STATISTICS 385: STOCHASTIC CALCULUS
HOMEWORK ASSIGNMENT 4
DUE NOVEMBER 23, 2016

Problem 1. *Normal Moments.* (A) Use the Itô formula and Brownian scaling to check that the even moments of the normal distribution are

$$E W_1^{2n} = (2n-1)(2n-3)\cdots 3\cdot 1. \tag{1}$$

(This is not meant to be hard. You should also obtain the formula by integrating against the standard normal density, if you haven't seen this before.)

(B) Show that the product on the right side of (1) is also the number of *perfect matchings* of the integers $1, 2, \ldots, 2n$. (A perfect matching is a partition of the set $\{1, 2, \ldots, 2n\}$ into n subsets of cardinality 2. These are sometimes called *dimer coverings*.)

(C)* Can you give a direct combinatorial or probabilistic explanation of why these two things are the same?

Problem 2. *Concentration Inequality.* Let M_t be an Itô integral process such that $M_0 = 0$ and $[M]_\infty \le 1$. (This means that $M_t = I_t(V)$ for some progressively measurable process V that satisfies $\int_0^\infty V_s^2\, ds \le 1$.) Prove that for every $x \ge 0$,

$$P\{\sup_{t\ge 0} M_t \ge x\} \le e^{-x^2/2}.$$

HINT: For every $\theta > 0$ the process

$$\exp\{\theta M_t - \theta^2 [M]_t/2\}$$

is a positive supermartingale.

Problem 3. *Local Time.* Recall from the lectures (see section 5.2 of the Notes) that the *local time* process $L_t = L_t^0$ is the unique adapted, nondecreasing process such that for every $t \ge 0$,

$$|W_t| = L_t + \int_0^t (W_s)\, dW_s.$$

Furthermore, for any even, C^∞ probability density $\psi(x)$ with support $[-1, 1]$ the local time L_t is given by

$$L_t = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t \psi(W_s/\varepsilon)\, ds.$$

(A) (Easy) Check that L_t increases only on the zero set of the Brownian motion W.
(B) Show that L_t is not identically zero.
(C) Use this to show that $\eta(a) := \inf\{t : L_t > a\} < \infty$ almost surely.
(D) Show that the process $\{\eta(a)\}_{a\ge 0}$ is a stable$-\frac{1}{2}$ subordinator. What are the jumps?

Problem 4. *More Local Time.* For any $\varepsilon > 0$ and any $t > 0$, define the number $N_t(\varepsilon)$ of *upcrossings* of the interval $[0, \varepsilon]$ by the process $|W|$ up to time t as follows. First, set $\alpha_0 = 0$, and for each $n = 0, 1, 2, \ldots$ define

$$\beta_n = \min\{t > \alpha_n : |W_t| = \varepsilon\} \quad \text{and}$$
$$\alpha_{n+1} = \min\{t > \beta_n : |W_t| = 0\};$$

then let $N_t(\varepsilon) = \max\{n : \beta_n \le t\}$. Prove that with probability one,

$$\lim_{\varepsilon \to 0} \varepsilon N_t(\varepsilon) = L_t \quad (\text{or maybe } 2^{\pm 1} L_t).$$

HINTS: One way to approach this is to use the representation

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \psi(W_s/\varepsilon)\, ds.$$

If you follow this approach, you might want to first show that the random variable $Y = \int_0^{\alpha_1} \mathbf{1}_{[0,\varepsilon]}(|W_s|)\, ds$ has finite exponential moments.

Problem 5. *Feynman-Kac Formula I.* Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a twice-continuously differentiable function that satisfies the second order ODE

$$\frac{1}{2}\varphi''(x) = k(x)\varphi(x) \tag{2}$$

for some continuous function $k : \mathbb{R} \to \mathbb{R}$, called the *potential function.* Assume that $k \ge 0$. (Thus, any positive solution φ of the differential equation will be convex.) Standard existence/uniqueness theorems for ordinary differential equations imply that if k is piecewise C^1, or more generally Lipschitz continuous, then (2) will have a unique solution for any specification of an initial value $\varphi(x_0)$ and initial first derivative $\varphi'(x_0)$. You may assume this fact as known.

