Problem 1. Normal Moments. (A) Use the Itô formula and Brownian scaling to check that the even moments of the normal distribution are

\[ EW_{2n}^2 = (2n - 1)(2n - 3) \cdots 3 \cdot 1. \]

(This is not meant to be hard. You should also obtain the formula by integrating against the standard normal density, if you haven’t seen this before.)

(B) Show that the product on the right side of (1) is also the number of perfect matchings of the integers 1, 2, \ldots, 2n. (A perfect matching is a partition of the set \{1, 2, \ldots, 2n\} into \(n\) subsets of cardinality 2. These are sometimes called dimer coverings.) (c) Can you give a direct combinatorial or probabilistic explanation of why these two things are the same?

Problem 2. Concentration Inequality. Let \( M_t \) be a continuous local martingale such that \( M_0 = 0 \) and \([M]_t \leq 1 \) for all \( t \geq 0 \). Prove that for every \( x \geq 0 \),

\[ P\{ \sup_{t \geq 0} M_t \geq x \} \leq e^{-x^2/2}. \]

HINT: First show that for every \( \theta > 0 \) the process

\[ \exp\{ \theta M_t - \theta^2 t/2 \} \]

is a positive supermartingale.

Problem 3. Brownian bridge. (A) Solve the stochastic differential equation

\[ dY_t = -\frac{Y_t}{1-t} d t + dW_t \]

for \( 0 \leq t \leq 1 \) with initial condition \( Y_0 = 0 \). HINT: Use the Itô formula to write a stochastic differential equation for the process \( f(t)Y_t \), where \( f(t) \) is an arbitrary \( C^2 \) function, and then look for a function \( f \) that will make the \( d t \) terms vanish.

(B) Check that the process you found in part (a) is a centered Gaussian process and that its covariance function is

\[ EY_s Y_t = \min(s, t) - s t. \]

This implies that \( Y_t \) is a Brownian bridge.

(C) Prove that the Brownian bridge is a semi-martingale, that is, it is the sum of a continuous local martingale and a progressively measurable, continuous function of bounded variation.

Note: Sampling without replacement and the Brownian bridge. In the same way that Brownian motion is a scaling limit of simple random walks, the Brownian bridge is a scaling limit of the “random walk” gotten by sampling without replacement from a box with \( n \) tickets marked +1 and \( n \) tickets marked −1. The sampling mechanism can be described in the following equivalent terms: Let
(Xₙ₁, Xₙ₂, ..., X₂n) be uniformly distributed on the set of all \( \binom{2n}{n} \) sequences \((x₁, x₂, ..., x₂n)\) with \( n \) entries +1 and \( n \) entries −1, and for each \( m \leq 2n \) define
\[
S_m = \sum_{i=1}^{m} X_i.
\]

It can be shown that for any fixed \( 0 < t₁ < \cdots < t_k < 1 \) the joint distribution of the random variables
\[
(S_{\lfloor nt₁ \rfloor}, S_{\lfloor nt₂ \rfloor}, ..., S_{\lfloor nt_k \rfloor}) / \sqrt{n}
\]
converges to that of \((Y₁, Y₂, ..., Yₖ)\), where \( Yₖ \) is the Brownian bridge. This fact gives a heuristic rationale for the drift term in the stochastic differential equation (2).

**Problem 4. Feynman-Kac Formula.** Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a twice-continuously differentiable function that satisfies the second order ODE
\[
\frac{1}{2} \varphi''(x) = k(x)\varphi(x)
\]
for some continuous function \( k : \mathbb{R} \to \mathbb{R} \), called the potential function. Assume that \( k \geq 0 \). (Thus, any positive solution \( \varphi \) of the differential equation will be strictly convex.) Standard existence/uniqueness theorems for ordinary differential equations imply that if \( k \) is piecewise \( C^1 \), or more generally Lipschitz continuous, then (3) will have a unique solution for any specification of an initial value \( \varphi(x₀) \) and initial first derivative \( \varphi'(x₀) \). You may assume this fact as known.

Assume that \( \varphi \) is a solution of (3), and for any two real numbers \( a < 0 < b \) set
\[
\tau(a) = \min\{t : W_t = a\}; \quad \tau(b) = \min\{t : W_t = b\};
\]
\[
T = T_{a,b} = \min(\tau(a), \tau(b)); \quad \text{and}
\]
\[
Z_t = \varphi(W_t) \exp\left\{ -\int_0^t k(W_s) \, ds \right\}.
\]

(A) Show that \( \{Z_{t∧T}\}_{t≥0} \) is an \( L^2 \)-bounded martingale relative to the filtration \( \mathcal{F} \), under any of the measures \( P^x \), where \( x \in [a, b] \). **Note:** \( P^x \) denotes the probability measure under which the process \( W_t \) is a Brownian motion started at \( W_0 = x \).

(B) Show that for every \( x \in [a, b] \),
\[
\varphi(x) = E^x \left\{ \varphi(W_T) \exp\left\{ -\int_0^T k(W_s) \, ds \right\} \right\}.
\]

Note: This is a time-independent version of the Feynman-Kac formula.

