Problem 1. First-passage time process. Let \( \{W_t\}_{t \geq 0} \) be a standard one-dimensional Wiener process, and for each \( a \geq 0 \) define \( \tau_a \) to be the first time \( t \) that \( W_t = a \). By the recurrence theorem for Brownian motion (to be proved in class next week) and the continuity of paths, each \( \tau_a \) is finite and well-defined.

(A) Use the strong Markov property for Brownian motion to show that the stochastic process \( \{\tau_a\}_{a \geq 0} \) has stationary, independent increments (i.e., it is a Lévy process).

(B) Check that the sample paths \( a \mapsto \tau_a \) are nondecreasing and left-continuous. (Note: A Lévy process with nondecreasing paths is called a subordinator.)

(C) Prove that with probability one, the set \( \{a \geq 0 : \tau_a+ - \tau_a > 0\} \) of jump discontinuities is countable and dense. Hint: You will have to make use of what you know about Brownian paths.

(D) Use Brownian scaling to show that for every \( a > 0 \), the distribution of \( a^2 \tau_a \) is the same as that of \( \tau_1 \).

(E) Use the results of (A) and (D) to show that \( \tau_1 \) has the same distribution as \( n^{-2} \sum_{i=1}^{n} \tau_1^{(i)} \), where \( \tau_1^{(i)} \) are independent, identically distributed random variables each with the same distribution as \( \tau_1 \). Then use this to show that for some constant \( C > 0 \) the Laplace transform of \( \tau_1 \) is given by

\[
E e^{-\lambda \tau_1} = \exp\{-C \sqrt{\lambda}\}.
\]

(See the section of the notes on Wald’s identities for a proof that \( C = \sqrt{2} \).)

Problem 2. Let \( \{W_t\}_{t \geq 0} \) be a standard one-dimensional Wiener process, and for each \( t \geq 0 \) let \( M_t = \max_{s \leq t} W_s \).

(A) Use the reflection principle to find the joint distribution of \( (W_t, M_t) \). (The answer is given in Corollary 5 of the notes; your job is to supply the derivation.)

(B) Use the result of part (A) to conclude that for every \( t \), the distribution of \( M_t - W_t \) is the same as that of \( |W_t| \).

(C) Formulate and prove a corresponding result for simple random walk on \( \mathbb{Z} \). Hint: You should be able to do this directly, by induction on the number of steps.

Problem 3. Let \( \{W_t\}_{t \geq 0} \) be a standard one-dimensional Wiener process, and for each pair \( 0 \leq s \leq t \) define \( M(s, t) \) to be the maximum value attained by \( W_r \) for \( s \leq r \leq t \).

(A) Show that with probability one, for every pair of rational \( 0 \leq s < t \),

\[
M(s, t) > \max(W_s, W_t).
\]
(B) Conclude that with probability one, the local maxima of the Brownian path \( t \to W_t \) are dense in \([0, \infty)\). Also, prove that the set of times \( t \) at which the Brownian path has local maxima are dense in \([0, \infty)\). Note: By definition, a local maximum occurs at any time \( t \) such that for some \( \varepsilon > 0 \),
\[
W_t \geq \max_{s \in [t-\varepsilon, t+\varepsilon]} W_s.
\]

(C) Prove that with probability one, for every rational pair \( 0 \leq s < t \) the maximum value \( M(s, t) \) of the Brownian path on the time interval \([s, t]\) is attained at a unique time \( r \in (s, t) \). Thus, with probability one, the local maxima of the Brownian path are distinct.