Look for a useful martingale.

4 Harmonic Functions

4.1 Representation by Brownian expectations

A function $f : D \to \mathbb{R}$ defined on a domain D of $\mathbb{R}^d$ is said to be *harmonic* in D if it is C^2 and satisfies the *Laplace equation*

$$\Delta f = 0 \quad \text{in } D.$$

Theorem 5. *Assume that $f : \bar{D} \to \mathbb{R}$ is continuous and bounded on the closure of a domain D and harmonic in D. Let $\tau = \tau_D$ be the first exit time from D, that is, $\tau = \inf\{t : W_t \notin D\}$ or $\tau = \infty$ if there is no such t. If $P^x\{\tau < \infty\} = 1$ for every $x \in D$ then*

$$f(x) = E^x f(W_\tau) \tag{10}$$

Note 1. We do not assume here that $E^x \tau_D < \infty$, as in Theorem 2, but only that the domain D is transient, i.e., that $P^x\{\tau_D < \infty\} = 1$. This is the case, for instance, if $D = H$ is the upper half-space

$$H := \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \, : \, x_d > 0\}$$

(this follows from the recurrence of 1D Brownian motion). Observe that the expected exit time from H is infinite.

Theorem 5 is a probabilistic form of the *Poisson integral formula* for harmonic functions. The distribution of W_τ under P^x (which we would usually call the *exit distribution*) is known in analysis as the *harmonic measure* or *Poisson kernel*

$$\omega(x; dy) := P^x\{W_\tau \in dy\}.$$

In order that this be a probability distribution (rather than merely a sub-probability distribution), the exit time τ must be finite with P^x-probability 1 for all $x \in \bar{D}$. Call a domain D that has this property a *transient domain*. Every *bounded* domain is transient (why?). In $\mathbb{R}^2$ any domain whose complement contains a ball is transient, because two-dimensional Brownian motion visits every ball w.p.1 (see section 3 below), but this isn't true in higher dimensions $d \geq 3$.

For domains D with *smooth* boundaries it can be shown that for each x the harmonic measure $\omega(x; dy)$ is absolutely continuous with respect to surface area measure on the boundary ∂D; the Radon-Nikodym derivative is known as the *Poisson kernel*. This kernel can be calculated explicitly for a number of important domains, including balls and halfspaces, and in two dimensions can be gotten for many more domains by *conformal mapping*. In any case the integral formula (10) can be rewritten as

$$f(x) = \int_{\partial D} f(y) \, \omega(x; dy). \tag{11}$$

Exercise 4. Let $d = 2$, and consider the domain $D = \mathbb{H}$, the upper half-plane. Show that the Poisson kernel is the *Cauchy distribution* by following the steps below.

(A) Let $W_t = (X_t, Y_t)$. Show that for any $\theta \in \mathbb{R}$ and any $t \geq 0$

$$E^{(x,y)} \exp\{i\theta X_t - \theta Y_t\} = e^{i\theta x - \theta y}.$$

(B) Conclude that for any $x, \theta \in \mathbb{R}$ and $y > 0$, under $P^{(x,y)}$ the process $\{e^{i\theta X_t - \theta Y_t}\}$ is a martingale.

(C) Using (B), show that for $x, \theta \in \mathbb{R}$ and $y > 0$,

$$E^{(x,y)} e^{i\theta X_\tau} = e^{i\theta x - |\theta| y}.$$

NOTE: Somewhere you should explain where the absolute value comes in.

(D) Use Fourier inversion to calculate the density of the random variable X_τ.

Proof of Theorem 5. Let D_n be an increasing sequence of domains, each satisfying the hypotheses of Theorem 2, whose union is D. (For instance, let D_n be the intersection of D with the ball of radius n centered at the origin.) Denote by τ_n the first exit time of the domain D_n; then

$$\tau_n \uparrow \tau_D,$$

and so by path-continuity of Brownian motion, $W(\tau_n) \to W_{\tau_D}$. For each $n \geq 1$, Theorem 2 implies that

$$E^x f(W_{\tau_n}) = f(x) \quad \text{for all } \; x \in D_n.$$

By the bounded convergence theorem, it follows that for every $x \in D_n$ the formula (10) holds. Since the domains D_n exhaust D, the formula must hold for all $x \in D$. $\square$

Example 1. Although the hypothesis that f is bounded can be relaxed, it cannot be done away with altogether. Consider, for example, the upper half-plane $D = \{(x, y) \in \mathbb{R}^2 \, : \, y > 0\}$ and the function

$$f(y) = y \quad \text{for} \;\; (x, y) \in \bar{D}.$$

This is obviously harmonic, but clearly the integral formula (10) fails at all points.

