Continuous Martingales I. Fundamentals

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1 Review: Discrete-Time Martingales

Recall that a *filtration* of a probability space $(\Omega, \mathscr{F}, P)$ is an indexed family $\mathbb{F} = \mathscr{F}_{t\, t\in J}$ of $\sigma-$algebras all contained in $\mathscr{F}$. The index set J is assumed to be totally ordered, and in virtually all applications will be a subset of $\mathbb{R}$; for any two indices $s, t \in J$ such that $s < t$ it must be the case that $\mathscr{F}_s \subset \mathscr{F}_t$. A *martingale* (respectively, *sub-martingale* or *super-martingale*) relative to the filtration $\mathbb{F}$ is an indexed family of integrable random variables $\{M_t\}_{t\in J}$ such that

(a) for each t the random variable M_t is measurable relative to $\mathscr{F}_t$, and
(b) if $s < t$ then $E(M_t \,|\, \mathscr{F}_s) = M_s$ (respectively, $E(M_t \,|\, \mathscr{F}_s) \ge M_s$ or $E(M_t \,|\, \mathscr{F}_s) \le M_s$).

For *discrete-time* martingales and sub-martingales the index set J is a subset of $\mathbb{Z}$; if $J = (-\infty, k]$ then a martingale $\{M_t\}_{t\in J}$ relative to the filtration $\mathscr{F} = \{\mathscr{F}_t\}_{t\in J}$ is usually called a *reverse martingale*. The key elements of the theory of discrete-time martingales are the *optional sampling theorem*, the *maximal* and *upcrossings* inequalities, and the *martingale convergence theorems.*

Theorem 1. *(Optional Sampling) Let $\{X_n\}_{n\ge0}$ be a martingale (respectively, submartingale) relative to a filtration $\{\mathscr{F}_n\}_{n\ge0}$, and let τ be a stopping time. Then the stopped sequence $\{X_{\tau\wedge n}\}_{n\ge0}$ is a martingale (respectively submartingale or supermartingale). Consequently, for any $n \in \mathbb{N}$,*

$$\begin{aligned} EX_{\tau\wedge n} &= EX_0 && \textit{(martingales)} \\ EX_{\tau\wedge n} &\le EX_0 && \textit{(supermartingales)} \\ EX_{\tau\wedge n} &\le EX_n && \textit{(submartingales).} \end{aligned}$$

Furthermore, if $\mathscr{F}_{\tau\wedge n}$ is the stopping $\sigma-$algebra for the stopping time $\tau\wedge n$ then

$$\begin{aligned} X_{\tau\wedge n} &= E(X_n \,|\, \mathscr{F}_{\tau\wedge n}) && \textit{(martingales)} \\ X_{\tau\wedge n} &\ge E(X_n \,|\, \mathscr{F}_{\tau\wedge n}) && \textit{(supermartingales)} \\ X_{\tau\wedge n} &\le E(X_n \,|\, \mathscr{F}_{\tau\wedge n}) && \textit{(submartingales).} \end{aligned}$$

Proposition 1. *(Maximal Inequality) Let* $\{X_n\}_{n\geq 0}$ *be a sub- or super-martingale relative to* $\{Y_n\}_{n\geq 0}$*, and for each* $n \geq 0$ *define*

$$M_n = \max_{0\leq m\leq n} X_m, \qquad \textit{and} \tag{1}$$

$$M_\infty = \sup_{0\leq m<\infty} X_m = \lim_{n\to\infty} M_n \tag{2}$$

Then for any scalar $\alpha > 0$ *and any* $n \geq 1$*,*

$$P\{M_n \geq \alpha\} \leq E(X_n \vee 0)/\alpha \qquad \text{if } \{X_n\}_{n\geq 0} \text{ is a submartingale, and} \tag{3}$$

$$P\{M_\infty \geq \alpha\} \leq EX_0/\alpha \qquad \text{if } \{X_n\}_{n\geq 0} \text{ is a nonnegative supermartingale.} \tag{4}$$

See the notes on discrete martingales for the upcrossings inequality; we won't need it here. The martingale convergence theorems are next.