Assume that φ is a solution of (2), and for any two real numbers $a < 0 < b$ set

$$\tau(a) = \min\{t : W_t = a\};$$
$$\tau(b) = \min\{t : W_t = b\};$$
$$T = T_{a,b} = \min(\tau(a), \tau(b)); \quad \text{and}$$
$$Z_t = \varphi(W_t) \exp\left\{-\int_0^t k(W_s)\, ds\right\}.$$

(A) Show that $\{Z_{t\wedge T}\}_{t\ge 0}$ is an L^2–bounded martingale relative to the filtration $\mathbb{F}$, under any of the measures P^x, where $x \in [a, b]$. NOTE: P^x denotes the probability measure under which the process W_t is a Brownian motion started at $W_0 = x$.

(B) Show that for every $x \in [a, b]$,

$$\varphi(x) = E^x\left\{\varphi(W_T) \exp\left\{-\int_0^T k(W_s)\, ds\right\}\right\}. \tag{3}$$

Note: This is a time-independent version of the *Feynman-Kac formula.*

(C) Use the result of (B) to solve for the Laplace transform $E^x e^{-\lambda T}$. NOTE: This Laplace transform was obtained by other martingale methods in the lecture notes on Brownian motion. The problem here is to get the same result by solving a second-order differential equation.

NOTE: The results of Problem 5 can be extended to higher dimensions to give Brownian path integral representations of the solutions of the PDE

$$\frac{1}{2}\Delta\varphi = k\varphi.$$

This PDE is the time-independent *Schrödinger equation*, which is of basic importance in quantum mechanics. The Feynman-Kac formula arose out of Feynman's attempt to give a *path integral* formulation of quantum mechanics.

Problem 6. *Feynman-Kac Formula II.* Let $k(x) = \theta \mathbf{1}_{(-\infty,0]}(x)$, where $\theta > 0$ is a positive constant. This function has a discontinuity at $x = 0$, so the results of Problem 5 do not apply directly. Nevertheless, one can use Problem 5 indirectly, as follows.

Let $0 \le k_1 \le k_2 \le k_3 \le \dots$ be a non-decreasing sequence of C^∞ functions such that $\lim k_n(x) = k(x)$, and for each n let φ_n be the solution of the equation

$$\frac{1}{2}\varphi_n''(x) = k_n(x)\varphi_n(x)$$

that satisfies the boundary conditions $\varphi_n(a) = \varphi_n(b) = 1$, for some fixed constants $a < 0 < b$.

(A) Show that for each $x \in [a,b]$ the sequence $\varphi_n(x)$ is non-increasing in n, and converges to a finite limit $\varphi(x)$. HINT: Rewrite the differential equation as an integral equation.

(B) Show that the limit function $\varphi(x)$ is continuous and twice differentiable (except at the argument $x = 0$), and satisfies the boundary value problem

$$\frac{1}{2}\varphi''(x) = k(x)\varphi(x) \quad \text{and} \quad \varphi(a) = \varphi(b) = 1.$$

(C) Show that for all $x \in [a,b]$

$$\varphi(x) = E^x \varphi(W_T) \exp\left\{-\theta \int_0^T \mathbf{1}_{(-\infty,0]}(W_s)\, ds\right\}.$$

(D) Solve the boundary value problem in part (B) to obtain an *explicit* formula for the Laplace transform of the total occupation time random variable

$$Y = \int_0^T \mathbf{1}_{(-\infty,0]}(W_s)\, ds.$$

Bonus Problems: The Hermite functions and Itô calculus

(These are optional, not to be handed in.) The *Hermite functions* $H_n(x,t)$ are polynomials in the variables x and t that satisfy the backward heat equation $H_t + H_{xx}/2 = 0$. The first few Hermite functions are

$$\begin{aligned} H_0(x,t) &= 1, \\ H_1(x,t) &= x, \\ H_2(x,t) &= x^2 - t, \\ H_3(x,t) &= x^3 - 3xt, \\ H_4(x,t) &= x^4 - 6x^2 t + 3t^2. \end{aligned} \tag{4}$$