5 The Dirichlet Problem

Given a transient domain D and a bounded, continuous function $f : \partial D \to \mathbb{R}$ on the boundary, is it always the case that f has a unique continuous extension $u : \bar{D} \to \mathbb{R}$ that is harmonic in D? (Henceforth we will call such a function a *harmonic extension* of f.) Are such harmonic extensions unique? These questions are purely analytic. However, Theorem 5 shows that the existence of harmonic functions in a domain is intimately tied up with the exit behavior of Brownian motion: in particular, it implies that if there is a bounded harmonic function with prescribed continuous boundary values then it must satisfy formula (10), so there can be no other harmonic extension.

5.1 Brownian motion and the mean value property

Theorem 6. *Let D be a transient domain and $f : \partial D \to \mathbb{R}$ a bounded, Borel measurable function on the boundary. For every $x \in D$ define*

$$u(x) = E^x f(W_\tau) \tag{12}$$

where W_t is $d-$dimensional Brownian motion started at x and $\tau = \tau_D$ is the first exit time from D. Then u is harmonic in D.

Proof. Let $\varepsilon > 0$ be sufficiently small that the ball $B_{2\varepsilon}(x)$ is contained in D, and let T_ε be the first time that the Brownian motion exits the ball $B_\varepsilon(x)$. By path-continuity, Brownian motion started at x must exit $B_\varepsilon(x)$ before it exits D, and at the exit time T_ε must be located on the sphere $\partial B_\varepsilon(x)$. Hence, by the Strong Markov property,

$$E^x f(W_\tau) = E^x E^{W_{T_\varepsilon}} f(W_\tau).$$

Because standard $d-$dimensional Brownian motion is rotationally symmetric, the distribution of the exit point $W(T_\varepsilon)$ under P^x must be the uniform distribution on the sphere $\partial B_\varepsilon(x)$. Thus, the function $u(x)$ must satisfy the *mean value property*

$$u(x) = \int_{\partial B_\varepsilon(x)} u(y)\,\sigma(dy) \tag{13}$$

where σ is the uniform distribution on $\partial B_\varepsilon(x)$ (i.e., normalized surface area measure).

We will prove in Lemma 1 below that any function that satisfies the mean value property in D must be C^∞. Given this, it is easy to show, using Dynkin's formula (9), that any such function must in fact be harmonic. Dynkin's formula implies that if u is C^∞ in an open domain containing the closed ball $\bar{B}_\varepsilon(x)$ then

$$\begin{aligned} u(x) &= E^x u(W(T_\varepsilon)) - \frac{1}{2}E^x \int_0^{T_\varepsilon} \Delta u(W_s)\,ds \\ &= \int_{\partial B_\varepsilon(x)} u(y)\,\sigma(dy) - \frac{1}{2}E^x \int_0^{T_\varepsilon} \Delta u(W_s)\,ds \end{aligned}$$

By the mean value property, the last integral in this equality must be zero for all sufficiently small $\varepsilon > 0$. Since Δu is continuous (by Lemma 1), it follows that $\Delta u(x) = 0$, as the following argument shows. If $\Delta u(x) > 0$ (it has to be either positive or negative), there would exist $\delta, \varepsilon > 0$ such that $\Delta u > \delta$ throughout $B_\varepsilon(x)$. But this would imply that

$$E^x \int_0^{T_\varepsilon} \Delta u(W_s)\,ds \geq \delta E^x T_\varepsilon > 0.$$

□

Lemma 1. *If $u : D \to \mathbb{R}$ satisfies the mean value property in D then it is C^∞.*