Martingale Convergence Theorem . *Let* $\{X_n\}$ *be an* L^1-*bounded submartingale relative to a sequence* $\{Y_n\}$*, that is, a submartingale such that* $\sup_n E|X_n| < \infty$*. Then with probability one the limit*

$$\lim_{n\to\infty} X_n := X_\infty \tag{5}$$

exists, is finite, and has finite first moment. Furthermore,

(a) $X_n \to X_\infty$ *in* L^1 *if and only if the sequence* $\{X_n\}_{n\geq 0}$ *is uniformly integrable, and*
(b) $X_n \to X_\infty$ *in* L^2 *if and only if the sequence* $\{X_n\}_{n\geq 0}$ *is bounded in* L^2*.*

Reverse Martingale Convergence Theorem . *Let* $\{X_n\}_{n\leq -1}$ *be a reverse martingale relative to a reverse filtration* $(\mathscr{F}_n)_{n\leq -1}$*. Then*

$$\lim_{n\to-\infty} X_n = E(X_{-1} \mid \cap_{n\leq -1} \mathscr{F}_n) \tag{6}$$

almost surely and in L^1*.*

2 Continuous-Time Processes: Progressive Measurability

Henceforth let $(\Omega, \mathscr{F}, P)$ be a fixed probability space and $\mathbb{F} := \{\mathscr{F}_t\}_{t\in J}$ a filtration indexed by J, where J is an interval, usually $J = [0,\infty)$. A family of random variables $\{X_t\}_{t\in J}$ indexed by elements of J is called a *stochastic process*; if each random variable X_t is measurable with respect to the corresponding $\sigma-$algebra $\mathscr{F}_t$ then the process $\{X_t\}_{t\in J}$ is said to be *adapted.*

Unfortunately, because the index set J is uncountable, the *sample paths* $t \mapsto X_t(\omega)$ of an adapted stochastic process need not be at all nice – in fact, they need not even be Lebesgue measurable. For interesting stochastic processes, such as Brownian motion, this problem does not arise, because the sample paths are *by assumption* continuous in the time parameter t. But even in the study of Brownian motion, interesting auxiliary processes with badly discontinuous sample paths occur – consider, for instance, the process $\mathbf{1}\{W_t > 0\}_{t\geq 0}$, where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion. Thus, sadly, we must dispense with some measure-theoretic technicalities before we go further with the theory of continuous-time martingales.

Exercise 1. Exhibit a filtered probability space with an adapted stochastic process $\{X_t\}_{t\geq 0}$ such that every sample path $t \mapsto X_t(\omega)$ is non-measurable. NOTE: You will need the Axiom of Choice.

A filtration $\mathbb{F}$ is said to be *complete* if each $\mathscr{F}_t$ contains all sets of measure 0, and is *right-continuous* if $\mathscr{F}_t = \cap_{s>t}\mathscr{F}_s$. A *standard filtration* is a filtration that is both complete and right-continuous. (The French would say that such a filtration "satisfies the usual assumptions".) An arbitrary filtration $\mathbb{F} := \{\mathscr{F}_t\}_{t\geq 0}$ can be augmented to yield a standard filtration:

$$\mathscr{F}_t^* = (\cap_{s>t}\mathscr{F}_s) \cup \mathscr{Z},$$

where $\mathscr{Z}$ is the collection of all sets of measure 0 in the enveloping $\sigma-$algebra $\mathscr{F}$.

Definition 1. Let $X = \{X_t\}_{t\in J}$ be an adapted stochastic process relative to the filtration $\mathbb{F}$. The process X is said to be *progressively measurable* (or just *progressive*) if for every subinterval $[r,s] \subset J$ the restriction $\{X_t\}_{t\in[r,s]}$ is (when considered as a function $X(t,\omega)$ on $\Omega \times [r,s]$) measurable relative to the $\sigma-$algebra $\mathscr{B}_{[r,s]} \times \mathscr{F}_s$.

Progressive measurability is the least we should expect for any stochastic process that we hope to integrate. If a stochastic process X is progressively measurable then for each $\omega \in \Omega$ the function $t \mapsto X(t,\omega)$ is a Borel measurable function of t, and so if X is (for instance) uniformly bounded then it can be integrated against a Lebesgue-Stieltjes measure on $\mathbb{R}$. Not every adapted process is progressively measurable – see the exercise above. Thus, checking that a process is progressively measurable is in principle more difficult than checking that it is adapted. However, there are some simple, useful sufficient conditions.