(C) Use the result of (B) to solve for the Laplace transform \( E^x e^{-λT} \). **Note:** This Laplace transform was obtained by other martingale methods in the lecture notes on Brownian motion. The problem here is to get the same result by solving a second-order differential equation.

**Note:** The results of Problem 4 can be extended to higher dimensions to give Brownian path integral representations of the solutions of the PDE
\[
\frac{1}{2} \Delta \varphi = k \varphi.
\]
This PDE is the time-independent Schrödinger equation, which is of basic importance in quantum mechanics. The Feynman-Kac formula arose out of Feynman’s attempt to give a path integral formulation of quantum mechanics.

**Bonus Problems: The Hermite functions and Itô calculus**

(These are optional, not to be handed in.) The Hermite functions $H_n(x, t)$ are polynomials in the variables $x$ and $t$ that satisfy the backward heat equation $H_t + H_{xx}/2 = 0$. The first few Hermite functions are

\[
H_0(x, t) = 1,
H_1(x, t) = x,
H_2(x, t) = x^2 - t,
H_3(x, t) = x^3 - 3xt,
H_4(x, t) = x^4 - 6x^2t + 3t^2.
\]

The easiest way to define them is by specifying their exponential generating function:

\[
\sum_{n=0}^{\infty} H_n(x, t) \frac{\theta^n}{n!} = \exp\{\theta x - \theta^2 t/2\}
\]

**Problem 5.** Use the generating function to show that the Hermite functions satisfy the two-term recursion relation

\[
H_{n+1} = 2xH_n - 2nH_{n-1}
\]

Conclude that every term of $H_{2n}$ is a constant times $x^{2m}t^{n-m}$ for some $0 \leq m \leq n$, and that the lead term is the monomial $x^{2n}$. Conclude also that each $H_n$ solves the backward heat equation.

**Problem 6.** Show that for each $n \geq 0$,

\[
H_{n+1}(W_t, t) = \int_0^t (n+1)H_n(W_s, s) dW_s = \cdots = (n+1)! \int_0^t \cdots \int_0^{t_n} dW_{n+1} dW_n \cdots dW_1
\]

and use this to show that $H_n(W_t, t)$ is a martingale. Hint: Use the Itô formula on $\exp\{\theta W_t - \theta^2 t/2\}$.

The standard Hermite polynomials are the one-variable polynomials $H_n(x) = H_n(2x, 1/2)$. Equivalently,

\[
\sum_{n=0}^{\infty} H_n(x) \frac{\theta^n}{n!} = \exp\{2\theta x - \theta^2\}
\]

**Problem 7.** (A) Show that the Hermite polynomials satisfy the second-order differential equations

\[
H''_n(x) - 2xH'_n(x) = -2nH_n(x)
\]

Thus, the Hermite polynomials are the eigenfunctions of the generator of the Ornstein-Uhlenbeck process (see next problem). Hint: First differentiate the generating function with respect to $x$ to obtain a first-order ODE for $H_n$. Then combine this with the two-term recurrence relation $H_{n+1} = 2xH_n - 2nH_{n-1}$ gotten by specialization from Problem 5 above.

(B) Show that

\[
\int_{-\infty}^{\infty} H_n(x)H_m(x) \frac{e^{-x^2}}{\sqrt{\pi}} dx = C_n^m \delta_{n,m}
\]
and evaluate the normalizing constants $C_n$. Here $\delta_{n,m}$ is the Kronecker delta, that is, 0 is $m \neq n$ and 1 if $n = m$. Conclude that the polynomials $H_n/C_n$ are an orthonormal basis for $L^2(\Phi)$, where $\Phi$ is the normal distribution with mean 0 and variance 1/2. Hint: Take the product of the generating function (7) with itself, but evaluated at two different values of $\theta$ (say, $\theta$ and $\xi$), and integrate this against the normal density with mean zero and variance 1/2.

**Problem 8.** The Ornstein-Uhlenbeck process is the diffusion process that satisfies
\[ dY_t = -Y_t \, dt + dW_t. \]

(A) Show that for each $n \geq 0$ the process
\[ M_n(t) := e^{nt} H_n(Y_t) \]
is a martingale.

(B) Deduce that for each function $f(y)$ in the linear span of the Hermite polynomials, and for each $t > 0$ and $x \in \mathbb{R}$,
\[ E^x f(Y_t) = \int p_t(x, y) f(y) \, dy \quad \text{where} \quad p_t(x, y) = \frac{e^{-y^2/2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} e^{-nt} H_n(x) H_n(y)/C_n^2. \]
This is the eigenfunction expansion of the transition probability kernel.

(C) Use the representation $Y_t = 2^{-1/2} e^{-t} W_{e^t}$ of the stationary Ornstein-Uhlenbeck process to obtain another formula for the transition probabilities:
\[ p_t(x, y) = \frac{1}{\sqrt{\pi}} \frac{1}{(1 - e^{-2t})^{-1/2}} \exp \left\{ -\frac{(xe^{-t} - y)^2}{1 - e^{-2t}} \right\}. \]