Proof. Fix $\varepsilon > 0$ small enough that the ball $B_\varepsilon(x)$ is entirely contained in D, and let $\varphi : \mathbb{R}^d \to [0, \infty)$ be a *radial*[2] C^∞ probability density with support contained in the ball $B_\varepsilon(0)$. Corollary 8 guarantees that there is such a probability density. For any $x \in D$ whose distance from ∂D is greater than ε, define

$$u * \varphi(x) = \int u(x-y)\varphi(y)\,dy. \tag{14}$$

This convolution is well-defined, because φ has support contained in $B_\delta(0)$ and u is bounded (and measurable) in D. Furthermore, since φ is a radial function, the convolution (14) is an average of averages over spheres centered at x. Consequently, since u satisfies the mean value property in D,

$$u(x) = u * \varphi(x).$$

But $u * \varphi$ is C^∞, because any convolution with a C^∞ density with compact support is C^∞ (see Lemma 3 of the Appendix). □

[2] A *radial function* is a function whose value at $y \in \mathbb{R}^d$ depends only on $|y|$.

5.2 Coupling

The term *coupling* is loosely used to denote constructions in probability theory in which several (often only two) stochastic processes or other random objects are built in such a way that comparisons between them can be easily made. Here we will use couplings that involve Brownian motions started at different initial points, with the aim of using the coupling to compare hitting probabilities and distributions.

Proposition 1. *For any two points $x, y \in \mathbb{R}^d$ there exist, on some probability space $(\Omega, \mathcal{F}, P)$, stochastic processes $\{W_t^x\}_{t\geq 0}$ and $\{W_t^y\}_{t\geq 0}$ such that*

(a) $\{W_t^x\}_{t\geq 0}$ is a $d-$dimensional Brownian motion with initial point x;
(b) $\{W_t^y\}_{t\geq 0}$ is a $d-$dimensional Brownian motion with initial point y; and
(c) with probability one, $W_t^x = W_t^y$ for all sufficiently large t.

Proof. Homework. □

Corollary 2. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$ then the coupling in Proposition 1 can be arranged so that*

$$P\{W_t^x = W_t^y \quad \text{for all} \ \ t \geq \varepsilon\} \geq 1 - \varepsilon. \tag{15}$$

Proof. Brownian scaling. □

5.3 The Dirichlet problem in a regular domain

Definition 3. Let D be a domain in $\mathbb{R}^d$ and let $f : \partial D \to \mathbb{R}$ be a bounded, continuous function on the boundary. A continuous function $u : \bar{D} \to \mathbb{R}$ is said to be a solution to the *Dirichlet problem* with boundary data f if $u = f$ on ∂D and if u is harmonic in D.

Even for bounded domains D the Dirichlet problem need not have a solution, as the following example shows.

Example 2. Let D be the open unit ball of $\mathbb{R}^3$ with the line segment $L = \{(x, 0, 0) : 0 \leq x < 1\}$ removed. Then the boundary ∂D is the union of the unit sphere S^2 with the segment L. A Brownian motion started at a point $x \in D$ will never hit the line segment L, because its two-dimensional projection doesn't hit points. Consequently, for each $x \in D$ the exit distribution ω_x will be the same for D as for the unit ball. This implies that the Dirichlet problem for D doesn't always have a solution. In particular, let $f : \partial D \to [0, 1]$ be a continuous function that is identically 0 on the sphere S^2 but positive on L. If $u : D \to \mathbb{R}$ were a harmonic extension, then the Poisson integral formula (11) would force $u(x) = 0$ at every $x \in D$. But this doesn't extend continuously, since f is positive on L.

This example is somewhat artificial in that part of the boundary is contained in the interior of $\bar{D}$. One might be tempted to conjecture that barring this sort of behavior would remove the obstruction, but Lebesgue showed that this isn't the only difficulty.