Lemma 1. *Limits of progressively measurable processes are progressively measurable. Moreover, any adapted process with continuous paths is progressively measurable. If* $\{X_t\}_{t\in J}$ *is an adapted process with right-continuous paths, then it is progressively measurable if the filtration* $\mathbb{F}$ *is standard.*

Proof. The first assertion is trivial, because limits of measurable functions are always measurable. Consider a process $\{X_t\}_{t\geq 0}$ with right-continuous paths (the case of left-continuous processes is similar). Fix $T > 0$; we must show that the function $X(t,\omega)$ is jointly measurable in t,ω relative to $\mathscr{B}_{[0,T]} \times \mathscr{F}_T$. For integers $m \geq 1$ and $0 \leq k \leq 2^m$ define

$$X_m(t,\omega) = X(kT/2^m,\omega) \quad \text{for } (k-1)T/2^m \leq t < kT/2^m.$$

Clearly, $X_m(t,\omega)$ is jointly measurable in (t,ω) relative to the product $\sigma-$algebra $\mathscr{B}_{[0,T]} \times \mathscr{F}_T$ (even though it is not adapted). Moreover, because the process $X(t,\omega)$ is right-continuous in t,

$$\lim_{m\to\infty} X_m(t,\omega) = X(t,\omega) \quad \text{for all } \ t \leq T, \omega \in \Omega. \tag{7}$$

Therefore, $X(t,\omega)$ is jointly measurable relative to $\mathscr{B}_{[0,T]} \times \mathscr{F}_T$. □

Example 1. Let W_t be a one-dimensional Wiener process, and $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ a standard, admissible filtration. Is the process $Y(t) = \mathbf{1}\{W_t > 0\}$ progressively measurable? To see that it is, consider the sequence of processes $Y_n(t) = (W_t \wedge 1)_+^{1/n}$ where the subscript + denotes positive part and $\wedge$ denotes min. For each integer $n \geq 1$ the process Y_n is progressively measurable (because it is adapted and has continuous paths), and $Y_n(t) \to Y(t)$ pointwise as $n \to$. As we will see later, there are some very good reasons for wanting to integrate the process $Y(t)$ against the underlying Wiener process W_t, as this leads to the important *Tanaka formula* in the theory of Brownian local time.

What happens when a process is evaluated at a stopping time?

Lemma 2. *Let $\tau \in J$ be a stopping time relative to a standard filtration $\mathbb{F}$ and let $\{X_t\}_{t\in J}$ be progressively measurable. Then $X_\tau \mathbf{1}_{\{\tau<\infty\}}$ is an $\mathscr{F}_\tau-$measurable random variable.*

Lemma 3. *Let $\{X_t\}_{t\in J}$ be progressively measurable. If $\int_J E|X_t|\,dt < \infty$ then X_s is almost surely integrable on J, and*

$$\int_{s\leq t} X_s\,ds$$

is continuous and progressively measurable.

The proofs of Lemmas 2–3 are relatively easy, and so they are left as exercises

3 Continuous-Time Martingales

3.1 Cadlag Versions

Theorem 2. *If $M = \{M_t\}_{t\geq 0}$ is a martingale relative to a standard filtration $\mathscr{F}$ then there is a version $\{M'_t\}_{t\geq 0}$ of M that has right-continuous sample paths with left limits.*

Paths $x(t)$ that are right- continuous with left limits are traditionally called *cadlag*. Recall that a *version* of a stochastic process $\{X_t\}_{t\geq 0}$ is a stochastic process $\{X'_t\}_{t\geq 0}$ such that for each $t \geq 0$,

$$X'_t = X_t \quad \text{almost surely.}$$

Because the implied sets of measure 0 depend on t, and since we are dealing with an uncountable set of time points t, it is not necessary the case that the sample paths of X and X' are the same except on a set of probability 0.

I will not prove Theorem 2, as it would take us too far afield.[1] Virtually all of the martingales one encounters in the study of stochastic calculus or Lévy processes are cadlag by construction (recall that Lévy processes have cadlag paths by definition), and so we won't have any need to quote Theorem 2.

