STATISTICS 385: STOCHASTIC CALCULUS
HOMEWORK ASSIGNMENT 2
DUE OCTOBER 24, 2016

Problem 1. *(Poisson kernel for the upper half-plane).* Let $W_t = (X_t, Y_t)$ be a standard 2-dimensional Brownian motion started at the origin $(0,0)$, and for each $a > 0$ let $\tau(a)$ be the first time t such that $Y_t = a$.

(a) Show that for each real θ the process $\exp\{i\theta X_t + \theta Y_t\}$ is a martingale.
(b) Use Doob's Optional Sampling Theorem to deduce a formula for $E e^{i\theta X_{\tau(a)}}$.
(c) Now use the Fourier inversion formula to identify the distribution of $X_{\tau(a)}$.

Problem 2. *(Coupling).* Show that for any two points $x, y \in \mathbb{R}^d$ there exist, on some probability space $(\Omega, \mathscr{F}, P)$, stochastic processes $\{W_t^x\}_{t\geq 0}$ and $\{W_t^y\}_{t\geq 0}$ such that

(a) $\{W_t^x\}_{t\geq 0}$ is a $d-$dimensional Brownian motion with initial point x;
(b) $\{W_t^y\}_{t\geq 0}$ is a $d-$dimensional Brownian motion with initial point y; and
(c) with probability one, $W_t^x = W_t^y$ for all sufficiently large t.

HINT: First, prove this for dimension $d = 1$ by running two independent Brownian motions from x and y until they meet, and then having them follow the same trajectory after meeting. Second, deduce the $d-$dimensional case from the $1-$dimensional case.

Problem 3. Use the result of problem 2 to prove the *Choquet-Deny* theorem, which states that the only bounded harmonic functions on $\mathbb{R}^d$ are the constant functions. HARDER: (Optional) Prove that if u is a harmonic function on $\mathbb{R}^d$ that grows sublinearly (that is, $\sup_{|x|\leq R} |u(x)| = o(R)$) then u is constant.

Problem 4. Let $\{W_t\}_{t\geq 0}$ be a standard two-dimensional Brownian motion. Prove that with probability one the origin 0 is *not* in the unbounded connected component of $\mathbb{R}^2 \setminus \{W_t : 0 \leq t \leq 1\}$. HINT: First verify that there is positive probability that a Brownian motion started at a point on the unit circle will loop around the origin before reaching either the circle of radius $1/2$ or the circle of radius 2.

Problem 5. Let W_t be a three-dimensional Brownian motion started at a point $x \neq 0$, let $Y_t = 1/|W_t|$, and for each $m = 0, 1, 2, \ldots$ let τ_m be the time of first passage to the ball of radius 2^{-m} centered at 0.

(a) Prove that the random variables $\{Y_t\}_{t\geq 0}$ are bounded in L^2.
(b) Prove that the process $\{Y_t\}_{t\geq 0}$ is *not* a martingale.
(c) Show that for each m the stopped process $\{Y_{t\wedge\tau_m}\}_{t\geq 0}$ is a martingale.
(d) Verify that $\tau_m \to \infty$ as $m \to \infty$.

This shows that the process $\{Y_t\}_{t\geq 0}$ is a *local* martingale that is not a martingale.

Problem 6. *(Extension of Dynkin's formula)* Let $u : \mathbb{R}_+ \times \mathbb{R}^d$ be a C^2 function with compact support in $(0,\infty) \times \mathbb{R}^d$., and let $\{W_t\}_{t\geq 0}$ be a standard d–dimensional Brownian motion. Show that

$$Eu(t, W_t) = u(0,0) + E\int_0^t ((\partial_s + \frac{1}{2}\Delta)u)(t, W_s)\,ds,$$

where ∂_s denotes partial derivative with respect to the first (time) coordinate and Δ is the Laplacian with respect to the last d (space) coordinates.