Example 3. ("Lebesgue's Thorn") Let D be the unit ball of $\mathbb{R}^3$ with the region $\Theta = \{(x, y, z) : x \geq 0 \ \text{ and } \ z^2 + y^2 \leq f(x)\}$ removed. Here $f : [0, 1] \to [0, .1]$ should be a nondecreasing, continuous function such that $f(0) = 0$. If $f(x) \to 0$ rapidly as $x \to 0$ then the tip of the thorn won't be "visible" to Brownian motion started at points very near the origin, that is, there will exist $\delta > 0$ such that

$$P^x\{|W(\tau_D)| = 1\} \geq \delta \tag{16}$$

for all points $x = (x_1, 0, 0)$ with $x_1 < 0$. If this is the case then there exist continuous functions on ∂D that do not extend continuously to harmonic functions on D. In particular, let $f = 1$ on the unit sphere, $0 \leq f \leq 1$ on ∂D, and $f = 0$ on $\partial D \cap \{x \in \mathbb{R}^{\nVdash} : |x| \leq 1/2\}$. Any harmonic extension of f would have to take values $\geq \delta$ on the line segment $\{(x_1, 0, 0)x_{1<0}\}$, and therefore would be discontinuous at the origin.

Exercise 5. Fill in the gap by showing that if f is suitably chosen then (16) holds.

Definition 4. A point $x \in \partial D$ in the boundary of a domain D is said to be a *regular point for D^c* if $P^x\{\tau_D = 0\} = 1$, where $\tau_D = \inf\{t > 0 : W_t \in D^c\}$. A domain D with nonempty boundary is said to be regular if all its boundary points are regular for D^c.

Theorem 7. *If D is a regular, transient domain then every bounded, continuous function $f : \partial D \to \mathbb{R}$ has a unique bounded, harmonic extension.*

Both hypotheses on the domain D (regularity and transience) are needed. Example 1 shows that a regular, transient domain may support several different harmonic extensions of a bounded, continuous function on the boundary, but Theorem 7 implies that only one of them can be bounded. Example 4 shows that the domain must in general be transient in order that the conclusion of Theorem 7 hold.

The Poisson integral formula (10) implies that if a continuous function f on ∂D has a bounded harmonic extension then it must be unique. So the real issue to be dealt with in proving Theorem 7 is the *existence* of a bounded harmonic extension. This will be deduced from the next two lemmas.

Lemma 2. *Suppose that D is a regular, transient domain and that $z \in \partial D$ is a regular boundary point for D^c. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in D$ is a point at distance less than δ from z then*

$$P^x\{|W_{\tau_D} - z| > \varepsilon\} < \varepsilon. \tag{17}$$

Proof. By hypothesis, if a Brownian motion is started at z then almost surely it will enter D^c immediately. Thus, for any $\varepsilon > 0$ and $\beta > 0$ there exists $\gamma > 0$ such that

$$P^z\{W_t \in D^c \quad \text{for some} \;\; t \in [\gamma, \beta]\} > 1 - \varepsilon.$$

Moreover, because sample paths of Brownian motion are continuous, if $\beta > 0$ is sufficiently small then for all x,

$$P^x\{\max_{t \leq \beta} |W_t - x| > \varepsilon\} < \varepsilon.$$

By Corollary 2, there exists $0 < \delta < \varepsilon$ so small that for any point $x \in D$ within distance δ of z a coupling between Brownian motions W_t^x and W_t^z started at x and z, respectively, can be arranged in such a way that with probability at least $1 - \varepsilon$,

$$W_t^x = W_t^z \quad \text{for all} \;\; t \geq \gamma.$$

This implies that for any such x,

$$P^x\{\tau_D \leq \beta\} > 1 - 2\varepsilon.$$

By our choice of β, it now follows that for any $x \in D$ at distance less than δ from z,

$$P^x\{|W_{\tau_D} - z| > 2\varepsilon\} < 3\varepsilon.$$

□

Proof of Theorem 7. Let $f : \partial D \to \mathbb{R}$ be bounded and continuous. For each $x \in D$, define $u(x) = E^x f(W_{\tau_D})$. Since D is a transient domain, this expectation is well-defined and finite. By Theorem 6, u is harmonic in D, and since f is bounded, u must also be bounded. Lemma 2 implies that if all points of the boundary are regular then u is a continuous extension of f. □

5.4 Maximum Principle

The usefulness of Brownian motion in studying harmonic functions is not limited to

Corollary 3. *(Weak Maximum Principle) Let D be a transient domain. If $f : \bar{D} \to \mathbb{R}$ is continuous and bounded on the closure of a domain D and harmonic in D, with bounded partial derivatives up to order 2, then it must attain its maximum value on the boundary ∂D.*