3.2 Optional Sampling and Maximal Inequality

Let $M = \{M_t\}_{t\geq 0}$ be a continuous-time martingale relative to a standard filtration $\mathbb{F}$ with cadlag sample paths. By Lemma 1, the process M is progressively measurable, and so for any finite stopping time τ the function M_τ is measurable.

Proposition 2. *(Optional Sampling) If τ is a* bounded *stopping time then M_τ is integrable and*

$$EM_\tau = EM_0.$$

Proof. Recall that any bounded stopping time τ can be approximated from above by stopping times $\tau_k \downarrow \tau$ each of which takes values in a finite set F_k. Since the restriction of M to a finite index set is a discrete-time martingale, the Optional Sampling formula for discrete time implies that for each $k = 1, 2, \dots,$

$$EM_{\tau_k} = EM_0.$$

Since M has right-continuous sample paths, $M_\tau = \lim M_{\tau_k}$ pointwise. Therefore, to prove the theorem, we need only show that the sequence M_{τ_k} is uniformly integrable. For this, we use the hypothesis that the stopping time τ is bounded, which implies that all of the stopping times τ_k can be uniformly bounded by a constant $T < \infty$. The martingale property then implies that for each k,

$$M_{\tau_k} = E(M_T \,|\, \mathscr{F}_{\tau_k}).$$

But a standard result of measure theory states that for any random variable $X \in L^1(\Omega, \mathscr{F}, P)$, the collection $\{E(X|\mathscr{G})\}_{\mathscr{G}\subset\mathscr{F}}$, where $\mathscr{G}$ ranges over all $\sigma-$algebras contained in $\mathscr{F}$, is uniformly integrable. (Exercise: If you haven't seen this, prove it.) □

Proposition 3. *(Maximal Inequality) Let $M_T^* = \sup_{t\leq T}|M_t|$. Then for every time $T > 0$ and every $\alpha > 0$,*

$$P\{M_T^* \geq \alpha\} \leq \frac{E|M_T|}{\alpha}. \tag{8}$$

Remark 1. It follows by the moment inequality (Hölder) that if $M_T \in L^p$ for some $p > 1$ then the upper bound can be replaced by $\alpha^{-1}\|M_T\|_p$. In this case the following stronger inequality can be proved:

$$\|M_T^*\|_p \leq \frac{p}{p-1}\|M_T\|_p. \tag{9}$$

[1] You can find a proof in Kallenberg, *Foundations of Modern Probability.*

Proof of Proposition 3. Let $\mathbb{D}_k[0,T]$ be the set of nonnegative dyadic rationals $m/2^k$ that are bounded above by T. The restriction of the martingale M_t to the index set $t \in \mathbb{D}_k[0,T]$ is a discrete-time martingale, so the maximal inequality for discrete time martingales implies that (8) holds if M^{*T} is replaced by

$$M^*_{T,k} = \max_{t \in \mathbb{D}_k[0,T] \cup \{T} |M_t|.$$

But because M has cadlag paths,

$$M^*_T = \uparrow \lim_{k\to\infty} M^*_{T,k},$$

and so the inequality (8) follows routinely. □

3.3 L^2−bounded martingales.

Let $\{M_t\}_{t\geq 0}$ be a cadlag martingale relative to the filtration $\mathbb{F}$. Say that the martingale $\{M_t\}_{t\geq 0}$ is *bounded in* L^p if $\sup_{t\geq 0} E|M_t|^p < \infty$.

Proposition 4. *If $\{M_t\}_{t\geq 0}$ is bounded in L^2 then the random variables M_t converge almost surely and in L^2 as $t \to \infty$, and so for every $t \geq 0$,*

$$M_t = E(M_\infty \,|\, \mathscr{F}_t) \quad \textit{where} \quad M_\infty = L^2 - \lim_{t\to\infty} M_t. \tag{10}$$

Remark 2. This is also true when the exponent 2 is replaced by any $p > 1$, but is not true for $p = 1$: the “double-or-nothing” martingale is L^1−bounded but is not uniformly integrable and does not converge in L^1.