The easiest way to define them is by specifying their *exponential generating function*:

$$\sum_{n=0}^{\infty} H_n(x,t)\frac{\theta^n}{n!} = \exp\{\theta x - \theta^2 t/2\} \tag{5}$$

Problem 7. Use the generating function to show that the Hermite functions satisfy the two-term recursion relation $H_{n+1} = xH_n - ntH_{n-1}$. Conclude that every term of H_{2n} is a constant times $x^{2m}t^{n-m}$ for some $0 \le m \le n$, and that the lead term is the monomial x^{2n}. Conclude also that each H_n solves the backward heat equation.

Problem 8. Show that for each $n \ge 0$,

$$H_{n+1}(W_t,t) = \int_0^t (n+1)H_n(W_s,s)\,dW_s = \cdots = (n+1)!\int_0^t\int_0^{t_1}\cdots\int_0^{t_n} dW_{t_{n+1}}\,dW_{t_n}\cdots dW_{t_1}$$

and use this to show that $H_n(W_t,t)$ is a martingale. HINT: Use the Itô formula on $\exp\{\theta W_t - \theta^2 t/2\}$.

The standard *Hermite polynomials* are the one-variable polynomials $H_n(x) = H_n(2x, 1/2)$. Equivalently,

$$\sum_{n=0}^{\infty} H_n(x)\frac{\theta^n}{n!} = \exp\{2\theta x - \theta^2\} \tag{6}$$

Problem 9. (A) Show that the Hermite polynomials satisfy the second-order differential equations

$$H_n''(x) - 2xH_n'(x) = -2nH_n(x)$$

Thus, the Hermite polynomials are the *eigenfunctions* of the generator of the Ornstein-Uhlenbeck process (see next problem). HINT: First differentiate the generating function with respect to x to obtain a first-order ODE for H_n. Then combine this with the two-term recurrence relation $H_{n+1} = 2xH_n - 2nH_{n-1}$ gotten by specialization from Problem 7 above.

(B) Show that

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)\frac{e^{-x^2}}{\sqrt{\pi}}\,dx = C_n^2\delta_{n,m}$$

and evaluate the normalizing constants C_n. Here $\delta_{n,m}$ is the Kronecker delta, that is, 0 is $m \ne n$ and 1 if $n = m$. Conclude that the polynomials H_n/C_n are an orthonormal basis for $L^2(\Phi)$, where Φ is the normal distribution with mean 0 and variance 1/2. HINT: Take the product of the generating

function (6) with itself, but evaluated at two different values of θ (say, θ and ξ), and integrate this against the normal density with mean zero and variance $1/2$.

Problem 10. The Ornstein-Uhlenbeck process is the diffusion process that satisfies

$$dY_t = -Y_t\,dt + dW_t.$$

(A) Show that for each $n \geq 0$ the process

$$M_n(t) := e^{nt} H_n(Y_t)$$

is a martingale.

(B) Deduce that for each function $f(y)$ in the linear span of the Hermite polynomials, and for each $t > 0$ and $x \in \mathbb{R}$,

$$E^x f(Y_t) = \int p_t(x,y) f(y)\,dy \quad \text{where} \quad p_t(x,y) = \frac{e^{-y^2/2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} e^{-nt} H_n(x) H_n(y)/C_n^2.$$

This is the *eigenfunction expansion* of the transition probability kernel.

(C) Use the representation $Y_t = 2^{-1/2} e^{-t} W_{e^{2t}}$ of the stationary Ornstein-Uhlenbeck process to obtain another formula for the transition probabilities:

$$p_t(x,y) = \frac{1}{\sqrt{\pi}} (1 - e^{-2t})^{-1/2} \exp\left\{ -\frac{(xe^{-t} - y)^2}{1 - e^{-2t}} \right\}.$$