Proof. This is an obvious consequence of the integral representation (**??**). □

Example 4. The hypothesis that D is transient is needed for this corollary. In section 3 below we will show that Brownian motion in dimensions $d \geq 3$ is transient, and in particular that if the initial point is outside the ball $B_1(0)$ of radius 1 centered at the origin then there is positive probability that the Brownian motion will never hit $\bar{B}_1(0)$. Thus, the domain $\bar{B}_1(0)^c$ is *not* a transient domain.[3] It will follow from this that if τ is the first exit time of $\bar{B}_1(0)^c$ then the function

$$u(x) := P^x\{\tau = \infty\}$$

is harmonic and positive on $\bar{B}_1(0)^c$ but identically 0 on the boundary. Thus, the Weak Maximum Principle fails for the region $\bar{B}_1(0)^c$.

Corollary 4. *(Uniqueness Theorem) Let D be a transient domain. Suppose that $f : \bar{D} \to \mathbb{R}$ and $g : \bar{D} \to \mathbb{R}$ are both continuous and bounded on the closure of a domain D and harmonic in D, with bounded partial derivatives up to order 2. If $f = g$ on ∂D then $f = g$ in D.*

Proof. Apply the maximum principle to the difference $f - g$. □

[3] The terminology is a bit unfortunate. The Brownian motion is itself transient, but the domain $\bar{B}_1(0)^c$ is not transient. In fact, it is the transience of the Brownian motion that causes $\bar{B}_1(0)^c$ not to be transient.

5.5 Harnack Principle

Proposition 2. *Let D be a transient domain. Then for any two points $x, x' \in D$ the exit distributions $\omega_x(dy) = \omega(x; dy)$ and $\omega_{x'}(dy) = \omega(x'; dy)$ are mutually absolutely continuous on ∂D, and the Radon-Nikodym derivative $d\omega'_x/d\omega_x$ is bounded away from 0 and ∞ on ∂D. Moreover, for each $x \in D$,*

$$\lim_{x' \to x} \sup_{y \in \partial D} \left| \frac{d\omega_{x'}(y)}{d\omega_x(y)} - 1 \right| = 0. \tag{18}$$

Proof. Fix $x \in D$ and let $r > 0$ be small enough that $\bar{B}_r(x) \subset D$. Let τ be the time of first exit from $\bar{D}$, and let $\tau(B_r(x))$ be the first exit time of the ball $\bar{B}_r(x)$. If a Brownian motion starts at x then it must exit the ball $\bar{B}_r(x)$ before exiting $\bar{D}$, and so $\tau(B_r(x)) \leq \tau$. By the rotational symmetry of Brownian motion, the exit distribution of the ball is the uniform distribution on the boundary sphere. Consequently, by the strong Markov property, for every bounded, continuous function $f : \bar{D} \to \mathbb{R}$,

$$\begin{aligned} E^x f(W_\tau) &= \int_{\partial D} f(y)\, \omega_x(dy) \\ &= E^x E^x(f(W_\tau) \,|\, \mathcal{F}_{\tau(B_r(x))}) \\ &= \frac{1}{|\partial B_r(x)|} \int_{x' \in \partial B_r(x)} \int_{y \in \partial D} f(y)\, \omega_{x'}(dy). \end{aligned}$$

This means that the exit distribution ω_x of D must be the average, relative to the uniform distribution on $\partial B_r(x)$, of the exit distributions $\omega_{x'}$ of D, where $x' \in \partial B_r(x)$. Since this also holds for every ball centered at x of radius $r' \leq r$, it follows that for any $0 \leq r' < r$,

$$\omega_x(dy) = \frac{1}{|A_x(r', r)|} \int_{x' \in A_x(r', r)} \omega_{x'}(dy)$$

where $A_x(r', r)$ denotes the annular region $B_r(x) - B_{r'(x)}$.

Now let x_* be a point of D very near x. The exit distribution ω_{x_*} is, by the same reasoning as above, the average of the exit distributions $\omega_{x'}$ where x' lies in an annular region centered at x_*. For x_* near x the annular region for x_* can be chosen so that it is contained in the annular region for x, and hence $\omega_{x_*} \ll \omega_x$. Similarly, $\omega_x \ll \omega_{x_*}$. Moreover, if x and x_* are very close then these annular regions can be chosen so as to have the same inner and outer radii and nearly 100% overlap. This implies the relation (18).