Proof of Proposition 4. The nice feature of L^2− martingales us that they have orthogonal increments: if $r \leq t$ and $s \geq 0$ then

$$E(M_{t+s} - M_t)(M_t - M_r) = EE(M_{t+s} - M_t \,|\, \mathscr{F}_t)(M_t - M_r) = 0.$$

Consider the embedded discrete-time martingale $\{M_k\}_{k\geq 0}$. Because this is L^2−bounded, the convergence theorem for discrete-time L^2−bounded martingales (see the notes on Discrete-Time Martingales, sec. 5) implies that $M_\infty := \lim M_k$ exists almost surely and in L^2, and

$$M_k = E(M_\infty \,|\, \mathscr{F}_k) \quad \text{for each } k \geq 0.$$

To see that the L^2 convergence extends from the discrete-time sequence $\{M_k\}_{k\geq 0}$ to the entire family $\{M_t\}_{t\geq 0}$, let $t \in [k, k+1]$ be an arbitrary real time, and observe that by the orthogonal increments property,

$$E(M_t - M_k)^2 \leq E(M_{k+1} - M_k)^2.$$

Since the sequence $\{M_k\}_{k\geq 0}$ converges in L^2 to M_∞, the norms $\|M_{k+1} - M_k\|_2 \to 0$ as $k \to \infty$. Thus, $M_t \to M_\infty$ in L^2 as $t \to \infty$ through the reals. □

Proposition 5. *Assume that the filtration $\mathbb{F}$ is standard. If τ is a stopping time and $\{M_t\}_{t\geq 0}$ is an L^2-bounded martingale then*

$$M_\tau = E(M_\infty \,|\, \mathscr{F}_\tau). \tag{11}$$

Consequently, if $\tau_0 \leq \tau_1 \leq \cdots$ are stopping times, then the sequence $\{M(\tau_n)\}_{n\geq 0}$ is a martingale relative to the discrete-time filtration $\{\mathscr{F}_{\tau_n}\}_{n\geq 0}$.

Proof. It suffices to prove the identity (11), because the tower property of conditional expectation will then imply that the sequence $\{M(\tau_n)\}_{n\geq 0}$ is a martingale, since the $\sigma-$algebras $\mathscr{F}_{\tau_n}$ increase with n. To prove (11), recall that τ is the decreasing limit of the discrete stopping times $\tau_m =$ smallest $k/2^m$ larger than τ. Since the filtration is standard, $\mathscr{F}_\tau = \cap_{n\geq 1}\mathscr{F}_{\tau_n}$, and so by the *reverse* martingale convergence theorem for discrete-time martingales,

$$E(M_\infty \,|\, \mathscr{F}_\tau) = \lim_{n\to\infty} E(M_\infty \,|\, \mathscr{F}_{\tau_n}).$$

Consequently, it suffices to prove the identity (11) for stopping times that take values in the discrete set $\{k/2^m\}_{k\geq 0}$. This is a routine EXERCISE. □

3.4 The space $\mathscr{M}_2$ of L^2-bounded *continuous* martingales

Definition 2. Let $\mathscr{M}_2$ be the linear space of all L^2-bounded martingales M_t with continuous sample paths and initial value $M_0 = 0$. Since each such martingale $M = (M_t)_{0\leq t<\infty}$ is L^2-closed, it has an L^2-limit M_∞. Define the $\mathscr{M}_2-$norm of M by

$$\|M\| = \|M_\infty\|_2. \tag{12}$$

Notation: Since we will often have to consider *sequences* of martingales, we will write $M(t)$ instead of M_t for the value of the martingale at time t, and we will use a subscript $M_n(t)$ to denote the nth of a sequence of martingales.

Proposition 6. *The space $\mathscr{M}_2$ is complete in the metric determined by the norm $\|\cdot\|$. That is, if $M_n = \{M_n(t)\}_{t\geq 0}$ is a sequence of martingales in $\mathscr{M}_2$ such that $M_n(\infty) \to M(\infty)$ in L^2, then the martingale*

$$M(t) := E(M(\infty)|\mathscr{F}_t) \tag{13}$$

has a version with continuous paths.

Remark 3. This will be of critical importance in the theory of the Itô integral, as it will guarantee that Itô integral processes have (versions with) continuous sample paths.

