STATISTICS 385: STOCHASTIC CALCULUS
HOMEWORK ASSIGNMENT 1
DUE OCTOBER 10, 2016

Problem 1. First-passage time process. Let $\{W_t\}_{t\geq 0}$ be a standard one-dimensional Wiener process, and for each $a \geq 0$ define τ_a to be the first time t that $W_t = a$. By the recurrence theorem for Brownian motion (to be proved in class next week) and the continuity of paths, each τ_a is finite and well-defined.

(A) Use the strong Markov property for Brownian motion to show that the stochastic process $\{\tau_a\}_{a\geq 0}$ has stationary, independent increments (i.e., it is a *Lévy process*).

(B) Check that the sample paths $a \mapsto \tau_a$ are nondecreasing and *left*-continuous. (Note: A Lévy process with nondecreasing paths is called a *subordinator*.)

(C) Prove that with probability one, the set $\{a \geq 0 : \tau_{a+} - \tau_a > 0\}$ of jump discontinuities is countable and dense. HINT: You will have to make use of what you know about Brownian paths.

(D) Use *Brownian scaling* to show that for every $a > 0$, the distribution of $a^2 \tau_a$ is the same as that of τ_1.

(E) Use the results of (A) and (D) to show that τ_1 has the same distribution as $n^{-2}\sum_{i=1}^n \tau_1^{(i)}$, where $\tau_1^{(i)}$ are independent, identically distributed random variables each with the same distribution as τ_1. Then use this to show that for some constant $C > 0$ the Laplace transform of τ_1 is given by

$$E e^{-\lambda\tau_1} = \exp\{-C\sqrt{\lambda}\}.$$

(See the section of the notes on Wald's identities for a proof that $C = \sqrt{2}$.)

Problem 2. Fix $-a < 0 < b$ and let $T = T_{a,b} = \min\{t > 0 : W_t = -a \text{ or } + b\}$ be the time of first exit from the interval $(-a, b)$.

(A) Calculate the Laplace transform of the distribution T, that is, for each $\lambda > 0$ evaluate $E e^{-\lambda T}$.

(B) Use your formula for the Laplace transform to evaluate ET and $\text{var}(T)$.

HINT: Use the third Wald identity for both $+\theta$ and $-\theta$.

Problem 3. For each $k = 0, 1, 2, \ldots$ define stopping times $T_{k,1}, T_{k,2}, \ldots$ as follows:

$$T_{k,0} = 0 \quad \text{and} \quad T_{k,m+1} = \min\{t > T_{k,m} : |W_t - W_{T_{k,m}}| = 2^{-k}\}.$$

These are the jump times of the kth level embedded simple random walk. For each $s > 0$ define $N_k(s)$ to be $\max\{m : T_{k,m} < s\}$; thus, $N_k(s)$ is the number of jumps made by the kth level embedded random walk by time s. Show that for each s,

$$P - \lim_{k\to\infty} N_k(s)/4^k = s.$$

Here $P - \lim$ denotes *convergence in probability*. In fact, the convergence holds *almost surely*; if you can prove this, all the better.

Problem 4. Let $\{W_t\}_{t\geq 0}$ be a standard one-dimensional Wiener process, and for each $t \geq 0$ let $M_t = \max_{s\leq t} W_s$.

(A) Use the reflection principle to find the joint distribution of (W_t, M_t). (The answer is given in Corollary 5 of the notes; your job is to supply the derivation.)

(B) Use the result of part (A) to conclude that for every t, the distribution of $M_t - W_t$ is the same as that of $|W_t|$.

(C) Formulate and prove a corresponding result for simple random walk on $\mathbb{Z}$. HINT: You should be able to do this directly, by induction on the number of steps.

Problem 5. Let $\{W_t\}_{t\geq 0}$ be a standard one-dimensional Wiener process, and for each pair $0 \leq s \leq t$ define $M(s,t)$ to be the maximum value attained by W_r for $s \leq r \leq t$.

(A) Show that with probability one, for every pair of *rational* $0 \leq s < t$,

$$M(s,t) > \max(W_s, W_t).$$

(B) Conclude that with probability one, the local maxima of the Brownian path $t \mapsto W_t$ are dense in $[0,\infty)$. Also, prove that the set of *times* t at which the Brownian path has local maxima are dense in $[0,\infty)$. NOTE: By definition, a local maximum occurs at any time t such that for some $\varepsilon > 0$,

$$W_t \geq \max_{s\in[t-\varepsilon,t+\varepsilon]} W_s.$$

(C) Prove that with probability one, for every rational pair $0 \leq s < t$ the maximum value $M(s,t)$ of the Brownian path on the time interval $[s,t]$ is attained at a unique time $r \in (s,t)$. Thus, with probability one, the local maxima of the Brownian path are distinct.