The foregoing argument shows that for every $x \in D$ there is a neighborhood of x in which the exit distributions are all mutually a.c., with Radon-Nikodym derivatives bounded above and below by $1/2$ and 2. Since D is pathwise connected, it follows that all exit distributions are mutually a.c, and that the Radon-Nikodym derivatives are bounded away from 0 and ∞.

$\square$

Corollary 5. *(Strong Maximum Principle) Let D be a transient domain, and let f be a bounded, harmonic function on D that extends continuously to the boundary ∂D. If f attains its maximum value at an interior point of D, then f must be constant on $\bar{D}$.*

Proof. Suppose that $f(x) = \max_{y\in\bar{D}} f(y)$. Then by the integral representation formula (11) it must be the case that $f(y) = f(x)$ for ω_x-almost every point $y \in \partial D$. But Proposition 2 implies that all exit distributions $\omega_{x'}$, where $x' \in D$, are mutually a.c. Consequently, for any $x' \in D$,

$$f(y) = f(x) \quad \text{for } \omega_{x'} - \text{almost every } \; y \in \partial D.$$

Therefore, by the Poisson integral representation (11), $f('x) = f(x)$ for every $x' \in D$. □

Corollary 6. *(Harnack Principle) For any domain D and compact subset $K \subset D$ there is a positive constant $C = C_{K,D}$ with the following property. For every harmonic function $u : D \to (0, \infty)$ and all $x, x' \in K$,*

$$\frac{u(x')}{u(x)} \leq C \tag{19}$$

Proof. The domain D is not assumed to be transient. However, since K is compact there is an open set $U \subset D$ containing K such that U has compact closure; this domain U must therefore be transient. Furthermore, any function u that is harmonic in D is continuous on $\bar{U}$, and so the Poisson integral formula applies in U. Now Proposition 2 implies that there is a finite constant $C \geq 1$ such that for all $x, x' \in K$ and $y \in D$,

$$\frac{\omega_{x'}(dy)}{\omega_x(dy)} \leq C.$$

This, together with the Poisson integral formula (11), implies (19).

□

Corollary 7. *Assume that $f : \bar{D} \to \mathbb{R}$ is continuous and bounded on the closure of a domain D and harmonic in D, with bounded partial derivatives up to order 2. If f has a local maximum at an interior point $x \in D$ then f is constant on D.*

Proof. Suppose that f has a local maximum at $x \in D$, and let $B_r(x)$ be a ball centered at x whose closure is contained in D. If $r > 0$ is sufficiently small then $f(x) \geq f(y)$ for every $y \in B_r(x)$, because $f(x)$ is a local max. Applying Theorem 5 for f in the ball $B_r(x)$ shows that on the boundary $\partial B_r(x)$ the function f must be identically equal to $f(x)$, because the exit distribution of Brownian motion started at the center of a ball is the uniform distribution on its boundary, by the rotational symmetry of Brownian motion. It follows that f must be constant on the closed ball $\bar{B}_r(x)$.

□

6 Harmonic Measure and Equilibrium Distribution

7 Appendix: Smoothing and Localization

The usual way to approximate a not-so-smooth function by a smooth (that is, infinitely differentiable) one is by *convolution* with a smooth probability density. The next lemma shows that such a convolution will be smooth.

Lemma 3. *(Smoothing) Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be of class C^k for some integer $k \geq 0$. Assume that φ has compact support.*[4] *Let μ be a finite measure on $\mathbb{R}^d$ that is supported by a compact subset of $\mathbb{R}^d$. Then the convolution*

$$\varphi * \mu(x) := \int \varphi(x-y)\, d\mu(y) \tag{20}$$

is of class C^k, has compact support, and its first partial derivatives satisfy

$$\frac{\partial}{\partial x_i} \varphi * \mu(x) = \int \frac{\partial}{\partial x_i} \varphi(x-y)\, d\mu(y) \tag{21}$$

Proof. (Sketch) The shortest proof is by induction on k. For the case $k = 0$ it must be shown that if φ is continuous, then so is $\varphi * \mu$. This follows by a routine argument from the dominated convergence theorem, using the hypothesis that the measure μ has compact support (exercise). That $\varphi * \mu$ has compact support follows easily from the hypothesis that φ and μ both have compact support.