Proof. This is a routine consequence of the maximal inequality. First, if the sequence $\{M_n\}$ is Cauchy in $\mathscr{M}_2$ then the sequence $\{M_n(\infty)\}$ is Cauchy in L^2, by definition of the norm. Consequently, the random variables $M_n(\infty)$ converge in L^2, because the space L^2

is complete. Let $M(\infty) = \lim_{n\to\infty} M_n(\infty)$. It follows that for each t the sequence $M_n(t)$ converges in L^2 to $M(t)$, where $M(t)$ is defined by (13). Without loss of generality, assume (by taking a subsequence if necessary) that $\|M_n - M_{n+1}\| < 4^{-n}$. Then by Doob's maximal inequality,

$$P\{\sup_{t\geq 0} |M_n(t) - M_{n+1}(t)| \geq 2^{-n}\} \leq 4^{-2n}/4^{-n} = 4^{-n}.$$

Since $\sum 4^{-n} < \infty$, the Borel-Cantelli lemma implies that w.p.1, for all sufficiently large n the maximal difference between M_n and M_{n+1} will be less than 2^{-n}. Since $\sum 2^{-n} < \infty$, it follows that w.p.1 the sequence $M_n(t)$ converges uniformly in t. Since each $M_n(t)$ is continuous in t, it follows that the limit $M(t)$ must also be continuous. □

3.5 Martingales of bounded variation

Proposition 7. *Let $M(t)$ be a continuous-time martingale with continuous paths. If the paths of $M(t)$ are of bounded variation on a time interval J of positive length, then $M(t)$ is constant on J.*

Proof. Recall that a function $F(t)$ is of bounded variation on J if there is a finite constant C such that for every finite partition $\mathscr{P} = \{I_j\}$ of J into finitely many nonoverlapping intervals I_j,

$$\sum_j |\Delta_j(F)| \leq C$$

where $\Delta_j(F)$ denotes the increment of F over the interval I_j. The supremum over all partitions of J is called the *total variation* of F on J, and is denoted by $\|F\|_{TV}$. If F is continuous then the total variation of F on $[a,b] \subset J$ is increasing and continuous in b. Also, if F is continuous and J is compact then F is uniformly continuous, and so for any $\varepsilon > 0$ there exists $\delta > 0$ so that if $\max_j |I_j| < \delta$ then $\max_j |\Delta_j(F)| < \varepsilon$, and so

$$\sum_j |\Delta_j(F)|^2 \leq C\varepsilon.$$

It suffices to prove Proposition 7 for intervals $J = [0,b]$. It also suffices to consider *bounded* martingales of *bounded* total variation on J. To see this, let $M(t)$ be an arbitrary continuous martingale and let $M_n(t) = M(t \wedge \tau_n)$, where τ_n is the first time that either $|M(t)| = n$ or such that the total variation of M on the interval $[0,t]$ reaches or exceeds n. (Observe that $\tau_n = 0$ on the event that the martingale is of unbounded variation on every interval $[0,\varepsilon]$.) Since $M(t)$ is continuous, it is bounded for t in any finite interval J, by Weierstrass' theorem. Consequently, for all sufficiently large n, the function $M_n(t)$ will coincide with $M(t)$; thus, if each $M_n(t)$ is continuous on J then so is $M(t)$.

Suppose then that $M(t)$ is a bounded, continuous martingale of total variation no larger than C on J. Denote by $\mathscr{D}_n$ the sequence of dyadic partitions of $[0,b]$, and let

$\Delta_j^n = |M(jb/2^n) - M((j-1)b/2^n)|$ be the absolute differences along the dyadic partitions; then as $n \to \infty$,

$$\sum_j |\Delta_j^n(M)|^2 \longrightarrow 0 \quad \text{almost surely.}$$

Because the increments of L^2 martingales over nonoverlapping time intervals are orthogonal, if $J = [0, b]$ then for each n,

$$E(M(b) - M(0))^2 = E\sum_j |\Delta_j^n(M)|^2.$$

This sum converges to zero a.s. as $n \to \infty$. The sums are uniformly bounded because M is uniformly bounded and has uniformly bounded total variation. Therefore, by the dominated convergence theorem,

$$E(M(b) - M(0))^2 = 0.$$

Since this equation holds for every sub-interval of $[0, b]$, it follows that with probability one, $M(t) = M(0)$ for every rational t, and so by path-continuity $M(t)$ is constant on $[0, b]$. $\square$