Assume now that the result is true for all integers $k \leq K$, and let φ be a function of class C^{K+1}, with compact support. All of the first partial derivatives $\varphi_i := \partial\varphi/\partial x_i$ of φ are of class C^K, and so by the induction hypothesis each convolution

$$\varphi_i * \mu(x) = \int \varphi_i(x-y)\, d\mu(y)$$

is of class C^K and has compact support. Thus, to prove that $\varphi * \mu$ has continuous partial derivatives of order $K+1$, it suffices to prove the identity (21). By the fundamental theorem of calculus and the fact that φ_i and φ have compact support, for any $x \in \mathbb{R}^d$

$$\varphi(x) = \int_{-\infty}^{0} \varphi_i(x + te_i)\, dt$$

where e_i is the ith standard unit vector in $\mathbb{R}^d$. Convolving with μ and using Fubini's theorem shows that

$$\begin{aligned}
\varphi * \mu(x) &= \int \int_{-\infty}^{0} \varphi_i(x - y + te_i)\, dt d\mu(y) \\
&= \int_{-\infty}^{0} \int \varphi_i(x - y + te_i)\, d\mu(y) dt \\
&= \int_{-\infty}^{0} \varphi_i * \mu(x + te_i)\, dt
\end{aligned}$$

This identity and the fundamental theorem of calculus (this time used in the reverse direction) imply that the identity (21) holds for each $i = 1, 2, \ldots, d$. □

Is there a smooth probability density with *compact support*? Yes:

Lemma 4. *There exists an even, C^∞ probability density $\psi(x)$ on $\mathbb{R}$ with support $[-1, 1]$.*

[4] The hypothesis that φ has compact support isn't really necessary, but it makes the proof a bit easier.

Proof. Let $U, U_1, U_2, \ldots$ be independent, identically distributed uniform-$[-1,1]$ random variables and set

$$Y = \sum_{n=1}^{\infty} U_n/2^n.$$

The random variable Y is certainly between -1 and 1, and its distribution is symmetric about 0, so if it has a density ψ the density must be an even function with support $[-1,1]$. That it does have a density follows because each of the summands $U_n/2^n$ has a density: in general, if V, W are independent random variables and if V has a density then so does $V+W$, and the density of $V+W$ has at least as many continuous derivatives as does that of V. (Just use the convolution formula, and differentiate under the integral.)

Thus, what remains to be shown is the density of Y is infinitely differentiable. For this we will use Fourier analysis. First, the Fourier transform of the uniform distribution on $[-1,1]$ is

$$Ee^{i\theta U} = \frac{\sin\theta}{\theta}, \quad \text{and this implies}$$
$$Ee^{i\theta U_n/2^n} = 2^n \frac{\sin(\theta/2^n)}{\theta}$$

Consequently,

$$E\exp\{i\theta\sum_{j=1}^{n} U_j/2^j\} = 2^{\binom{n}{2}}\theta^{-n}\prod_{j=1}^{n}\sin(\theta/2^j) := \psi_n(\theta)$$

This function decays like θ^{-n} as $|\theta| \to \infty$, and so for $n \geq 2$ it is integrable. Therefore, the Fourier inversion theorem implies that for any $n \geq 2$ the distribution of the random variable $\sum_1^n U_j/2^j$ has a density f_n. Furthermore, $|\theta|^{n-2}\psi_n(\theta)$ is integrable, so Fourier theory implies that the density f_n has $n-2$ bounded, continuous derivatives. Finally, since the random variable Y is gotten by adding an independent random variable to the finite sum $\sum_1^n U_j/2^j$, it follows that the distribution of Y has a density, and that it has at least $n-2$ bounded, continuous derivatives.

□

Say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is a *radial* function if it is a function only of the distance to the origin, that is,

$$f(x) = f(|x|).$$

Corollary 8. *For any $\varepsilon > 0$ and $d \geq 1$ there exists a C^∞* radial *probability density on $\mathbb{R}^d$ that is supported by the ball of radius ε centered at the origin.*

Proof. By Lemma 4, there is an even, C^∞ probability density ψ on the real line supported by the interval $[-1,1]$. Fix $\delta > 0$, and define $g : \mathbb{R}^d \to \mathbb{R}_+$ by

$$g(x) = \psi(|x|^2/\delta^2).$$

This function is C^∞ (why?), it is obviously nonnegative, and its support is contained in the ball of radius δ centered at the origin. Moreover, its integral over $\mathbb{R}^d$ is *positive*, because ψ must be bounded away from 0 on some subinterval of $(0,1)$. Thus, we can set

$$f(x) = g(x)/\int_{\mathbb{R}^d} g(y)\,dy.$$

$\square$

Arguments in stochastic calculus often turn on some kind of *localization*, in which a smooth function f is replaced by a smooth function $\tilde{f}$ with compact support such that $f = \tilde{f}$ in some specified compact set. Such a function $\tilde{f}$ can be obtained by setting $\tilde{f} = fv$, where v is a smooth function with compact support that takes values between 0 and 1, and is such that $v = 1$ on a specified compact set. The next lemma implies that such functions exist.

Lemma 5. *(Mollification) For any open, relatively compact[5] sets $D, G \subset \mathbb{R}^k$ such that the closure $\bar{D}$ of D is contained in G, there exists a C^∞ function $v : \mathbb{R}^d \to \mathbb{R}$ such that*

$$\begin{aligned} v(x) &= 1 \quad \forall\, x \in \bar{D}; \\ v(x) &= 0 \quad \forall\, x \notin G; \\ 0 \le v(x) &\le 1 \quad \forall\, x \in \mathbb{R}^d \end{aligned}$$

Proof. By hypothesis, $\bar{D} \subset G$ and $\bar{G}$ is compact. Hence, there exists $\varepsilon > 0$ such that no two points $x \in \partial D$ and $y \in \partial G$ are at distance less than 4ε. Let F be the set of points x such that $\text{dist}(x, \partial D) \le 2\varepsilon$, and let φ be a C^∞ probability density on $\mathbb{R}^d$ with support contained in the ball of radius ε centered at the origin. Define

$$f = \varphi * \mathbf{1}_F.$$

By Lemma 3, the function f is C^∞. By construction, $f = 1$ on D and $f = 0$ outside G. (Exercise: Explain why.) $\square$

The existence of C^∞ probability densities with compact support has another consequence that is of importance in stochastic calculus: it implies that continuous functions can be arbitrarily well-approximated by C^∞ functions.

Corollary 9. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function with compact support, and let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a C^∞ probability density that vanishes outside the ball of radius 1 centered at the origin. For each $\varepsilon > 0$, define*

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi(x/\varepsilon). \tag{22}$$

Then $\varphi_\varepsilon * f$ converges to f uniformly *as $\varepsilon \to 0$:*

$$\lim_{\varepsilon \to 0} \|\varphi_\varepsilon * f - f\|_\infty = 0. \tag{23}$$

Proof. If Y is a $d-$dimensional random vector with density φ, then for any $\varepsilon > 0$ the random vector εY has density φ_ε. Consequently, for every $\varepsilon > 0$ and every $x \in \mathbb{R}^d$,

$$\varphi_\varepsilon * f(x) = Ef(x + \varepsilon Y).$$

Since f has compact support, it is bounded and uniformly continuous. Clearly, $\varepsilon Y \to 0$ as $\varepsilon \to 0$, so the bounded convergence theorem implies that the expectation converges to $f(x)$ for each x. To see that the convergence is *uniform* in x, observe that

$$\varphi_\varepsilon * f(x) - f(x) = E\{f(x + \varepsilon Y) - f(x)\}.$$

[5] *Relatively compact* means that the closure is compact.

Since f is uniformly continuous, and since $|Y| \leq 1$ with probability 1, for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that for all $\varepsilon < \varepsilon_\delta$,

$$|f(x + \varepsilon Y) - f(x)| < \delta$$

for all x, with probability one. Uniform convergence in (23) now follows immediately. $\square$